On the commutativity of the Clifford and “extension of scalars” functors

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Abstract

We introduce sheaves of $\mathcal{A}$-modules of fractions (or just $\mathcal{A}$-*modules of fractions*), on a topological space X, with denominator a monoid-subsheaf $\mathcal{S}$ of $\mathcal{A}$; as a side worth noting result, we remark (Theorem 2.4) that there is an isomorphism between the functors $\mathcal{S}^{-1}$ and $(\mathcal{S}^{-1}\mathcal{A})\otimes -$. Moreover, we discuss the classical problem related to the commutativity of the functors: *Clifford functor Cl* and *algebra extension functor of the ground algebra K of a quadratic K-module* (M, q). As a particular case, we show (Corollary 3.5) that given a sheaf $\mathcal{A}$ of algebras on a topological space X and $\mathcal{S}$ as above, the functor $\mathcal{C}l_{\mathcal{S}^{-1}\mathcal{A}}$ commutes with the functor $\mathcal{S}^{-1}\mathcal{C}l_{\mathcal{A}}$.

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Introduction

Abstract Differential Geometry (ADG in short) ([9, 10, 11]) offers a *sheaf-theoretic* approach to classical Differential Geometry in the sense that one does not require the usual notion of differentiability in order to define fundamental concepts such as connections on principal bundles, curvature, characteristic classes, and cohomologies. In fact, ADG centers around *locally free sheaves of $\mathcal{A}$-modules* over topological spaces unlike classical Differential Geometry, which centers around vector bundles over differential manifolds. Succinctly, (differential) functions that define *differentiability* on a manifold are replaced instead by an *arbitrary sheaf $\mathcal{A}$ of algebras*, based on the underlying topological space of the manifold in question; that is, *differentiability is now characterized by the picked sheaf $\mathcal{A}$.* This sheaf of algebras may in some cases contain a tremendous amount of singularities.

Yet, a particular instance of Abstract Differential Geometry that also interests us consists of putting within this framework some recent results pertaining to *sheaves of Clifford $\mathcal{A}$-algebras* (in short, *Clifford $\mathcal{A}$-algebras*) of quadratic $\mathcal{A}$-modules, defined on arbitrary topological spaces. As is mentioned in [5, p. 287], the motivating idea of constructing *the* Clifford $\mathcal{A}$-algebra for a quadratic $\mathcal{A}$-module $(\mathcal{E}, q)$, given on any topological space X, is underpinned by the need of expressing the $\mathcal{A}$-*quadratic morphism* q as a square of an $\mathcal{A}$-morphism φ on $\mathcal{E}$. (Our emphasis on *the* is justified by the fact that Clifford $\mathcal{A}$-algebras are unique up to isomorphisms. See [14].) In fact, let $\mathcal{K}$ be a unital associative $\mathcal{A}$-algebra; a sheaf morphism $\varphi : \mathcal{E} \longrightarrow \mathcal{K}$ is called a *Clifford $\mathcal{A}$-morphism* if

$$\varphi(\_)^2 = ev(q, \_) \cdot 1(\_),$$

where: (a) $ev : \mathcal{M}or(\mathcal{E}, \mathcal{A}) \oplus \mathcal{E} \longrightarrow \mathcal{A}$ is the evaluation $\mathcal{A}$-morphism, viz

$$ev_U(\psi, s) := \psi_U(s),$$

for any open U in X and sections $s \in \mathcal{E}(U)$, $\psi \in \mathcal{M}or(\mathcal{E}, \mathcal{A})(U)$ (some standard convention: $\mathcal{M}or(\mathcal{E}, \mathcal{A})$ stands for the *sheaf of morphisms of $\mathcal{E}$ into $\mathcal{A}$*), and (b) $1 \in \mathcal{M}or_X(\mathcal{E}, \mathcal{K})$ is the constant morphism $1_V(t) = 1_{\mathcal{K}(V)}$ for every open $V \subseteq X$ and section $t \in \mathcal{E}(V)$. We also recall (see [14]) that by a *Clifford $\mathcal{A}$-algebra of a quadratic $\mathcal{A}$-module* $(\mathcal{E}, q)$, defined on a topological space X, we mean any pair $(\mathcal{C}, \varphi_{\mathcal{C}})$, where $\mathcal{C}$ is a unital associative $\mathcal{A}$-algebra on X and $\varphi_{\mathcal{C}}$ is a Clifford $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{C}$, satisfying the following properties:

(1) $\mathcal{C}$ is generated by the sub-$\mathcal{A}$-algebra $\varphi_{\mathcal{C}}(\mathcal{E})$ and the unital line sub-$\mathcal{A}$-algebra $1_{\mathcal{C}}$ of $\mathcal{C}$.

(2) Every Clifford $\mathcal{A}$-morphism $\varphi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$, where $\mathcal{K}$ is an associative and unital $\mathcal{A}$-algebra, factors through the Clifford $\mathcal{A}$-morphism $\varphi_{\mathcal{C}}$, i.e., there exist a 1-respecting $\mathcal{A}$-morphism $\Phi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{K})$ such that $\varphi = \Phi \circ \varphi_{\mathcal{C}}$.

As a general remark, we assume that *all the sheaves of $\mathcal{A}$-algebras throughout the paper are of characteristic* 0, in order to avoid pathological cases such as [6, p. 106, Example(3.1.3)], that is, cases where canonical $\mathcal{A}$-morphisms $\mathcal{A} \longrightarrow \mathcal{C}$ and $\varphi_{\mathcal{C}} : \mathcal{E} \longrightarrow \mathcal{C}$ may not be injective. Equivalently, Clifford $\mathcal{A}$-algebras are also defined as follows: Let $\mathcal{A}_{\mathcal{C}l}(\mathcal{E}, q) \equiv \mathcal{A}_{\mathcal{C}l}(\mathcal{E})$, where $(\mathcal{E}, q)$ is a quadratic $\mathcal{A}$-module on a topological space X, be the category whose objects are the Clifford $\mathcal{A}$-morphisms $\varphi \in \mathcal{H}om^{\mathcal{C}l}_{\mathcal{A}}(\mathcal{E}, \mathcal{K}) \subseteq \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$, with $\mathcal{K}$ being any associative and unital $\mathcal{A}$-algebra sheaf on X, and such that, given two objects $\varphi \in \mathcal{H}om^{\mathcal{C}l}_{\mathcal{A}}(\mathcal{E}, \mathcal{K})$ and $\psi \in \mathcal{H}om^{\mathcal{C}l}_{\mathcal{A}}(\mathcal{E}, \mathcal{L})$, a morphism $u : \varphi \longrightarrow \psi$ is an $\mathcal{A}$-morphism of $\mathcal{A}$-algebras $\mathcal{K}$ and $\mathcal{L}$ such that $\psi = u \circ \varphi$. If $\mathcal{A}_{\mathcal{C}l}(\mathcal{E}, q)$ contains an initial universal object ρ (which is unique up to $\mathcal{A}$-isomorphism), its target is called the *Clifford $\mathcal{A}$-algebra, associated with* $(\mathcal{E}, q)$; we shall denote it by $\mathcal{C}l_{\mathcal{A}}(\mathcal{E}, q) \equiv \mathcal{C}l(\mathcal{E}, q) \equiv \mathcal{C}l(\mathcal{E})$. A useful result pervading some arguments in the paper states the following [15]: *Let $(\mathcal{E}, q)$ be a quadratic $\mathcal{A}$-module on a topological space X, and consider the sheaf morphism $\Phi \equiv \otimes \circ \Delta - q : \mathcal{E} \longrightarrow \mathcal{T}\mathcal{E}$, where $\mathcal{T}\mathcal{E}$ is the tensor algebra sheaf of $\mathcal{E}$ (see* [12] *for $\mathcal{T}\mathcal{E}$) and Δ is the diagonal $\mathcal{A}$-morphism $\mathcal{E} \longrightarrow \mathcal{E} \oplus \mathcal{E}$ with $\Delta_U(s) := (s, s)$ for any open $U \subseteq X$ and section $s \in \mathcal{E}(U)$. Moreover, let $\mathcal{I}(\mathcal{E}, q)$ be the two-sided ideal sheaf in $\mathcal{T}\mathcal{E}$, generated by the presheaf $\Phi(\mathcal{E})$, and let*

$$\mathcal{C}l(\mathcal{E}, q) := \mathcal{T}(\mathcal{E})/\mathcal{I}(\mathcal{E}, q).$$

Then, the $\mathcal{A}$-morphism

$$\rho : \mathcal{E} \longrightarrow \mathcal{C}l(\mathcal{E}, q)$$

is an initial universal object in the category $\mathcal{A}_{\mathcal{C}l}(\mathcal{E}, q)$.

As for an example, consider a quadratic line $\mathcal{A}$-module $\mathcal{E}$ on a topological space X; for any open set U in X, $T(\mathcal{E}(U))$ is isomorphic to $\mathcal{A}(U)[e]$, where e is a generator of $\mathcal{E}(U)$. It is clear that $I(\mathcal{E}(U), q_U)$ is isomorphic to the ideal generated by $e^2 - q_U(e)$; thus $Cl(\mathcal{E}(U), q_U)$ is a free $\mathcal{A}(U)$-module of basis $(1, e)$. Hence, $\mathcal{C}l(\mathcal{E}) \simeq \mathcal{A}^2$.

Section 1 contains nothing essentially new, though we state the results in a novel way. A similar treatment of some of these results may be found in [7]. Section 2 is concerned with sheaves of $\mathcal{A}$-modules of fractions on arbitrary topological spaces ; its main result (Theorem 2.2) stipulates that given a sheaf $\mathcal{A}$ of unital and commutative algebras on a topological space X and $\mathcal{S}$ a sheaf of submonoids in $\mathcal{A}$, the sheaf $\mathcal{S}^{-1}\mathcal{A}$ is an algebra sheaf on X. In Theorem 2.4, we show that the functor $\mathcal{S}^{-1} : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow (\mathcal{S}^{-1}\mathcal{A})\text{-}\mathcal{M}od_X$ is exact and equivalent to the functor $\mathcal{S}^{-1}\mathcal{A} \otimes -$, i.e. $\mathcal{S}^{-1}\mathcal{E} \simeq \mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$. Finally, in Section 3, we establish the commutativity between the Clifford functor and the extension of the algebra sheaf $\mathcal{A}$ of scalars of an $\mathcal{A}$-module $\mathcal{E}$, which, in turn, gives rise to the isomorphism depicted by the diagram of Corollary 3.6.

1 Preliminaries on basic algebra sheaves

Here we lay down useful results, pertaining to changes of basic algebra sheaves in the category $\mathcal{A}\text{-}\mathcal{M}od_X$ of sheaves of $\mathcal{A}$-modules on a fixed topological space X. For instance, given a sheaf morphism $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ of *unital* and *commutative* algebra sheaves $\mathcal{A} \equiv (\mathcal{A}, \tau_{\mathcal{A}}, X)$ and $\mathcal{B} \equiv (\mathcal{B}, \tau_{\mathcal{B}}, X)$ and an $\mathcal{A}$-module $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$; $\mathcal{E}$ may be made in a natural way into a $\mathcal{B}$-module as follows: for all $x \in X$, $b \in \mathcal{B}_x$ and $e \in \mathcal{E}_x$, the product be is by definition $\varphi_x(b)e$, i.e., $\mathcal{E}_x$ is also a $\mathcal{B}_x$-module. And clearly, the "*exterior module multiplication in* $\mathcal{E}$", viz. the map

$$\mathcal{B} \circ \mathcal{E} \longrightarrow \mathcal{E} : \quad (b, e) \longmapsto be \equiv \varphi_x(b)e \in \mathcal{E}_x \subseteq \mathcal{E},$$

with $\tau_{\mathcal{B}}(b) = \pi(e) = x \in X$, *is continuous.* (For the sake of convenience, we have used the notation $\mathcal{B} \circ \mathcal{E} := \{(b, e) \in \mathcal{B} \times \mathcal{E} : \ \tau_{\mathcal{B}}(b) = \pi(e)\}$ in conformity with [9, p. 87, (1.1)].) Such an algebra sheaf morphism $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ is called an *extension* of the algebra sheaf $\mathcal{B}$, even though φ is not necessarily injective. Our terminology is different from the terminology of Mallios, [9, p. 260ff], which states the following: given two $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ on X, any short exact $\mathcal{A}$-sequence of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow 0$$

is called an $\mathcal{A}$-*extension* of $\mathcal{E}$ by $\mathcal{F}$.

Now, let us suppose that $\mathcal{E}$ is a free $\mathcal{B}$-module of finite rank n on X; we may derive from $\mathcal{E}$ two free $\mathcal{A}$-modules, called the *extensions* of $\mathcal{E}$, which are $\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}$ and $\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})$; in this vein, see [13] for *complexification* of $\mathcal{A}$-modules, which is defined to be the process of obtaining new free $\mathcal{A}$-modules by *enlarging the* $\mathbb{R}$-*algebra sheaf* $\mathcal{A}$ *to a* $\mathbb{C}$-*algebra sheaf,* denoted $\mathcal{A}_{\mathbb{C}}$. Indeed, for any $x \in X$, an element $a \in \mathcal{A}_x$ multiplies an element $a'\otimes e \in (\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E})_x = \mathcal{A}_x\otimes_{\mathcal{B}_x}\mathcal{E}_x = \mathcal{A}_x\otimes_{\mathcal{B}_x}(\mathcal{B}_x)^n = (\mathcal{A}_x)^n$ (the last three equalities actually stand for $\mathcal{B}_x$-isomorphisms; to corroborate this fact, see [9, p.123, (3.18); p.130, (5.9); p.131, (5.18)]) or an element $z \in \mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})_x = \mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{B}^n)_x = (\mathcal{A}^*)^n_x = (\mathcal{A}^*_x)^n = \mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x,\mathcal{B}^n_x)$, with these equalities being valid within $\mathcal{B}_x$-isomorphisms, in the following way:

$$a(a'\otimes e) = (aa')\otimes e \quad \text{and} \quad (az)(a') = z(aa').$$

Yet, still under the assumption that $\mathcal{E}$ is a free $\mathcal{B}$-module of finite rank on a topological space X, and $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ a sheaf morphism of unital and commutative algebra sheaves $\mathcal{A}$ and $\mathcal{B}$, the next lemma is related to the canonical $\mathcal{B}$-morphisms, $\mathcal{E} \longrightarrow \mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}$ and $\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E}) \longrightarrow \mathcal{E}$, given, for any $x \in X$, $e \in \mathcal{E}_x$ and $z \in \mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})_x \simeq \mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x,\mathcal{E}_x)$, by $e \longmapsto 1_{\mathcal{A}_x}\otimes e$ and $z \longmapsto z(1_{\mathcal{A}_x})$, respectively; the former is not always surjective, whereas the latter is not always injective. When $\mathcal{A} = \mathcal{B}$, these $\mathcal{B}$-morphisms are bijective for *any* given $\mathcal{B}$-module $\mathcal{E}$, *not necessarily free*, and both $\mathcal{B}\otimes_{\mathcal{B}}\mathcal{E}$ and $\mathcal{H}om_{\mathcal{B}}(\mathcal{B},\mathcal{E})$ are $\mathcal{B}$-isomorphic to $\mathcal{E}$.

Lemma 1.1 *Let $\mathcal{A}$, $\mathcal{B}$ be unital and commutative algebra sheaves on a topological space X, $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ a surjective sheaf morphism, and $\mathcal{E} \equiv (\mathcal{E},\pi,X)$ a locally free $\mathcal{B}$-module of rank n (i.e. a vector sheaf). Then, $\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}$ is canonically $\mathcal{B}$-isomorphic to the quotient $\mathcal{E}/(\ker\varphi)\mathcal{E}$, and $\mathcal{H}om_{\mathcal{B}}(\mathcal{A},\mathcal{E})$ is $\mathcal{B}$-isomorphic to the sub-$\mathcal{B}$-module of $\mathcal{E}$, whose stalks consist of elements $z \in \mathcal{E}_x$ ($\pi(z) = x \in X$) such that $(\ker\varphi)_x z = (\ker(\varphi_x))z = 0_x$.*

Proof. Let $\iota : \ker\varphi \longrightarrow \mathcal{B}$ be the natural injection, then, clearly,

$$0 \longrightarrow \ker\varphi \xrightarrow{\iota} \mathcal{B} \xrightarrow{\varphi} \mathcal{A} \longrightarrow 0 \tag{1}$$

is exact. Tensoring (1) with the vector sheaf $\mathcal{E}$ yields an exact $\mathcal{B}$-sequence (see [9, p.131, Theorem 5.1]), viz.

$$0 \longrightarrow \ker\varphi\otimes_{\mathcal{B}}\mathcal{E} \longrightarrow \mathcal{B}\otimes_{\mathcal{B}}\mathcal{E} \longrightarrow \mathcal{A}\otimes_{\mathcal{B}}\mathcal{E} \longrightarrow 0. \tag{2}$$

Note that, for $x \in X$,

$$(\ker \varphi \otimes_{\mathcal{B}} \mathcal{E})_x = \ker \varphi_x \otimes_{\mathcal{B}_x} \mathcal{E}_x = (\ker \varphi_x)\mathcal{E}_x = ((\ker \varphi)\mathcal{E})_x, \tag{3}$$

within $\mathcal{B}_x$-isomorphisms (*the second* $\mathcal{B}_x$*-isomorphism in* (3) *is a classical result*; see, for instance, the proof of [6, p.18, Lemma 1.9.1]). $(\ker \varphi)\mathcal{E}$ is the $\mathcal{B}$-module, obtained by sheafifying the presheaf

$$U \longmapsto \langle (\ker \varphi_U)\mathcal{E}(U)\rangle,$$

where $\langle (\ker \varphi_U)\mathcal{E}(U)\rangle$ is the $\mathcal{B}(U)$-module generated by the set $(\ker \varphi_U)\mathcal{E}(U)$, that is, the set of $t \in \mathcal{E}(U)$ such that $t = \alpha \cdot s$, with $\alpha \in \ker \varphi_U$ and $s \in \mathcal{E}(U)$. The restriction maps for this presheaf are obvious. It follows from (3) that

$$\ker \varphi \otimes_{\mathcal{B}} \mathcal{E} = (\ker \varphi)\mathcal{E}, \tag{4}$$

within $\mathcal{B}$-isomorphism. Since $\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E} = \mathcal{E}$ within $\mathcal{B}$-isomorphism, it follows, taking also account of (4), that $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E} = \mathcal{E}/(\ker \varphi)\mathcal{E}$ within $\mathcal{B}$-isomorphism.

For any $x \in X$, the following sequence of $\mathcal{B}_x$-modules, namely

$$0 \longrightarrow \mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x) \xrightarrow{\mu} \mathrm{H}om_{\mathcal{B}_x}(\mathcal{B}_x, \mathcal{E}_x) \xrightarrow{\nu} \mathrm{H}om_{\mathcal{B}_x}(\ker(\varphi_x), \mathcal{E}_x)\ , \tag{5}$$

where $\mu := \mathrm{H}om_{\mathcal{B}_x}(\varphi_x^*, \mathcal{E}_x)$ and $\nu := \mathrm{H}om_{\mathcal{B}_x}(\iota_x^*, \mathcal{E}_x)$, is exact. ($\mu$ and ν are given by: for $f \in \mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x)$, $\mu(f) = \mathrm{H}om_{\mathcal{B}_x}(\varphi_x^*, \mathcal{E}_x)(f) := f \circ \varphi_x$; similarly, for $g \in \mathrm{H}om_{\mathcal{B}_x}(\mathcal{B}_x, \mathcal{E}_x)$, $\nu(g) = \mathrm{H}om_{\mathcal{B}_x}(\iota_x^*, \mathcal{E}_x)(g) := g \circ \iota_x$.) . For (5), see, for instance, [3, p.227, Theorem 1]. The exactness of (5) implies that $\mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x)$ is $\mathcal{B}_x$-isomorphic to the sub-$\mathcal{B}_x$-module of $\mathrm{H}om_{\mathcal{B}_x}(\mathcal{B}_x, \mathcal{E}_x) \simeq \mathcal{E}_x$ consisting of z such that $\mathrm{H}om_{\mathcal{B}_x}(\iota_x^*, \mathcal{E}_x)(z) = 0_x$, i.e., $\ker(\varphi_x)z = 0_x$. Obviously, if $\mathcal{U} \equiv (U_\alpha)_{\alpha\in I}$ is a *local frame* of $\mathcal{E}$, i.e., for all $\alpha \in I$, $\mathcal{E}|_{U_\alpha} = \mathcal{B}^n|_{U_\alpha}$, within $\mathcal{B}|_{U_\alpha}$-isomorphism, then $\mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{E})(U_\alpha) = (\mathcal{A}^*|_{U_\alpha})^n$, within $\mathcal{B}|_{U_\alpha}$-isomorphism; consequently, for any $x \in X$, $\mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{E})_x = (\mathcal{A}_x^*)^n$ within $\mathcal{B}_x$-isomorphism. On the other hand, $\mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x) = \mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{B}_x^n) = (\mathcal{A}_x^*)^n$, within $\mathcal{B}_x$-isomorphism (cf. [9, p.299, (5.8)]). Thus, $\mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{E})_x = \mathrm{H}om_{\mathcal{B}_x}(\mathcal{A}_x, \mathcal{E}_x)$, within $\mathcal{B}_x$-isomorphism. Hence, for all $x \in X$, the corresponding stalk $\mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{E})_x$ is $\mathcal{B}_x$-isomorphic to the sub-$\mathcal{B}_x$-module of $\mathcal{E}_x$ consisting of z such that $\ker(\varphi_x)z = 0_x$. ■

For the purpose of the sequel, while keeping with the notations of Lemma 1.1, with the exception that the $\mathcal{B}$-module $\mathcal{E}$ is not necessarily locally free, we shall need the following $\mathcal{A}$-isomorphisms:

$$\mathcal{T}_{\mathcal{A}}(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}) = \mathcal{A}\otimes_{\mathcal{B}}\mathcal{T}_{\mathcal{B}}(\mathcal{E}) \tag{6}$$

and

$$\mathcal{S}_{\mathcal{A}}(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}) = \mathcal{A}\otimes_{\mathcal{B}}\mathcal{S}_{\mathcal{B}}(\mathcal{E}), \tag{7}$$

where $\mathcal{T}_{\mathcal{B}}(\mathcal{E})$ and $\mathcal{S}_{\mathcal{B}}(\mathcal{E})$ are *tensor algebra* and *symmetric algebra sheaves of $\mathcal{E}$ on X,* respectively; they both are *sheaves of $\mathcal{B}$-algebras* (or *$\mathcal{B}$-algebras*, for short) on X. These two notions are defined in a way similar to the way the notion of exterior algebra of a $\mathcal{B}$-module $\mathcal{E}$ on a topological space X is defined. For convenience, recall (cf. [9, pp.307-315]) that the *exterior algebra of $\mathcal{E}$*, yet denoted $\bigwedge\mathcal{E}$, is given by

$$\bigwedge\mathcal{E} := \oplus_{n=0}^{\infty}\bigwedge{}^{n}\mathcal{E},$$

where for every integer $n \geq 2$, the *n-th exterior power of $\mathcal{E}$*, i.e. $\bigwedge^{n}\mathcal{E}$, is defined as the *sheafification of the presheaf of $\mathcal{B}(U)$-algebras*

$$U \longmapsto \bigwedge{}^{n}(\mathcal{E}(U)) \equiv \bigwedge{}^{n}_{\mathcal{B}(U)}(\mathcal{E}(U)),$$

where U ranges over the open subsets of X. As per the classical theory, for $n = 0, 1$, one sets

$$\bigwedge{}^{0}(\mathcal{E}(U)) := \mathcal{B}(U) \quad \text{and} \quad \bigwedge{}^{1}(\mathcal{E}(U)) := \mathcal{E}(U),$$

so that

$$\bigwedge{}^{0}\mathcal{E} = \mathcal{B} \quad \text{and} \quad \bigwedge{}^{1}\mathcal{E} = \mathcal{E},$$

within $\mathcal{B}$-isomorphisms.

Note that the exterior algebra $\bigwedge\mathcal{E}$ can be constructed (ibid.) as the sheaf generated by the presheaf of $\mathcal{B}(U)$-algebras, given by the correspondence

$$U \longmapsto \oplus_{n=0}^{\infty}\bigwedge{}^{n}(\mathcal{E}(U)) \equiv \bigwedge\mathcal{E}(U),$$

where U is open in X. For the tensor algebra sheaf $\mathcal{T}\mathcal{E}$ of $\mathcal{E}$ on X, one defines the n-th ($n \geq 2$) *tensor power of $\mathcal{E}$*, denoted $\mathcal{T}^{n}\mathcal{E}$, as the *sheafification of the $\Gamma(\mathcal{A})$-presheaf* $(U \to T^{n}\mathcal{E}(U))_{X\supseteq U,\ open}$ (the restriction maps of this presheaf are obvious). On account of [9, p.101, (1.54)], it follows that $\mathcal{T}^{n}\mathcal{E}$ *is also a $\mathcal{B}$-module on X, for any given $\mathcal{B}$-module $\mathcal{E}$ on X and every integer $n \in \mathbb{N}$,* *with* $n \geq 2$. Furthermore, again by following the classical pattern, we set, for any open U in X,

$$T^0\mathcal{E}(U) := \mathcal{B}(U) \quad \text{and} \quad T^1\mathcal{E}(U) := \mathcal{E}(U),$$

so that one obtains

$$\mathcal{T}^0\mathcal{E} = \mathcal{B} \quad \text{and} \quad \mathcal{T}^1\mathcal{E} = \mathcal{E},$$

within $\mathcal{B}$-isomorphisms. In a similar way, one defines the symmetric algebra sheaf $\mathcal{SE}$ of $\mathcal{E}$ on X. We now show that the $\mathcal{B}$-algebra $\mathcal{T}_{\mathcal{B}}(\mathcal{E})$ may be constructed *equivalently* as the sheaf generated by the presheaf $T\mathcal{E} \equiv ((T\mathcal{E})(U) := T(\mathcal{E}(U)))_{X \supseteq U,\ open}$ of $\mathcal{B}(U)$-algebras, given by

$$U \longmapsto \oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)) \equiv T(\mathcal{E}(U)),$$

where $U \subseteq X$ is open, along with the obvious restriction maps. Indeed, with every open $U \subseteq X$, one associates the following (canonical) $\mathcal{B}(U)$-morphism

$$T\mathcal{E}(U) \equiv \oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)) \xrightarrow{\phi_U} (\mathcal{T}_{\mathcal{B}}\mathcal{E})(U) := \oplus_{n=0}^{\infty}(\mathcal{T}_{\mathcal{B}}^n\mathcal{E})(U); \tag{8}$$

therefore one obtains a morphism $\overline{\phi}$ of the sheaves, generated by the presheaves of $\mathcal{B}(U)$-algebras, which appear in the two members of (8). It suffices to prove that $\overline{\phi}$ is a fiber-wise $\mathcal{B}$-isomorphism. To this end, we observe the following $\mathcal{B}_x$-isomorphisms

$$\begin{aligned}(\mathbf{S}(T\mathcal{E}))_x &= (\oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)))_x = \varinjlim_{x \in U}(\oplus_{n=0}^{\infty} T^n(\mathcal{E}(U)))\\ &\equiv \varinjlim_{x\in U}(T(\mathcal{E}(U))) = T(\varinjlim_{x\in U}\mathcal{E}(U)) = T(\mathcal{E}_x) = \oplus_{n=0}^{\infty}(T^n(\mathcal{E}(U)))_x\\ &= \oplus_{n=0}^{\infty}(\varinjlim_{x\in U} T^n(\mathcal{E}(U))) = (\oplus_{n=0}^{\infty}\mathcal{T}^n\mathcal{E})_x \equiv (\mathcal{T}\mathcal{E})_x,\end{aligned}$$

for every $x \in X$.

Since, by virtue of [9, p.130, (5.9)],

$$\mathcal{T}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E})_x = (\mathcal{T}_{\mathcal{A}})_x(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E})_x = \mathcal{T}_{\mathcal{A}_x}(\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{E}_x) = \mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{T}_{\mathcal{B}_x}(\mathcal{E}_x),$$

within $\mathcal{A}_x$-isomorphisms (see [6, p.18] for the last $\mathcal{A}_x$-isomorphism), it follows, on the basis of [9, p.68, Theorem 12.1], that (6) is fulfilled. Likewise, one obtains (7).

Now, the next theorem is a connection between the functor $\mathcal{H}om$ and the bifunctor $\otimes$, which is classically called the *adjoint associativity of $\mathcal{H}om$ and tensor product.* This result is also established by Kashiwara and Schapira [7, p.439, Proposition 18.2.3(ii)] for presheaves and sheaves constructed on a *site X with values in a certain category with suitable properties.* We recall that a *site X is a small category $\mathcal{C}_X$ endowed with a Grothendieck topology.* The adjunction associativity formula suggests the following: *Let $\mathcal{R}$ be a sheaf of rings on a site X, and k_X a sheaf of k-algebras on X, where k denotes a commutative unital ring. If we denote by $PSh(\mathcal{R})$ the category of presheaves of $\mathcal{R}$-modules, then, given $F \in PSh(\mathcal{R}^{op})$, $G \in PSh(\mathcal{R})$ and $H \in PSh(k_X)$, there is an isomorphism*

$$\mathcal{H}om_{k_X}(F \otimes_{\mathcal{R}} G, H) \simeq \mathcal{H}om_{\mathcal{R}}(G, \mathcal{H}om_{k_X}(F, H)), \tag{9}$$

functorial in F, G and H. (The notations used are those of [7, p.439, Proposition 18.2.3(ii)].) For the purpose of *a version of the adjunction associativity formula* in our setting, let us notice that *given algebra sheaves $\mathcal{K}$ and $\mathcal{L}$ on a given topological space X, an $(\mathcal{K}, \mathcal{L})$-bimodule $\mathcal{E}$ on X and a left $\mathcal{K}$-module $\mathcal{F}$ on X, the sheaf $\mathcal{H}om_{\mathcal{K}}(\mathcal{E}, \mathcal{F})$ is a left $\mathcal{L}$-module.* We assume that all the sheaves involved in Theorem 1.2 are defined on a given topological space X.

Theorem 1.2 *Let $\mathcal{A}$, $\mathcal{B}$ be unital and commutative algebra sheaves, $\mathcal{E}$ a locally free left $\mathcal{B}$-module of rank m, $\mathcal{G}$ a left $\mathcal{A}$-module. Moreover, let $\mathcal{F}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule such that as a left $\mathcal{A}$-module, $\mathcal{F}$ is locally free and of rank n. Then,*

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}, \mathcal{G}) \tag{10}$$

within isomorphism of group sheaves.

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be local frames of $\mathcal{E}$ and $\mathcal{F}$, respectively. That $\mathcal{W} \equiv \mathcal{U} \cap \mathcal{V} := \{U \cap V : \ U \in \mathcal{U}, \ V \in \mathcal{V}\}$ is a common local frame of $\mathcal{E}$ and $\mathcal{F}$ is clear. So, if $U \in \mathcal{W}$, then, applying [9, p. 137, (6.22), (6.23), (6.24′)], one has the following $\mathcal{B}|_U$-isomorphisms:

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))|_U = \mathcal{H}om_{\mathcal{B}|_U}(\mathcal{B}^m|_U, \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{A}^n, \mathcal{G}|_U)),$$

that is,

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))|_U = \mathcal{G}^{mn}|_U. \tag{11}$$

In the same way, one shows that

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}, \mathcal{G})|_U = \mathcal{G}^{mn}|_U \tag{12}$$

within a $\mathcal{A}|_U$-isomorphism. On the other hand, for any open subset W of X, one has the following morphism

$$\mathrm{H}om_{\mathcal{B}|_W}(\mathcal{E}|_W, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})|_W) \xrightarrow{\varphi_W} \mathrm{H}om_{\mathcal{A}|_W}(\mathcal{E}|_W \otimes_{\mathcal{B}|_W} \mathcal{F}|_W, \mathcal{G}|_W),$$

which is given by

$$\varphi_W(\alpha)(s \otimes t) := (\alpha_Z(s))_Z(t) \equiv \alpha(s)(t),$$

where $\alpha \in \mathrm{H}om_{\mathcal{B}|_W}(\mathcal{E}|_W, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})|_W)$, $s \in (\mathcal{E}|_W)(Z) = \mathcal{E}(Z)$, $t \in \mathcal{F}(Z)$, with Z a subopen of W. Clearly, the family $\varphi \equiv (\varphi_W)_{X \supseteq W,\ open}$ yields an $\mathcal{A}$-morphism. We shall indeed show that the sheafification $\mathbf{S}(\varphi) \equiv \widetilde{\varphi}$ of φ is an $\mathcal{A}$-isomorphism. For this purpose, we notice that, by virtue of (11) and (12),

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))_x = \mathcal{G}^{mn}_x = \mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}, \mathcal{G})_x, \tag{13}$$

for any $x \in X$. The equalities in (13) are valid up to group isomorphisms. Furthermore, as

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))_x = \mathrm{H}om_{\mathcal{B}_x}(\mathcal{E}_x, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x)$$

and

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F}, \mathcal{G})_x = \mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x \otimes_{\mathcal{B}_x} \mathcal{F}_x, \mathcal{G}_x),$$

for any $x \in X$, φ_x is an $\mathcal{B}_x$-isomorphism (see [1, p. 198, Theorem 15.6]). Whence, by [9, p. 68, Theorem 12.1], φ is an $\mathcal{A}$-isomorphism, and the proof is complete. ■

When $\mathcal{E}$ is a locally free $\mathcal{B}$-module of finite rank as in Theorem 1.2, and $\mathcal{F}$ an $\mathcal{A}$-module, clearly one has the following canonical $\mathcal{A}$-isomorphisms

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{F}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{B}} \mathcal{A}, \mathcal{F}) = \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}). \tag{14}$$

Moreover, by [9, p.130, (5.14) and (5.15)], one has

$$\mathcal{E} \otimes_{\mathcal{B}} \mathcal{F} = (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}) \otimes_{\mathcal{A}} \mathcal{F} \tag{15}$$

within $\mathcal{A}$-isomorphism.

The fact that the category $\mathcal{A}$-$\mathcal{M}od_X$ of sheaves of $\mathcal{A}$-modules, where $\mathcal{A}$ is commutative and unital, on a fixed topological space X is an *abelian category* [9, p.158] heralds the following.

Definition 1.3 (i) An object $\mathcal{P} \in \mathcal{A}$-$\mathcal{M}od_X$ is *projective* if the functor

$$\mathrm{H}om_{\mathcal{A}}(\mathcal{P}, \cdot) : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow \mathcal{A}(X)\text{-}\mathrm{M}od, \tag{16}$$

where $\mathcal{A}(X)$-Mod is the category of modules over the algebra $\mathcal{A}(X)$, is exact.
(ii) An object $\mathcal{F} \in \mathcal{A}$-$\mathcal{M}od_X$ is *flat* if the functor $\mathcal{E} \longmapsto \mathcal{F} \otimes_{\mathcal{A}} \mathcal{E}$ is exact.
(iii) Given a unital and commutative algebra sheaf $\mathcal{B}$ on X, an *extension* $\mathcal{B} \longrightarrow \mathcal{A}$ is called *flat* if $\mathcal{A}$ is a flat $\mathcal{B}$-module.

Lemma 1.4 *Let $\mathcal{A}$, $\mathcal{B}$ be unital and commutative algebra sheaves on a given topological space X, and $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ a unity-preserving sheaf morphism. For any locally free $\mathcal{B}$-module $\mathcal{E}$ of finite rank on X, $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}$ is $\mathcal{A}$-projective if $\mathcal{E}$ is $\mathcal{B}$-projective. On the other hand, for any $\mathcal{B}$-module $\mathcal{F}$ on X, the $\mathcal{A}$-module $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}$ is $\mathcal{A}$-flat if $\mathcal{F}$ is $\mathcal{B}$-flat.*

Proof. By the $\mathcal{A}$-isomorphism (14), one has

$$\begin{aligned}\mathrm{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{F}) &= \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{F})(X) \\ &= \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})(X) = \mathrm{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}).\end{aligned}$$

Therefore, if $\mathrm{H}om_{\mathcal{B}}(\mathcal{E}, \cdot)$ is exact, then $\mathrm{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \cdot)$ is exact, which means that $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}$ is $\mathcal{A}$-projective whenever $\mathcal{E}$ is $\mathcal{B}$-projective. The remaining assertion is also easy to prove. ■

Now, as in Lemma 1.4, we assume that $\mathcal{E}$ is a locally free $\mathcal{B}$-module of rank n, and $\mathcal{F}$ any $\mathcal{B}$-module, both on the same topological space X. Moreover, let us consider the following canonical $\mathcal{A}$-morphism

$$\Phi \equiv (\Phi_x)_{x \in X} : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}) \tag{17}$$

such that, for all $x \in X$,

$$\Phi_x(a \otimes z)(a' \otimes e) := aa' \otimes z(e), \tag{18}$$

where $a,\ a' \in \mathcal{A}_x,\ z \in \mathcal{H}om_{\mathcal{B}}(\mathcal{E},\mathcal{F})_x = \varinjlim_{U\ni x} \mathrm{H}om_{\mathcal{B}|_U}((\mathcal{B}|_U)^n, \mathcal{F}|_U) = \mathcal{F}_x^n = \mathrm{H}om_{\mathcal{B}_x}(\mathcal{B}_x^n, \mathcal{F}_x)$ (with U a *local gauge* of $\mathcal{E}$) and $e \in \mathcal{E}_x$. Observe the following $\mathcal{A}_x$-isomorphisms

$$\begin{aligned}
\mathcal{H}om_{\mathcal{A}}(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{E}, \mathcal{A}\otimes_{\mathcal{B}}\mathcal{F})_x &= \varinjlim_{U\ni x} \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{E}|_U, \mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{F}|_U)\\
&= \varinjlim_{U\ni x} \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{A}|_U \otimes_{\mathcal{B}|U} (\mathcal{B}|_U)^n, \mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{F}|_U)\\
&= \varinjlim_{U\ni x} \mathrm{H}om_{\mathcal{A}|_U}((\mathcal{A}|_U)^n, \mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{F}|_U)\\
&= \varinjlim_{U\ni x} (\mathcal{A}|_U \otimes_{\mathcal{B}|U} \mathcal{F}|_U)^n\\
&= (\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{F}_x)^n
\end{aligned}$$

(U is a local gauge of $\mathcal{E}$); therefore, Φ is well defined.

Lemma 1.5 *Let $\mathcal{E}$ be a locally free $\mathcal{B}$-module of rank n on a topological space X such that every stalk $\mathcal{E}_x$ is $\mathcal{B}_x$-projective. Then, $\mathcal{E}$ is $\mathcal{B}$-projective.*

Proof. In fact, let

$$0 \longrightarrow \mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}'' \longrightarrow 0$$

be a $\mathcal{B}$-exact sequence. Since exactness is transferred to stalks of sheaves (cf. [9, p.113, (2.34)]) and, for any $x \in X$ and $\mathcal{B}$-module $\mathcal{F}$ on X, $\mathrm{H}om_{\mathcal{B}_x}(\mathcal{E}_x, \mathcal{F}_x) = \mathcal{F}_x^n = \mathcal{H}om_{\mathcal{B}}(\mathcal{E},\mathcal{F})_x$, one has

$$0 \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E},\mathcal{S}')_x \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E},\mathcal{S})_x \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E},\mathcal{S}'')_x \longrightarrow 0. \quad (19)$$

The exactness of (19) follows from the fact that $\mathcal{E}_x$, $x \in X$, is $\mathcal{B}_x$-projective (see, for instance, [3, p.231, Proposition 4]). On the other hand, since *any complex $\mathcal{G}' \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{G}''$ of sheaves is exact if and only if, for any $x \in X$, the induced complex $\mathcal{G}'_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{G}''_x$ is exact* (cf. [16, p.99, Proposition 5.3.4], yet, [9, p. 113, (2.34)]), one obtains the following exact $\mathcal{B}$-sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E},\mathcal{S}') \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E},\mathcal{S}) \longrightarrow \mathcal{H}om_{\mathcal{B}}(\mathcal{E},\mathcal{S}'') \longrightarrow 0;$$

whence the vector sheaf $\mathcal{E}$ is $\mathcal{B}$-projective. ■

Keeping with the notations of Lemma 1.4, we have.

Theorem 1.6 *Suppose that every stalk $\mathcal{E}_x$ of the vector sheaf $\mathcal{E}$ is $\mathcal{B}_x$-projective, then the canonical sheaf morphism* (17) *is an $\mathcal{A}$-isomorphism.*

Proof. Since every $\mathcal{E}_x$ is projective, it follows, by means of [6, p.19, Proposition 1.9.7], that

$$\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})_x = \mathcal{H}om_{\mathcal{A}_x}(\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{E}_x, \mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{F}_x)$$

within $\mathcal{A}_x$-isomorphism; whence the $\mathcal{A}$-morphism Φ is an $\mathcal{A}$-isomorphism (see, for instance, [9, p.68, Theorem 12.1]). ■

Definition 1.7 [7, p.446, Definition 18.5.1] An $\mathcal{A}$-module $\mathcal{E}$ on a topological space X is said to be *locally finitely presented* if there is an open covering $\mathcal{U} \equiv (U_\alpha)_{\alpha \in I}$ of X such that, for every $\alpha \in I$, the $\mathcal{A}$-sequence

$$(\mathcal{A}|_{U_\alpha})^m = \mathcal{A}^m|_{U_\alpha} \longrightarrow \mathcal{A}^n|_{U_\alpha} \longrightarrow \mathcal{E}|_{U_\alpha} \longrightarrow 0, \tag{20}$$

where m, $n \in \mathbb{N}$, is exact.

We shall state a *useful property of flat $\mathcal{A}$-extensions,* which stipulates that *under certain conditions the functors $\otimes$ and $\mathcal{H}om$ commute.*

First, let us recall the following result; cf. [17, p.114, ($\star$) and subsequent remarks].

Lemma 1.8 *Let $(X, \mathcal{A})$ be an algebraized space and $\mathcal{E}$ a finitely presented $\mathcal{A}$-module on X. Then, for any $\mathcal{A}$-module $\mathcal{F}$ on X and $x \in X$, the natural morphism*

$$(\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F}))_x \longrightarrow \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{E}_x, \mathcal{F}_x) \tag{21}$$

is an $\mathcal{A}_x$-isomorphism.

Theorem 1.9 *Let $\mathcal{A}$, $\mathcal{B}$ be unital commutative algebra sheaves on a topological space X, and $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ a flat extension. For any locally finitely presented $\mathcal{B}$-module $\mathcal{E}$ on X,*

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}) \tag{22}$$

within $\mathcal{A}$-isomorphism, for any $\mathcal{B}$-module $\mathcal{F}$ on X.

Proof. That (22) holds for free $\mathcal{B}$-modules of finite rank is obvious. In fact, suppose that $\mathcal{E} \simeq \mathcal{B}^n$ $(n \in \mathbb{N})$, then, one has,

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}^n = (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})^n, \tag{23}$$

with the preceding equalities being valid within $\mathcal{A}$-isomorphisms. On the other hand,

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{A}^n, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}) \simeq (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})^n. \tag{24}$$

Fix $x \in X$; if $a \in \mathcal{A}_x$, $z \in \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})_x = \mathcal{F}_x^n$,

$$\Phi_x(a \otimes z)(a' \otimes e) := aa' \otimes z(e) = 0$$

for all $a' \in \mathcal{A}_x$ and $e \in \mathcal{E}_x$, $a \otimes z = 0$; this implies that Φ_x is injective. Moreover, since both $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})$ and $\mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})$ are free as finite direct sums of $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}$ (cf. (23) and (24)), it follows that Φ_x is bijective. Hence, Φ is an $\mathcal{A}$-isomorphism (see, for instance, [9, p.68, Theorem 12.1]).

Now, let us suppose that $\mathcal{E}$ is properly locally finitely presented; so, for every $x \in X$, there are an open set $U \subseteq X$ and locally free $\mathcal{B}$-modules $\mathcal{E}_1$ and $\mathcal{E}_0$ of finite rank such that

$$\mathcal{E}_1|_U \simeq \mathcal{B}^m|_U \longrightarrow \mathcal{E}_0|_U \simeq \mathcal{B}^n|_U \longrightarrow \mathcal{E}|_U \longrightarrow 0$$

is exact. One thus obtains, by virtue of Lemma 1.8, the following diagram, for every $x \in X$,

$$0 \longrightarrow \mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_x, \mathcal{F}_x) \longrightarrow \mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_{0x}, \mathcal{F}_x) \longrightarrow \mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_{1x}, \mathcal{F}_x),$$

with $\mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_{0x}, \mathcal{F}_x) = \mathrm{H}om(\mathcal{A}_x \otimes \mathcal{E}_{0x}, \mathcal{A}_x \otimes \mathcal{F}_x)$ and $\mathcal{A}_x \otimes \mathrm{H}om(\mathcal{E}_{1x}, \mathcal{F}_x) = \mathrm{H}om(\mathcal{A}_x \otimes \mathcal{E}_{1x}, \mathcal{A}_x \otimes \mathcal{F}_x)$ within $\mathcal{A}_x$-isomorphisms. On the other hand, since

$$0 \longrightarrow \mathrm{H}om(\mathcal{A}_x \otimes \mathcal{E}_x, \mathcal{A}_x \otimes \mathcal{F}_x) \longrightarrow \mathrm{H}om(\mathcal{A}_x \otimes \mathcal{E}_{0x}, \mathcal{A}_x \otimes \mathcal{F}_x),$$

it follows

$$\begin{aligned} \mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F})_x &= \mathcal{A}_x \otimes_{\mathcal{B}_x} \mathrm{H}om_{\mathcal{B}_x}(\mathcal{E}_x, \mathcal{F}_x) \\ &= \mathrm{H}om_{\mathcal{A}_x}(\mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{E}_x, \mathcal{A}_x \otimes_{\mathcal{B}_x} \mathcal{F}_x) = \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F})_x \end{aligned}$$

within $\mathcal{A}_x$-isomorphisms. Since the last $\mathcal{A}_x$-isomorphisms hold for any $x \in X$,

$$\Phi : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{E}, \mathcal{A} \otimes_{\mathcal{B}} \mathcal{F}),$$

where Φ_x is given by (18), is an $\mathcal{A}$-isomorphism. ■

2 $\mathcal{A}$-modules of fractions

Definition 2.1 Let X be a topological space, $\mathcal{A} \equiv (\mathcal{A}, \pi, X)$ a sheaf of unital and commutative algebras, and $\mathcal{S} \equiv (\mathcal{S}, \pi|_{\mathcal{S}}, X)$ a sheaf of submonoids in $\mathcal{A}$. A **sheaf of algebras of fractions of $\mathcal{A}$ by $\mathcal{S}$** is a sheaf of algebras, denoted $\mathcal{S}^{-1}\mathcal{A}$, such that, for every point $x \in X$, the corresponding stalk $(\mathcal{S}^{-1}\mathcal{A})_x$ is an *algebra of fractions of $\mathcal{A}_x$ by $\mathcal{S}_x$*; in other words,

$$(\mathcal{S}^{-1}\mathcal{A})_x = \mathcal{S}_x^{-1}\mathcal{A}_x \tag{25}$$

for all $x \in X$.

Explicitly, fix x in X; the stalk $\mathcal{S}_x$ is a submonoid of the unital and commutative algebra $\mathcal{A}_x$. The algebra of fractions of $\mathcal{A}_x$ by $\mathcal{S}_x$ is defined (see [8, pp. 107- 111]) by considering the equivalence relation, on the set $\mathcal{A}_x \times \mathcal{S}_x$:

$$(r, s) \sim (r', s')$$

provided there exists an element $t \in \mathcal{S}_x$ such that

$$t(s'r - sr') = 0.$$

The equivalence class containing a pair (r, s) is denoted by $\frac{r}{s}$, and the set of all equivalence classes is denoted by $\mathcal{S}_x^{-1}\mathcal{A}_x$. The set $\mathcal{S}_x^{-1}\mathcal{A}_x$ becomes an algebra by virtue of the operations

$$\frac{r}{s} + \frac{r'}{s'} := \frac{s'r + sr'}{ss'}$$

and

$$\frac{r}{s}\frac{r'}{s'} := \frac{rr'}{ss'}.$$

Theorem 2.2 *$\mathcal{S}^{-1}\mathcal{A}$ is an algebra sheaf on X.*

Proof. Let us consider the projection map

$$q : \mathcal{A} \circ \mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{A} \tag{26}$$

given by

$$q_x(r, s) := \frac{r}{s}, \tag{27}$$

for every $x \in X$, $r \in \mathcal{A}_x$ and $s \in \mathcal{S}_x$. ($\mathcal{A} \circ \mathcal{S}$ is the subsheaf of the sheaf $\mathcal{A} \times \mathcal{S}$, given by $\mathcal{A} \circ \mathcal{S} := \{(a, s) \in \mathcal{A} \times \mathcal{S} :\ \pi(a) = \pi|_{\mathcal{S}}(s)\}$.) By considering the topology coinduced by q on $\mathcal{S}^{-1}\mathcal{A}$, that is, $U \subseteq \mathcal{S}^{-1}\mathcal{A}$ is open if and only if $q^{-1}(U)$ is open in $\mathcal{A} \circ \mathcal{S}$, with $\mathcal{A} \circ \mathcal{S}$ carrying the relative topology from $\mathcal{A} \times \mathcal{S}$, we quickly show that the *map*

$$\sigma : \mathcal{S}^{-1}\mathcal{A} \longrightarrow X$$

such that

$$\begin{tikzcd} \mathcal{A} \circ \mathcal{S} \arrow[rr, "q"] \arrow[dr, "\tau"'] & & \mathcal{S}^{-1}\mathcal{A} \arrow[dl, "\sigma"] \\ & X & \end{tikzcd},$$

where τ is the obvious projection, *is a local homeomorphism*; hence $\mathcal{S}^{-1}\mathcal{A} \equiv (\mathcal{S}^{-1}\mathcal{A}, \sigma, X)$ *is a sheaf of algebras on X.* Indeed, for any open U in X, we clearly have that $\sigma^{-1}(U)$ is open in $\mathcal{S}^{-1}\mathcal{A}$, which implies that σ is continuous. To show that σ is a local homeomorphism, consider a point $z \in \mathcal{S}^{-1}\mathcal{A}$ and let V be an open neighborhood of z in $\mathcal{S}^{-1}\mathcal{A}$. Then, $q^{-1}(z) \subseteq q^{-1}(V)$, with

$q^{-1}(V)$ open in $\mathcal{A}\circ\mathcal{S}$. Next, let $u \in q^{-1}(z)$ and W an open neighborhood of u such that $\tau|_W$ is homeomorphic. (The projection τ is a local homeomorphism for the following reason: *Given two sheaves of $\mathcal{A}$-modules $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{E}' \equiv (\mathcal{E}', \pi', X)$ on X, the triple $(\mathcal{E} \oplus \mathcal{E}', \sigma, X)$, where $\mathcal{E} \oplus \mathcal{E}' := \{(z, z') \in \mathcal{E} \times \mathcal{E}' : \ \pi(z) = \pi'(z')\}$ and $\sigma : \mathcal{E} \oplus \mathcal{E}' \longrightarrow X : \ (z, z') \longmapsto \sigma(z, z') := \pi(z) = \pi'(z')$, is a sheaf on X, viz. σ is a local homeomorphism.* See [9, p. 120].) Since $\sigma(z) \in X$ and σ is continuous, there exists an open neighborhood O of $\sigma(z)$ in X such that $z \in \sigma^{-1}(O)$. That $\sigma|_{\sigma^{-1}(O\cap\tau(W))}$ is homeomorphic is clear. Indeed, let us first show that σ is bijective on $\sigma^{-1}(O \cap \tau(W))$. To this end, consider $z_1 \neq z_2$ in $\sigma^{-1}(O \cap \tau(W))$; so $q^{-1}(z_1) \cap q^{-1}(z_2) = \emptyset$, whence $\tau(q^{-1}(z_1) \cap W) \cap \tau(q^{-1}(z_2) \cap W) = \emptyset$. Consequently, $\sigma(z_1) \neq \sigma(z_2)$; hence, $\sigma|_{\sigma^{-1}(O\cap\tau(W))}$ is injective. For surjectiveness, let $\alpha \in O \cap \tau(W)$. Then, $\sigma(q(\tau^{-1}(\alpha))) = \alpha$, with $q(\tau^{-1}(\alpha)) \in \sigma^{-1}(O \cap \tau(W))$. Finally, let V be open in $\sigma^{-1}(O \cap \tau(W))$. It follows that $q^{-1}(V)$ is open, and since

$$q^{-1}(V) \subseteq q^{-1}(\sigma^{-1}(O \cap \tau(W))) = \tau^{-1}(O \cap \tau(W)) \subseteq W,$$

$\tau(q^{-1}(V))$ is open. But

$$\tau(q^{-1}(V)) = (\sigma \circ q)(q^{-1}(V)) = \sigma(V),$$

therefore $\sigma(V)$ is open in $O\cap\tau(W)$. Thus, as required, σ is a homeomorphism on $\sigma^{-1}(O \cap \tau(W))$. ■

Given a sheaf $\mathcal{A}$ of unital and commutative algebras on a given topological space X, let $\mathfrak{C}$ be the category of morphisms $\varphi : \mathcal{A} \longrightarrow \mathcal{P}$ of sheaves of *unital* algebras such that $\varphi(\mathcal{S}) \subseteq \mathcal{P}^{\bullet}$, where $\mathcal{P}^{\bullet}$ is the subsheaf of units of $\mathcal{P}$; so, for any point $x \in X$ and element $z \in \mathcal{S}_x$, $\varphi_x(z)$ is invertible in $\mathcal{P}_x$. If $\varphi : \mathcal{A} \longrightarrow \mathcal{P}$ and $\psi : \mathcal{A} \longrightarrow \mathcal{Q}$ are two objects in $\mathfrak{C}$, a *morphism* u of φ to ψ is a sheaf morphism $u : \mathcal{P} \longrightarrow \mathcal{Q}$ making the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & \mathcal{P} \arrow[dl, "u"] \\ & \mathcal{Q} & \end{tikzcd}$$

commute. The sheaf morphism $\mathcal{A} \longrightarrow 0$ is a final universal object in $\mathfrak{C}$, and in this case, if every $\mathcal{S}_x$ contains 0_x, then the morphism $\mathcal{A} \longrightarrow 0$ is the unique object of $\mathfrak{C}$.

Lemma 2.3 *The sheaf morphism*

$$\varphi_{\mathcal{S}} : \mathcal{A} \longrightarrow \mathcal{S}^{-1}\mathcal{A} \tag{28}$$

such that $(\varphi_{\mathcal{S}})_x(r) := \frac{r}{1_x} \equiv \frac{r}{1}$, *for every* $x \in X$ *and* $r \in \mathcal{A}_x$, *is a universal initial object in* $\mathfrak{C}$.

Proof. Clearly, $\varphi_{\mathcal{S}}$ can be decomposed as

$$\varphi_{\mathcal{S}} = q \circ \iota, \tag{29}$$

where $\iota : \mathcal{A} \longrightarrow \mathcal{A} \circ \mathcal{S}$ is the injection given by

$$\iota_x(r) = (r, 1),$$

for any $r \in \mathcal{A}_x \subseteq \mathcal{A}$ $(x \in X)$, and q is the projection (26). Thus, $\varphi_{\mathcal{S}}$ is continuous. Since, in addition, $\varphi_{\mathcal{S}}$ is "*fiber preserving*", it is a morphism of sheaves of algebras $\mathcal{A}$ and $\mathcal{S}^{-1}\mathcal{A}$.

Now, let $\varphi : \mathcal{A} \longrightarrow \mathcal{P}$ be an object of $\mathfrak{C}$. It is clear that if $r, r' \in \mathcal{A}_x$, $s, s' \in \mathcal{S}_x$, where $x \in X$, and $\frac{r}{s} = \frac{r'}{s'}$,

$$\varphi_x(r)\varphi_x(s)^{-1} = \varphi_x(r')\varphi_x(s')^{-1};$$

so that we can define a map

$$\psi : \mathcal{S}^{-1}\mathcal{A} \longrightarrow \mathcal{P}$$

such that

$$\psi_x(\frac{r}{s}) = \varphi_x(r)\varphi_x(s)^{-1},$$

for all $\frac{r}{s} \in (\mathcal{S}^{-1}\mathcal{A})_x$. It is trivially verified that, for every $x \in X$, ψ_x is the unique algebra homomorphism such that $\psi_x \circ (\varphi_{\mathcal{S}})_x = \varphi_x$. By virtue of (29), we have that, for every open U in $\mathcal{P}$,

$$\varphi^{-1}(U) = (\varphi_{\mathcal{S}})^{-1}(\psi^{-1}(U)) = \iota^{-1}q^{-1}(\psi^{-1}(U)),$$

with $\varphi^{-1}(U)$ open in $\mathcal{A}$. But $\iota(\mathcal{A})$ is open in $\mathcal{A} \circ \mathcal{S}$, therefore $q^{-1}(\psi^{-1}(U))$ is open in $\mathcal{A} \circ \mathcal{S}$; so $\psi^{-1}(U)$ is open in $\mathcal{S}^{-1}\mathcal{A}$, whence ψ is continuous. We deduce that ψ is the unique sheaf morphism such that $\psi \circ \varphi_{\mathcal{S}} = \varphi$, which means that $\varphi_{\mathcal{S}}$ is the required initial universal object. ■

In keeping with the above notations, let, now, $\mathcal{E} \equiv (\mathcal{E}, \rho, X)$ be an $\mathcal{A}$-module on a topological space X, and $\mathfrak{D}$ the category whose objects are the $\mathcal{A}$-morphisms $\varphi : \mathcal{E} \longrightarrow \mathcal{P}$ from $\mathcal{E}$ into any $(\mathcal{S}^{-1}\mathcal{A})$-module $\mathcal{P}$; given an $\mathcal{A}$-morphism $\varphi' : \mathcal{E} \longrightarrow \mathcal{P}'$, a morphism from φ to φ' is an $(\mathcal{S}^{-1}\mathcal{A})$-morphism $u : \mathcal{P} \longrightarrow \mathcal{P}'$ such that $\varphi' = u \circ \varphi$. If $\mathfrak{D}$ contains an initial universal object $\varphi_{\mathcal{E}}^{\mathcal{S}} \equiv \varphi_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{M}$, then $\mathcal{M}$, any $(\mathcal{S}^{-1}\mathcal{A})$-module, is called the **sheaf of $(\mathcal{S}^{-1}\mathcal{A})$-modules of fractions of $\mathcal{E}$ with denominator in $\mathcal{S}$** and is denoted by $\mathcal{S}^{-1}\mathcal{E}$.

We need to show that $\varphi_{\mathcal{E}}$ exists in the category $\mathfrak{D}$. For this purpose, we set, on every stalk $\mathcal{E}_x \times \mathcal{S}_x$, the following equivalence relation: Two elements (e, s) and (e', s') of $\mathcal{E}_x \times \mathcal{S}_x$, $x \in X$, are said to be equivalent if there exists an element $t \in \mathcal{S}_x$ such that

$$t(s'e - se') = 0;$$

the set of all equivalence classes in $\mathcal{E}_x \times \mathcal{S}_x$ is called the *module of fractions of the module $\mathcal{E}_x$ with denominator in $\mathcal{S}_x$* (see, for instance, [2, pp. 60-70], [6, pp. 21-25]), and is denoted by $\mathcal{S}_x^{-1}\mathcal{E}_x$. The equivalence class containing the pair (e, s) in $\mathcal{S}_x^{-1}\mathcal{E}_x$ is denoted by $\frac{e}{s}$. It is easy to see that every $\mathcal{S}_x^{-1}\mathcal{E}_x$ becomes an $\mathcal{S}_x^{-1}\mathcal{A}_x$-module under the operations

$$\frac{e_1}{s_1} + \frac{e_2}{s_2} := \frac{s_2 e_1 + s_1 e_2}{s_1 s_2}$$

and

$$\frac{p}{q}\frac{e}{s} := \frac{pe}{qs},$$

where $\frac{e}{s}, \frac{e_1}{s_1}, \frac{e_2}{s_2} \in \mathcal{S}_x^{-1}\mathcal{E}_x$ and $\frac{p}{q} \in \mathcal{S}_x^{-1}\mathcal{A}_x$.

As for the sheaf of algebras of fractions $\mathcal{S}^{-1}\mathcal{A}$ above, one shows that the space

$$\mathcal{S}^{-1}\mathcal{E} := \sum_{x \in X} \mathcal{S}_x^{-1}\mathcal{E}_x,$$

endowed with the final topology determined by the natural map

$$q : \mathcal{E} \circ \mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{E}$$

is a sheaf of $(\mathcal{S}^{-1}\mathcal{A})$-modules. (Again we have assumed the notation $\mathcal{E}\circ\mathcal{S} := \{(e,s)\in\mathcal{E}\times\mathcal{S} :\ \rho(e)=\pi(s)\}$.) Moreover, the mapping

$$\varphi_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{E} \tag{30}$$

such that

$$(\varphi_{\mathcal{E}})_x(e) := \frac{e}{1_x} = \frac{e}{1},$$

is an $\mathcal{A}$-morphism; similarly to the proof of Lemma 2.3, one shows that $\varphi_{\mathcal{E}}$ *is an initial universal object in* $\mathfrak{D}$.

Every sheaf $\mathcal{S}$ of submonoids in a sheaf $\mathcal{A}$ of unital and commutative algebras over a topological space X yields a functor from the category $\mathcal{A}$-$\mathcal{M}od_X$ of $\mathcal{A}$-modules into the category $(\mathcal{S}^{-1}\mathcal{A})$-$\mathcal{M}od_X$ of $(\mathcal{S}^{-1}\mathcal{A})$-modules; more accurately, with every $(\mathcal{A}$-$\mathcal{M}od_X)$-object $\mathcal{E}$, we associate the $(\mathcal{S}^{-1}\mathcal{A})$-module $\mathcal{S}^{-1}\mathcal{E}$, and with every $\mathcal{A}$-morphism $\psi : \mathcal{E} \longrightarrow \mathcal{F}$ we associate the $(\mathcal{S}^{-1}\mathcal{A})$-morphism $\mathcal{S}^{-1}\psi : \mathcal{S}^{-1}\mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$, which is obtained in the following way: because of the universal property of $\mathcal{S}^{-1}\mathcal{E}$, the $\mathcal{A}$-morphism

$$\mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$$

can be factorized in a unique way through $\mathcal{S}^{-1}\mathcal{E}$, that is,

$$\mathcal{S}^{-1}\psi : \mathcal{S}^{-1}\mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{F}$$

is unique and satisfies the equation

$$\mathcal{S}^{-1}\psi \circ \varphi^{\mathcal{S}}_{\mathcal{E}} \equiv \mathcal{S}^{-1}\psi \circ \varphi_{\mathcal{E}} = \varphi_{\mathcal{F}} \circ \psi \equiv \varphi^{\mathcal{S}}_{\mathcal{F}} \circ \psi. \tag{31}$$

Explicitly, (31) implies that, for any x and element $\frac{e}{s} \in (\mathcal{S}^{-1}\mathcal{E})_x$,

$$(\mathcal{S}^{-1}\psi)_x(\frac{e}{s}) := \frac{\psi_x(e)}{s}. \tag{32}$$

Theorem 2.4 *The functor* $\mathcal{S}^{-1} : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow (\mathcal{S}^{-1}\mathcal{A})\text{-}\mathcal{M}od_X$, $\mathcal{S}^{-1}(\mathcal{E}) := \mathcal{S}^{-1}\mathcal{E}$, $\mathcal{S}^{-1}(\psi) := \mathcal{S}^{-1}\psi$, *for all* $(\mathcal{A}\text{-}\mathcal{M}od_X)$*-object* $\mathcal{E}$ *and* $\mathcal{A}$*-morphism* ψ, *is exact. Moreover, there is a one-to-one correspondence between functors* $\mathcal{S}^{-1}$ *and* $\mathcal{S}^{-1}\mathcal{A}\otimes \_ : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow (\mathcal{S}^{-1}\mathcal{A})\text{-}\mathcal{M}od_X$, *such that* $\mathcal{E} \longmapsto \mathcal{S}^{-1}\mathcal{A}\otimes\mathcal{E}$, *and* $\psi \longmapsto \mathcal{S}^{-1}\mathcal{A}\otimes\psi := 1_{\mathcal{S}^{-1}\mathcal{A}}\otimes\psi$, *for all* $\mathcal{A}$*-module* $\mathcal{E}$ *and* $\mathcal{A}$*-morphism* ψ.

Proof. Let us consider an exact sequence $\mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{E}''$ in the category $\mathcal{A}$-$\mathcal{M}od_X$. Since $\psi \circ \varphi = 0$ and, for any $x \in X$ and element $\frac{e'}{s'} \in (\mathcal{S}^{-1}\mathcal{E}')_x$, on applying (32), one has

$$(\mathcal{S}^{-1}\psi \circ \mathcal{S}^{-1}\varphi)_x(\frac{e'}{s'}) = (\mathcal{S}^{-1}\psi)_x(\frac{\varphi_x(e')}{s'}) = \frac{\psi_x(\varphi_x(e'))}{s'} = \frac{(\psi \circ \varphi)_x(e')}{s'} = 0;$$

if follows

$$\mathcal{S}^{-1}\psi \circ \mathcal{S}^{-1}\varphi = 0,$$

i.e., im $\mathcal{S}^{-1}\varphi \subseteq \ker \mathcal{S}^{-1}\psi$. Now, let us show , for every $x \in X$, the inclusion

$$\ker(\mathcal{S}^{-1}\psi)_x \simeq (\ker \mathcal{S}^{-1}\psi)_x \subseteq (\mathrm{im}\ \mathcal{S}^{-1}\varphi)_x \simeq \mathrm{im}(\mathcal{S}^{-1}\varphi)_x$$

(cf. [9, pp. 108, 109; (2.11), (2.13)]); in other words, we must prove that every fraction $\frac{e}{s} \in \ker(\mathcal{S}^{-1}\psi)_x$ is contained in $\mathrm{im}(\mathcal{S}^{-1}\varphi)_x$. Since $\mathcal{S}^{-1}\mathcal{E}$ is an $(\mathcal{S}^{-1}\mathcal{A})$-module, claiming that $(\mathcal{S}^{-1}\psi)_x(\frac{e}{s}) = 0$ implies that $(\mathcal{S}^{-1}\psi)_x(\frac{e}{1}) = 0$; whence $\psi_x(e) \in \ker(\varphi_{\mathcal{E}''})_x$, where $\varphi_{\mathcal{E}''}$ is the canonical mapping $\mathcal{E}'' \longrightarrow \mathcal{S}^{-1}\mathcal{E}''$. By the classical result (cf. [6, p. 22]), which states that, *given a unital and commutative ring K, a multiplicative subset S of K, and a K-module M, an element $x \in M$ belongs to the kernel of the canonical morphism $M \longrightarrow S^{-1}M$, $x \longmapsto \frac{x}{1}$, if and only if there exist $t \in S$ such that $tx = 0$,* we have that $\psi_x(e) \in \ker(\varphi_{\mathcal{E}''})_x$ if and only if there exists $t \in \mathcal{S}_x$ such that $t\psi_x(e) = \psi_x(te) = 0_x \equiv 0$. Therefore there exists $e' \in \mathcal{E}'_x$ such that $\varphi_x(e') = te$, whence $\frac{e}{s} = (\mathcal{S}^{-1}\varphi)_x(\frac{e'}{st})$ as required.

The proof of the second part is just as straightforward. In fact, there is an $(\mathcal{S}^{-1}\mathcal{A})$-morphism $\mathcal{S}^{-1}\mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$ resulting from the universal property of $\mathcal{S}^{-1}\mathcal{E}$; more accurately, for every $x \in X$ and $e \in \mathcal{E}_x$, we have the following commutative diagram

$$\begin{tikzcd} e \arrow[r] \arrow[dr] & \frac{e}{1} \arrow[d] \\ & 1_x \otimes e. \end{tikzcd} \tag{33}$$

On the other hand, the tensor product $\mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$ yields an $(\mathcal{S}^{-1}\mathcal{A})$-morphism $\mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E} \longrightarrow \mathcal{S}^{-1}\mathcal{E}$ such that, for any $x \in X$ and elements $\frac{r}{s} \in (\mathcal{S}^{-1}\mathcal{A})_x$ and $e \in \mathcal{E}_x$, $\frac{r}{s} \otimes e$ is mapped onto $\frac{re}{s}$. The vertical arrow in (33) yields an $(\mathcal{S}^{-1}\mathcal{A})$-morphism which maps $\frac{re}{s} \in (\mathcal{S}^{-1}\mathcal{E})_x$ onto $\frac{r}{s} \otimes e \in (\mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E})_x = (\mathcal{S}^{-1}\mathcal{A})_x \otimes \mathcal{E}_x$

(the preceding equality actually stands for an $\mathcal{A}_x$-isomorphism, cf. [9, p. 130, (5.9)]). Clearly, the $(\mathcal{S}^{-1}\mathcal{A})_x$-morphisms $\frac{re}{s} \longmapsto \frac{r}{s} \otimes e$ and $\frac{r}{s} \otimes e \longmapsto \frac{re}{s}$ are inverse isomorphisms. By [9, p. 68, Theorem 12.1], $\mathcal{S}^{-1}\mathcal{E} = \mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$ within $(\mathcal{S}^{-1}\mathcal{A})$-isomorphism, and if we denote this isomorphism by $\mathcal{S}_{\mathcal{E}}^{-1}$, clearly, it follows that $\mathcal{S}_{\mathcal{E}''}^{-1} \circ \psi = (1_{\mathcal{S}^{-1}\mathcal{A}} \otimes \psi) \circ \mathcal{S}_{\mathcal{E}}^{-1}$, i.e., the $\mathcal{S}_{\mathcal{E}}^{-1}$'s form an equivalence transformation. ■

Corollary 2.5 *The algebra sheaf extension $\mathcal{A} \longrightarrow \mathcal{S}^{-1}\mathcal{A}$ is flat.*

Proof. Indeed, the exactness of the functor $\mathcal{E} \longmapsto \mathcal{S}^{-1}\mathcal{A} \otimes \mathcal{E}$ follows immediately from the exactness of the functor $\mathcal{E} \longmapsto \mathcal{S}^{-1}\mathcal{E}$ (cf. Theorem 2.4). ■

Corollary 2.6 *For all $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$, one has*

$$\mathcal{S}^{-1}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}) = \mathcal{S}^{-1}\mathcal{E} \otimes_{\mathcal{S}^{-1}\mathcal{A}} \mathcal{S}^{-1}\mathcal{F} \tag{34}$$

within $(\mathcal{S}^{-1}\mathcal{A})$-isomorphism.

Proof. Indeed, by an easy calculation, one has:

$$\begin{aligned}(\mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{S}^{-1}\mathcal{A}} (\mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{F}) &= \left[(\mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{S}^{-1}\mathcal{A}} \mathcal{S}^{-1}\mathcal{A}\right] \otimes_{\mathcal{A}} \mathcal{F} \\ &= \left[\mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} (\mathcal{E} \otimes_{\mathcal{S}^{-1}\mathcal{A}} \mathcal{S}^{-1}\mathcal{A})\right] \otimes_{\mathcal{A}} \mathcal{F} \\ &= (\mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} \mathcal{F} \\ &= \mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}),\end{aligned}$$

valid within $(\mathcal{S}^{-1}\mathcal{A})$-isomorphisms. ■

Relation (34) shows that the *functors $\mathcal{S}^{-1}$ and $\otimes$ commute.* In Theorem 2.7 below, we show that *the functor $\mathcal{S}^{-1}$ commutes with the functor $\mathcal{H}om$ under certain conditions.* See, for instance, [6, p. 19, Proposition 1.9.7] and [2, p. 76, Proposition 19] for the classical case.

Theorem 2.7 *For all $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ on a topological space X, the $(\mathcal{S}^{-1}\mathcal{A})$-morphism*

$$\vartheta : \mathcal{S}^{-1}\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{E}, \mathcal{S}^{-1}\mathcal{F}), \tag{35}$$

given by

$$\vartheta_x(f/s)(e/t) := f(e)/st, \tag{36}$$

where $x \in X$, $s, t \in \mathcal{S}_x$, $e \in (\mathcal{S}^{-1}\mathcal{E})_x$, $f \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})_x$, *is an* $(\mathcal{S}^{-1}\mathcal{A})$-*isomorphism, whenever* $\mathcal{E}$ *is a locally finitely presented* $\mathcal{A}$-*module.*

Proof. On the basis of Lemma 1.8, since $\mathcal{E}$ is a locally finitely presented $\mathcal{A}$-module, one has

$$\begin{aligned} \left(\mathcal{S}^{-1}\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})\right)_x &= (\mathcal{S}^{-1}\mathcal{A})_x \otimes_{\mathcal{A}_x} \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})_x \\ &= \mathcal{S}_x^{-1}\mathcal{A}_x \otimes_{\mathcal{A}_x} \mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x, \mathcal{F}_x) \\ &= \mathcal{S}_x^{-1}\mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x, \mathcal{F}_x), \end{aligned}$$

therefore elements of the $(\mathcal{S}_x^{-1}\mathcal{A}_x)$-module $\mathcal{S}_x^{-1}\mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x, \mathcal{F}_x)$ are of the form f/s, with $f \in \mathrm{H}om_{\mathcal{A}_x}(\mathcal{E}_x, \mathcal{F}_x)$ and $s \in \mathcal{S}_x \subseteq \mathcal{A}_x$. By Theorem 2.4, the functor $\mathcal{S}^{-1}$ is exact, therefore, $\mathcal{S}^{-1}\mathcal{E}$ is a locally finitely presented $(\mathcal{S}^{-1}\mathcal{A})$-module on X, so that

$$\mathcal{H}om_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{E}, \mathcal{S}^{-1}\mathcal{F})_x = \mathrm{H}om_{(\mathcal{S}^{-1}\mathcal{A})_x}\left((\mathcal{S}^{-1}\mathcal{E})_x, (\mathcal{S}^{-1}\mathcal{F})_x\right);$$

hence, (36) is well defined, and ϑ is clearly an $(\mathcal{S}^{-1}\mathcal{A})$-morphism. By virtue of Theorem 1.9 and Corollary 2.5, (35) is an $(\mathcal{S}^{-1}\mathcal{A})$-isomorphism whenever $\mathcal{E}$ is a locally finitely presented $\mathcal{A}$-module on X. ■

3 Change of the algebra sheaf of scalars

We are now in the position to use the definition of *sheaves of Clifford* $\mathcal{A}$-*algebras* (*Clifford* $\mathcal{A}$-*algebras* in short) *of quadratic* $\mathcal{A}$-*modules*, set over arbitrary topological spaces as quotient sheaves of tensor algebra sheaves over certain ideal sheaves to show that direct limits commute with the *Clifford functor* $\mathcal{C}l : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow \mathcal{A}\text{-}\mathcal{A}lg_X$, where $\mathcal{A}\text{-}\mathcal{M}od_X$ and $\mathcal{A}\text{-}\mathcal{A}lg_X$ stand for the categories of sheaves of $\mathcal{A}$-modules and $\mathcal{A}$-algebras on X, respectively. Finally, we draw our attention on to the change of the algebra sheaf $\mathcal{A}$ of scalars in any Clifford $\mathcal{A}$-algebra on a topological space X, and Clifford $\mathcal{A}$-algebras of orthogonal sums of $\mathcal{A}$-modules.

Theorem 3.4, which is concerned with the *commutativity of the Clifford functor* $\mathcal{C}l$ *with the extension functor through tensor product*, is proved by means of Lemma 3.1. [6, p. 54, Lemma 2.1.3] is a classical counterpart of Lemma 3.1.

Lemma 3.1 *Let* $\mathcal{E}$ *be a free* $\mathcal{A}$*-module on a topological space* X*, and* $\mathcal{F}$ *any* $\mathcal{A}$*-module, also on* X*. For any open subset* U *of* X*, let* $(e_i^U \equiv e_i)_{i\in I}$ *be a basis of* $\mathcal{E}(U)$ *and* $(t_{i,j}^U \equiv t_{i,j})_{i,j\in I}$ *be a family of sections in* $\mathcal{F}(U)$ *such that* $t_{i,j} = t_{j,i}$*. Then, there exists a unique* $\mathcal{A}|_U$*-quadratic morphism* $q \in \mathrm{Q}uad_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{F}|_U) \equiv \mathcal{Q}uad_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(U)$ *such that*

$$q_V(e_i|_V) = t_{i,i}|_V, \quad i \in I, \tag{37}$$

and

$$(B_q)_V(e_i|_V, e_j|_V) = t_{i,j}|_V, \quad i \neq j \text{ in } I, \tag{38}$$

where B_q *is the associated* $\mathcal{A}|_U$*-bilinear morphism of* q*.*

Proof. If g is an $\mathcal{A}$-bilinear morphism $\mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{F}$, the sheaf morphism $\varphi : \mathcal{E} \longrightarrow \mathcal{F}$ such that

$$\varphi_V(s) = g_V(s, s),$$

for any open $V \subseteq X$ and section $s \in \mathcal{E}(V)$, is $\mathcal{A}$-quadratic; clearly, the associated $\mathcal{A}$-bilinear morphism is the sheaf morphism $B_\varphi \equiv (B_{\varphi,V})_{X\supseteq V,\ open}$ such that

$$B_{\varphi,V}(s, t) = g_V(s, t) + g_V(t, s),$$

for all $s, t \in \mathcal{E}(V)$. Next, define a total order on the indexing set I and let $g : \mathcal{E}|_U \oplus \mathcal{E}|_U \longrightarrow \mathcal{F}|_U$ be the $\mathcal{A}|_U$-bilinear morphism such that

$$g_V(e_i|_V, e_j|_V) = t_{i,j}|_V, \quad i, j \in I \text{ with } i \leq j,$$

and

$$g_V(e_i|_V, e_j|_V) = 0, \quad i, j \in I \text{ with } i > j.$$

The $\mathcal{A}|_U$-quadratic morphism $q : \mathcal{E}|_U \longrightarrow \mathcal{F}|_U$ such that

$$q_V(s) = g_V(s, s),$$

for any open set $V \subseteq X$ and section $s \in (\mathcal{E}|_U)(V)$, satisfies the conditions of the lemma.

Now, let us prove the uniqueness of q. To this end, suppose that there is another $\mathcal{A}|_U$-quadratic $\overline{q} : \mathcal{E}|_U \longrightarrow \mathcal{F}|_U$ satisfying (8) and (6), that is,

$$(\overline{q})_V(e_i|_V) = t_{i,i}|_V, \quad i \in I,$$

and

$$(B_{\overline{q}})_V(e_i|_V, e_j|_V) = t_{i,j}|_V, \quad i \neq j \text{ in } I,$$

for any subopen V of U. It follows that, for any $i, j \in I$ and subopen $V \subseteq U$,

$$\big((q)_V - (\overline{q})_V\big)(e_i|_V) = 0$$

and

$$\big(B_{q-\overline{q}}\big)_V(e_i|_V, e_j|_V) = B_q\big((e_i|_V, e_j|_V)\big) - B_{\overline{q}}\big((e_i|_V, e_j|_V)\big) = 0.$$

By an easy calculation, one shows that, for any $s \in \mathcal{E}(U)$

$$\big(B_{q-\overline{q}}\big)_V(s|_V, s|_V) = 2(q - \overline{q})|_V(s|_V) = 0,$$

whence

$$q = \overline{q}.$$

■

Here is a very useful result of this section.

Theorem 3.2 *Let $\mathcal{A}$, $\mathcal{A}'$ be unital algebras sheaves on a topological space X, $\varphi : \mathcal{A} \longrightarrow \mathcal{A}'$ a sheaf morphism, $\mathcal{E}$ and $\mathcal{F}$ two $\mathcal{A}$-modules on X, and $q : \mathcal{E} \longrightarrow \mathcal{F}$ an $\mathcal{A}$-quadratic sheaf morphism. Then, there exists a unique $\mathcal{A}'$-quadratic morphism $q' : \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{E} \longrightarrow \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{F}$ such that*

$$q' \circ (1 \otimes \mathrm{id}_{\mathcal{E}}) = 1 \otimes (q \circ \mathrm{id}_{\mathcal{E}}), \tag{39}$$

where $1 \in \mathrm{End}_{\mathcal{A}'}\mathcal{A}'$ is the constant endomorphism of the underlying sheaf of sets of $\mathcal{A}'$ such that, for every $s \in \mathcal{A}'(U)$, where U is open in X, $1_U(s) := 1_{\mathcal{A}'(U)} \equiv 1 \in \mathcal{A}'(U)$.

Section-wise, (39) means that, for every open set U in X and sections $r \in \mathcal{A}'(U)$, $s \in \mathcal{E}(U)$,

$$\begin{aligned}[q'_U \circ (1_U \otimes (\mathrm{id}_{\mathcal{E}})_U)](r \otimes s) &:= q'_U(1 \otimes s) \\ &= 1 \otimes q_U(s) := [1_U \otimes (q_U \circ (\mathrm{id}_{\mathcal{E}})_U](r \otimes s). \end{aligned} \tag{40}$$

Proof. It is clear that q' is unique; we therefore simply need to prove its existence. Suppose that $\mathcal{E}$ is free. For a fixed open set U in X, we let $(s_i)_{i\in I}$ be a basis of $\mathcal{E}(U)$ and set $t_{i,i} = q_U(s_i) \equiv q(s_i)$ for all $i \in I$, $t_{i,j} = (B_q)_U(s_i, s_j)$ for all $i, j \in I$ such that $i \neq j$. Since $\mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U)$ is a free $\mathcal{A}'(U)$-module with basis $(1 \otimes s_i)_{i\in I}$, according to Lemma 3.1, there exists a unique $\mathcal{A}'(U)$-quadratic mapping $q'_U : \mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U) \longrightarrow \mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)$ such that $q'_U(1 \otimes s_i) = 1 \otimes t_{i,i}$ for all $i \in I$ and $(B_{q'_U})(1 \otimes s_i, 1 \otimes s_j) = 1 \otimes t_{i,j}$ for all $i, j \in I$ with $i \neq j$. Obviously, q'_U satisfies (40). Since q'_U is unique and U is arbitrary, the family $(q'_U)_{X\supseteq U,\ open}$ yields an $\mathcal{A}'$-quadratic morphism $q' : \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{E} \longrightarrow \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{F}$ satisfying the required condition (40).

Suppose now that $\mathcal{E}$ is not free. As above, fix an open set U in X; $\mathcal{E}(U)$ being an $\mathcal{A}(U)$-module is isomorphic to a quotient $\mathcal{A}(U)$-module of a free $\mathcal{A}(U)$-module. By [6, p. 57, Theorem 2.2.3], one shows that there exists a unique $\mathcal{A}'(U)$-quadratic morphism

$$q'_U : \mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U) \longrightarrow \mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)$$

such that

$$q'_U(1' \otimes s) = 1' \otimes q_U(s)$$

for any $s \in \mathcal{E}(U)$. Besides, we note that the collections $(\mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U))_{X\supseteq U,\ open}$ and $(\mathcal{A}'(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)_{X\supseteq U,\ open}$ induce the presheaves of modules $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E})$ and $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{F})$, respectively; if τ^* denotes the set of pairs (U, V), where both U and V are open in X and such that $V \subseteq U$, we shall let $(\mu^U_V)_{(U,V)\in\tau^*}$ and $(\sigma^U_V)_{(U,V)\in\tau^*}$ denote the restriction maps of the preceding presheaves, that is, $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E})$ and $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{F})$, respectively. Since, for any $(U, V) \in \tau^*$,

$$\sigma^U_V \circ q'_U = q'_V \circ \mu^U_V,$$

it follows that the family $(q'_U)_{X\supseteq U,\ open}$ yields the sought presheaf morphism of $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{E})$ into $\Gamma(\mathcal{A}') \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{F})$. It is obvious that $(q'_U)_{X\supseteq U,\ open}$ is unique. ■

The relevance of Lemma 3.3 will be evident in the proof of Theorem 3.4, below.

Lemma 3.3 *If* $(\mathcal{E}, q)$ *be a quadratic* $\mathcal{A}$*-module on a topological space* X*, then, for every* $x \in X$*,*

$$\mathcal{C}l(\mathcal{E})_x = Cl(\mathcal{E}_x) \tag{41}$$

within $\mathcal{A}_x$*-isomorphism, where* $Cl(\mathcal{E}_x)$ *is the usual Clifford algebra associated with the quadratic* $\mathcal{A}_x$*-module* $(\mathcal{E}_x, q_x)$*.*

Proof. If $\mathcal{I}(\mathcal{E}, q) \equiv \mathcal{I}(\mathcal{E})$ is the two-sided ideal sheaf in the tensor algebra (sheaf) $T(\mathcal{E})$ determined by the presheaf $J(\mathcal{E}, q)$, where, for any open set U in X, $J(\mathcal{E}, q)(U)$ is a two-sided ideal of the tensor algebra $T(\mathcal{E}(U))$ generated by elements of the form

$$s \otimes s - q_U(s) \equiv s \otimes s - q(s),$$

with s running through $\mathcal{E}(U)$, then, by [15, Theorem 2.1], the Clifford $\mathcal{A}$-algebra of $\mathcal{E}$, denoted by $\mathcal{C}l(\mathcal{E}) \equiv \mathcal{C}l(\mathcal{E}, q) \equiv \mathcal{C}l_{\mathcal{A}}(\mathcal{E})$, is given by

$$\mathcal{C}l(\mathcal{E}) := \mathcal{T}(\mathcal{E})/\mathcal{I}(\mathcal{E}).$$

On account of [9, p. 115, (2.50)], for every $x \in X$,

$$\mathcal{C}l(\mathcal{E})_x = \mathcal{T}(\mathcal{E})_x/\mathcal{I}(\mathcal{E})_x = T(\mathcal{E}_x)/I(\mathcal{E}_x) = Cl(\mathcal{E}_x),$$

where the preceding equalities actually stand for $\mathcal{A}_x$-isomorphisms. ■

As is known (cf. [6, p.110, Proposition 3.1.9]), let K and K' be unital commutative algebras, $f : K \longrightarrow K'$ an algebra morphism, which respects 1, and (E, q) an object in the category ${}_K\widetilde{Mod}$ of quadratic K-modules. Moreover, let $S : {}_K\widetilde{Mod} \longrightarrow {}_{K'}\widetilde{Mod}$ be such that

$$S(M, q) \equiv S(M) = K' \otimes_K M \equiv K' \otimes_K (M, q),$$

then

$$Cl_{K'} \circ S \simeq S \circ Cl_K; \tag{42}$$

that is, for every quadratic K-module (M,q),

$$Cl_{K'}(K' \otimes (M,q)) = K' \otimes Cl_K(M,q) \tag{43}$$

within K'-isomorphism. We shall see in Theorem 3.4 that the isomorphism (42) also holds for categories of sheaves of quadratic modules. For the classical case, see, for instance, [4, p. 46, Proposition 3.1.4].

Theorem 3.4 *Let $\mathcal{A}$, $\mathcal{B}$ be unital commutative algebra sheaves on a topological space X, $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ a morphism of algebra sheaves, and $(\mathcal{E}, q)$ a quadratic $\mathcal{A}$-module on X. The Clifford algebra sheaf $\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E})$ of the quadratic $\mathcal{B}$-module $\mathcal{B} \otimes_{\mathcal{A}} (\mathcal{E}, q) \equiv \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E} \equiv \mathcal{B} \otimes \mathcal{E}$, obtained by extending $\mathcal{A}$ to $\mathcal{B}$ via φ, is canonically isomorphic to the $\mathcal{B}$-algebra $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}l(\mathcal{E})$, that is,*

$$\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}) = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}l_{\mathcal{A}}(\mathcal{E}) \tag{44}$$

within $\mathcal{B}$-isomorphism.

Proof. First, let's observe the following fact. As in Lemma 3.3, let $\mathcal{I}(\mathcal{E}, q)$ be the two-sided ideal sheaf in the tensor $\mathcal{A}$-algebra $\mathcal{T}(\mathcal{E})$, generated by the presheaf $(\otimes \circ \Delta - q)(\mathcal{E})$ of sets, which is such that, for any open U in X, and section $s \in \mathcal{E}(U)$,

$$(\otimes \circ \Delta - q)_U(s) := s \otimes s - q_U(s) \equiv s \otimes s - q(s).$$

(Δ is the diagonal $\mathcal{A}$-morphism $\mathcal{E} \longrightarrow \mathcal{E} \oplus \mathcal{E}$ with $\Delta_U(s) := (s, s)$.) The Clifford $\mathcal{A}$-algebra $\mathcal{C}l(\mathcal{E})$ is generated by the presheaf $(Cl(\mathcal{E}(U)), \rho^U_V)_{(U,V)\in\tau^*}$, where

$$Cl(\mathcal{E}(U)) = T(\mathcal{E}(U))/I(\mathcal{E}(U)) \tag{45}$$

and

$$\rho^U_V(s + I(\mathcal{E}(U))) = \lambda^U_V(s) + I(\mathcal{E}(V));$$

assuming that $(T(\mathcal{E}(U)), \lambda^U_V)_{(U,V)\in\tau^*}$ is a generating presheaf of the tensor $\mathcal{A}$-algebra $\mathcal{T}(\mathcal{E})$. Indeed, (45) is guaranteed by the definition of quotient $\mathcal{A}$-modules and the completeness of the presheaves $(T(\mathcal{E}(U)), \lambda^U_V)_{(U,V)\in\tau^*}$ and $(I(\mathcal{E}(U)), \lambda^U_V)_{(U,V)\in\tau^*}$. Now, consider a point $x \in X$; by Lemma 3.3,

$$\big(\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E})\big)_x = Cl_{\mathcal{B}_x}(\mathcal{B}_x \otimes_{\mathcal{A}_x} \mathcal{E}_x)$$

and

$$\big(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}l(\mathcal{E})\big)_x = \mathcal{B}_x \otimes_{\mathcal{A}_x} Cl(\mathcal{E}_x),$$

within $\mathcal{B}_x$-isomorphisms. By [6, p.110, Proposition 3.1.9], which is summarized by the isomorphism (43),

$$Cl_{\mathcal{B}_x}(\mathcal{B}_x \otimes_{\mathcal{A}_x} \mathcal{E}_x) = \mathcal{B}_x \otimes_{\mathcal{A}_x} Cl(\mathcal{E}_x) \tag{46}$$

within $\mathcal{A}_x$-isomorphism. We denote the $\mathcal{A}_x$-isomorphism by φ_x. Next, let $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$ be the Clifford maps that make $\mathcal{C}l_{\mathcal{A}}(\mathcal{E})$ and $\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E})$, respectively, into sheaves of Clifford algebras; then $(\rho_{\mathcal{A}})_x$ and $(\rho_{\mathcal{B}})_x$ are Clifford sheaf morphisms associated with Clifford algebras $Cl_{\mathcal{A}_x}(\mathcal{E}_x)$ and $Cl_{\mathcal{B}_x}(\mathcal{B}_x \otimes_{\mathcal{A}_x} \mathcal{E}_x)$, respectively. The $\mathcal{A}_x$-isomorphism maps every $(\rho_{\mathcal{B}})_x(\lambda \otimes z)$ ($\lambda \in \mathcal{B}_x$ and $z \in \mathcal{E}_x$) onto $\lambda \otimes (\rho_{\mathcal{A}})_x(z)$. Since the presheaf, written loosely as $(\mathcal{B}(U) \otimes_{\mathcal{A}(U)} Cl(\mathcal{E}(U)))$ because its restriction maps are obvious, is a monopresheaf, it follows, from [9, p.68, Theorem 12.1], that the family $(\varphi_x)_{x\in X}$ of $\mathcal{A}_x$-isomorphisms yields the required $\mathcal{A}$-isomorphism of $\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E})$ onto $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}l(\mathcal{E})$. ■

Corollary 3.5 *Let $\mathcal{A}$ be a unital commutative algebra sheaf on a topological space X, and $\mathcal{S}$ a sheaf of submonoids in $\mathcal{A}$. Then, for any quadratic $\mathcal{A}$-module $\mathcal{E}$ on X, one has*

$$\mathcal{C}l_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{E}) = \mathcal{C}l_{\mathcal{S}^{-1}\mathcal{A}}(\mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E}) = \mathcal{S}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{C}l_{\mathcal{A}}(\mathcal{E}) = \mathcal{S}^{-1}\mathcal{C}l_{\mathcal{A}}(\mathcal{E}),$$

valid within $(\mathcal{S}^{-1}\mathcal{A})$-isomorphisms.

One more corollary, indeed, follows from Theorem 3.4, but, for the sake of self containedness of the paper, we recall the $\mathcal{A}$-*quadratic morphisms into a fixed target $\mathcal{A}$-module $\mathcal{F}$ constitute the objects of a category, denoted* $\mathcal{C}_{\mathcal{A}}(\mathcal{F})$; given two $\mathcal{C}_{\mathcal{A}}(\mathcal{F})$-objects $q \equiv (\mathcal{E}, q, \mathcal{F})$ and $q' \equiv (\mathcal{E}', q', \mathcal{F})$, a morphism between them is an $\mathcal{A}$-morphism $\varphi : \mathcal{E} \longrightarrow \mathcal{E}'$ such that $q' \circ \varphi = q$.

Let's assume the notations of Theorem 3.4. Then, we have

Corollary 3.6 *The sheaf isomorphism of Theorem 3.4 results into an isomorphism of functors $\mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes \_)$, $\mathcal{B} \otimes \mathcal{C}l(\_) : \mathcal{C}_{\mathcal{A}}(\mathcal{A}) \longrightarrow \mathcal{C}_{\mathcal{A}}(\mathcal{A})$ of the category $\mathcal{C}_{\mathcal{A}}(\mathcal{A})$ of $\mathcal{A}$-quadratic morphisms into $\mathcal{A}$. Specifically, for any morphism*

$\varphi : (\mathcal{E}, q, \mathcal{A}) \equiv (\mathcal{E}, q) \longrightarrow (\mathcal{E}', q') \equiv (\mathcal{E}', q', \mathcal{A})$, *the diagram*

$$\begin{tikzcd} \mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes (\mathcal{E}, q)) \arrow[r] \arrow[d, "\mathcal{C}l(\mathcal{B}\otimes\varphi)"'] & \mathcal{B} \otimes \mathcal{C}l(\mathcal{E}, q) \arrow[d, "\mathcal{B}\otimes\mathcal{C}l(\varphi)"] \\ \mathcal{C}l_{\mathcal{B}}(\mathcal{B} \otimes (\mathcal{E}', q')) \arrow[r] & \mathcal{B} \otimes \mathcal{C}l(\mathcal{E}', q') \end{tikzcd}$$

commutes. (*The horizontal arrows in the diagram are isomorphisms.*)

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