DISCONTINUOUS GALERKIN FINITE ELEMENT DISCRETIZATION FOR STEADY STOKES FLOWS WITH THRESHOLD SLIP BOUNDARY CONDITION

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Abstract. This work is concerned with the discontinuous Galerkin finite approximations for the steady Stokes equations driven by slip boundary condition of "friction" type. Assuming that the flow region is a bounded, convex domain with a regular boundary, we formulate the problem and its discontinuous Galerkin approximations as mixed variational inequalities of the second kind with primitive variables. The well posedness of the formulated problems are established by means of a generalization of the Babuška-Brezzi theory for mixed problems. Finally, a priori error estimates using energy norm for both the velocity and pressure are obtained.

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1. Introduction. We consider steady flows of incompressible viscous fluids modeled by the Stokes system

$$-\operatorname{div}\nu\boldsymbol{D}(\boldsymbol{v})+\nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{v} = 0 \qquad \text{in } \Omega, \tag{1.2}$$

where Ω is the flow region, a bounded domain in $\mathbb{R}^2$, while $\boldsymbol{D}(\boldsymbol{v})=\frac{1}{2}[\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T]$. The motion of the incompressible fluid is described by the velocity $\boldsymbol{v}(\boldsymbol{x})$ and pressure $p(\boldsymbol{x})$. In (1.1) $\boldsymbol{f}$ is the external body force per unit volume depending on $\boldsymbol{x}$, and ν is the positive parameter representing the kinetic viscosity. Equations (1.1) and (1.2) are supplemented by boundary conditions which constituted the novelty of our study. In order to describe the motion of the fluid at the boundary, we assume that the boundary of Ω, say, $\partial\Omega$ is made of two components S (say the outer wall) and Γ (the inner wall), and it is required that $\overline{\partial\Omega}=\overline{S\cup\Gamma}$, with $S\cap\Gamma=\emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$\boldsymbol{v}=0 \quad \text{on } \Gamma. \tag{1.3}$$

We have chosen to work with a homogeneous condition on the velocity in order to avoid the technical arguments linked to the Hopf lemma (see [10], Chapter 4, Lemma 2.3). On S, we first assume the impermeability condition

$$v_N = \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } S, \tag{1.4}$$

where $\boldsymbol{n}$ is the outward unit normal on the boundary $\partial\Omega$, and v_N is the normal component of the velocity, while $\boldsymbol{v_T} = \boldsymbol{v} - v_N\boldsymbol{n}$ is its tangential component. In addition to (1.4) we also impose on S, a "friction type" boundary condition [5, 6, 7, 8, 16], which is the main ingredient of this work. The "friction type" boundary condition can be formulated with the knowledge of a positive function $g : S \longrightarrow (0, \infty)$ called threshold slip or barrier function, and the use of sub-differential to link quantities of interest. It is written as

$$-(\boldsymbol{\sigma n})_{\boldsymbol{T}} \in \ g\partial|\boldsymbol{v_T}| \quad \text{on } S, \tag{1.5}$$

where $(\boldsymbol{\sigma n})_{\boldsymbol{T}}$ is the tangential component of the Cauchy tensor $\boldsymbol{\sigma}$ given by $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{D}(\boldsymbol{v})$, and $\partial|\cdot|$ is the sub-differential of the real-valued function $|\cdot|$, with $|\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v}$. We recall that if $\mathcal{X}$ is a Hilbert space with $x_0 \in \mathcal{X}$, and $y \in \mathcal{X}'$. Then

$$y \in \partial\Psi(x_0) \text{ means that } \Psi(x) - \Psi(x_0) \geq y \cdot (x - x_0) \quad \forall x \in \mathcal{X}. \tag{1.6}$$

It should be observed that (1.5) can also be written as [4]

$$\left.\begin{array}{l} |(\boldsymbol{\sigma n})_{\boldsymbol{T}}| \leq g, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{T}}| < g \Rightarrow \boldsymbol{v_T} = \boldsymbol{0}, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{T}}| = g \Rightarrow \boldsymbol{v_T} \neq \boldsymbol{0}, \ -(\boldsymbol{\sigma n})_{\boldsymbol{T}} = g\dfrac{\boldsymbol{v_T}}{|\boldsymbol{v_T}|} \end{array}\right\} \text{ on } \quad S. \tag{1.7}$$

Clearly, that condition express the fact that the tangential part of the traction is proportional but opposite to the tangential velocity, while $g/|\boldsymbol{v_T}|$ can be viewed here as the coefficient of "friction". So, it is the Coulomb's friction law, but also called in this context Navier's slip condition [16]. It should be mentioned that more advanced slip conditions can be formulated, (such slip condition will be examined in our future work) we refer the reader to [16]. The friction type boundary conditions (1.4) and (1.5) are applied to many situations, for example, in the melt spinning, the friction between the air and the "skin" of the fluid should be taken into consideration in the case the draw velocity of the fibre is important. The main concern in this research is to analyze numerically (1.1)–(1.5) using the discontinuous Galerkin finite element method. This work can be viewed as a first attempt to solve a flow problem driven by slip boundary condition of friction type using the discontinuous Galerkin methods. There are two major difficulties associated with problem (1.1)–(1.5), namely: the incompressibility condition, and the nonlinear slip boundary condition. It should be observed that the nonlinearity of the problem comes from the boundary condition (1.5), while the velocity $\boldsymbol{v}$ is related to the pressure via the incompressibility condition $\operatorname{div} \boldsymbol{v} = 0$. Thus the pressure is viewed as a Lagrange multiplier. Hence the problem (1.1)–(1.5) can be formulated as a mixed variational problem, which can be shown to be equivalent to many others variational problems [16].

It is well known that solving nonlinear variational problems is not a trivial task [13].

There exist many finite element discretizations for solving variational inequalities [13, 14, 19], steady Stokes and Navier-Stokes problems [10, 13] and mixed variational inequalities [9, 15, 17, 18]. The finite element methods presented in [15, 18] are motivated by problems in plasticity, while the analysis in [17] uses the penalty approach in the Stokes equations to circumvent the incompressibility constraint. Using a different type of slip boundary condition R. Verfurth [20] has analyzed the problem by relaxing the constraint (1.4) at the expense of an additional unknown. In [9] a solution technique and the convergence of an algorithm for solving the Stokes equations with leak and slip boundary conditions is presented, but the mathematical analysis of the finite element method presented is not discussed. Our framework for analyzing the finite element schemes of (1.1)–(1.5) is based on a suitable extension of the mixed finite theory of Babuška-Brezzi [2], reminiscent of those used in, e.g., [15, 18] for the analysis of problems in plasticity. We formulate and analyze the discontinuous Galerkin finite element approximations associated to (1.1)–(1.5) without penalization by considering the mixed variational approach in which the velocity and pressure satisfy the Babuška-Brezzi (BB) condition [2]. The discontinuous Galerkin finite element schemes are constructed on a regular decomposition of the domain [3] and a complete mathematical analysis is discussed.

The outline of our work is as follows. First we re-formulate (1.1)–(1.5) in terms of variational inequalities and prove the unique solvability of the variational problem in Section 2. In Section 3, we introduce some notations pertaining to the discontinuous Galerkin approximations. Next, we formulate the discontinuous Galerkin scheme, show the unique solvability and establish convergence of the approximate solution using energy norms for both the velocity and pressure. Some conclusions are drawn in Section 4.

2. Variational formulations/Slip boundary conditions.

2.1. Notation. We shall use the standard Sobolev spaces [1]. For any non-negative integer s and $r \geq 1$, the usual Sobolev space on a domain $E \subset \mathbb{R}^2$ is

$$W^{s,r}(E) = \{\phi \in L^r(E) : \quad \text{for all } |m| \leq s, \quad \partial^m \phi \in L^r(E)\},$$

where $\partial^m \phi$ are the partial derivatives of ϕ of order m. The norm in $W^{s,r}(E)$ is denoted by $\|\cdot\|_{s,r,E}$ and the semi norm by $|\cdot|_{s,r,E}$. The L^2 inner-product is denoted by $(\cdot,\cdot)_E$ and by $(\cdot,\cdot)$ if $E = \Omega$. For the Hilbert space $H^s(E) = W^{s,2}(E)$, the norm is denoted by $\|\cdot\|_{s,E}$. Throughout this work, boldface characters denote vector quantities. We introduce the spaces

$$V := \{\boldsymbol{v} \in H^1(\Omega)^2; \quad \boldsymbol{v}|_\Gamma = 0, \quad \boldsymbol{v}\cdot\boldsymbol{n}|_S = 0\}, \quad M := L_0^2(\Omega). \tag{2.1}$$

We assume that

$$\nu > 0, \ \boldsymbol{f} \in L^2(\Omega)^2, \ g \in L^\infty(S), \ g \geq 0 \text{ a.e. on } S\ . \tag{2.2}$$

2.2. Variational formulation. In this paragraph, we formulate variational models associated with (1.1)–(1.5); and we also indicate how existence and uniqueness of the solutions is obtained.

The first variational formulation of (1.1)–(1.5) is classical, and readily obtained by making use of Green's formula and (1.6). It reads

$$\begin{cases} \text{Find } \ (\boldsymbol{v}, p) \in V \times M \ \text{ such that} \\ \widetilde{a}(\boldsymbol{v}, \boldsymbol{w} - \boldsymbol{v}) + \widetilde{b}(\boldsymbol{w} - \boldsymbol{v}, p) + j(\boldsymbol{w}) - j(\boldsymbol{v}) \geq \ell(\boldsymbol{w} - \boldsymbol{v}) \text{ for all } \ \boldsymbol{w} \in V, \\ \widetilde{b}(\boldsymbol{v}, q) = 0 \text{ for all } \ q \in L^2(\Omega), \end{cases} \tag{2.3}$$

with

$$\widetilde{a}(\boldsymbol{v}, \boldsymbol{w}) := \nu(\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{w})), \quad \widetilde{b}(\boldsymbol{v}, q) := -(\operatorname{div} \boldsymbol{v}, q),$$
$$\ell(\boldsymbol{v}) := (\boldsymbol{f}, \boldsymbol{v}), \quad j(\boldsymbol{v}) := \int_S g|\boldsymbol{v}_{\boldsymbol{\tau}}| ds.$$

We also know that $\boldsymbol{v}$ is the solution of the optimization problem

$$\begin{cases} \text{Find } \boldsymbol{v} \in V_{\text{div}}(\Omega) \text{ such that} \\ J(\boldsymbol{v}) \leq J(\boldsymbol{w}) \ \text{ for all } \ \boldsymbol{w} \in V_{\text{div}}(\Omega), \end{cases} \tag{2.4}$$

where

$$J(\boldsymbol{w}) = \frac{1}{2}\widetilde{a}(\boldsymbol{w}, \boldsymbol{w}) + j(\boldsymbol{w}) - \ell(\boldsymbol{w})$$
$$V_{\text{div}}(\Omega) = \{\boldsymbol{v} \in V, \quad \operatorname{div} \boldsymbol{v}|_\Omega = 0\}.$$

One readily observes that (2.4) is equivalent to

$$\begin{cases} \text{Find } \ \boldsymbol{v} \in V_{\text{div}}(\Omega) \text{ such that} \\ \widetilde{a}(\boldsymbol{v}, \boldsymbol{w} - \boldsymbol{v}) + j(\boldsymbol{w}) - j(\boldsymbol{v}) \geq \ell(\boldsymbol{w} - \boldsymbol{v}) \ \text{ for all } \ \boldsymbol{w} \in V. \end{cases} \tag{2.5}$$

Let us point out that from a numerical point of view, the solution of (2.4) is difficult to compute, the main difficulty being to define an internal approximation of the set $V_{\text{div}}(\Omega)$. It is one of the reasons why mixed finite element methods have been introduced, and very popular.
Another equivalent model is the saddle point problem which can be formulated as follows:

$$\begin{cases} \text{Find } (\boldsymbol{v}, p) \in V \times M \text{ such that} \\ \mathcal{L}(\boldsymbol{v}, q) \leq \mathcal{L}(\boldsymbol{v}, p) \leq \mathcal{L}(\boldsymbol{w}, p) \ \text{ for all } \ \boldsymbol{w}, q \in V \times M, \end{cases} \tag{2.6}$$

where

$$\mathcal{L}(\boldsymbol{w}, q) = \frac{1}{2}\widetilde{a}(\boldsymbol{w}, \boldsymbol{w}) + j(\boldsymbol{w}) - \ell(\boldsymbol{w}) + b(\boldsymbol{w}, q).$$

For the well-posedness analysis of (2.3), instead of Babuska-Brezzi's theorem [2], we need its extension to mixed variational inequalities obtained by Han and Reddy [15, 18], which entails showing that

(a) the bilinear and linear forms $\widetilde{a}(\cdot,\cdot)$; $\widetilde{b}(\cdot,\cdot)$ and ℓ are continuous;

(b) j is convex and lower semi-continuous;

(c) there exists a constant $\alpha > 0$ such that

$$\widetilde{a}(\boldsymbol{v},\boldsymbol{v}) \geq \alpha \|\boldsymbol{v}\|_1^2 \quad \text{for all} \quad \boldsymbol{v} \in Z, \tag{2.7}$$

with $Z = \{\boldsymbol{w} \in V \ ; \ \widetilde{b}(\boldsymbol{w}; q) = 0 \ \text{ for all } q \in M\}$;

(d) there exists a constant $\beta > 0$ such that

$$\beta \|q\|_{M/Z^T} \leq \sup_{\boldsymbol{v}\in V} \frac{\widetilde{b}(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_1} \quad \text{for all} \quad \boldsymbol{v} \in V \tag{2.8}$$

with $Z^T = \{q \in M \ ; \ \widetilde{b}(\boldsymbol{w}, q) = 0 \ \text{ for all } \boldsymbol{w} \in V\}$.

We then state that (following Han and Reddy [15])

PROPOSITION 2.1. *Problem (2.3) has a unique solution, which satisfies the estimate*

$$\|\boldsymbol{v}\|_1 + \|p\|_0 \leq C(\Omega,\nu)(\|\boldsymbol{f}\| + \|g\|_{L^2(S)}). \tag{2.9}$$

3. Discontinuous Galerkin approximations.

3.1. Notation and preliminaries. Assume that Ω is a convex domain with polygonal boundary, so that it can be entirely covered by finite number of triangles, say, K. Let h_K be the diameter of a triangle K and ρ_K the diameter of its inscribed circle. We let $h = \max\{h_K, K \ \text{ triangle in the partition of } \ \Omega\}$, and we denote by $\mathcal{T}_h$ the corresponding triangulation. We assume that the triangulation $\mathcal{T}_h$ is regular, that is (see Ciarlet [3]), there exists a parameter $\sigma > 0$ such that

$$\frac{h_K}{\rho_K} = \sigma_K \leq \sigma \qquad \text{for all} \quad K \in \mathcal{T}_h. \tag{3.1}$$

The set of all the edges of $\mathcal{T}_h$ is denoted by $\mathcal{E}_h$, the set of all edges that lie on S is $\mathcal{E}_h^S$, and the set of all edges that do not lie on S is denoted by $\mathcal{E}_h^\ddagger$. Next, we define the average and jump operators. To this end, let K^+ and K^- be two adjacent elements of $\mathcal{T}_h$, and $\boldsymbol{x}$ be an arbitrary point of the interior edge $E = \partial K^+ \cap \partial K^-$. For scalar-, vector-, and matrix-valued functions q, $\boldsymbol{v}$, and $\underline{\boldsymbol{\tau}}$, respectively, that are smooth inside each element each $K^\pm$, respectively, we define the following averages at $\boldsymbol{x} \in E = \partial K^+ \cap \partial K^-$:

$$\{q\} = \frac{1}{2}(q^+ + q^-), \quad \{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-), \quad \{\underline{\boldsymbol{\tau}}\} = \frac{1}{2}(\underline{\boldsymbol{\tau}}^+ + \underline{\boldsymbol{\tau}}^-).$$

Similarly, the jumps at $\boldsymbol{x} \in E = \partial K^+ \cap \partial K^-$ are given by

$$
\begin{aligned}
[q] &= q^+ \boldsymbol{n}_{K^+} + q^- \boldsymbol{n}_{K^-}, \\
[\boldsymbol{v}] &= \boldsymbol{v}^+ \cdot \boldsymbol{n}_{K^+} + \boldsymbol{v}^- \cdot \boldsymbol{n}_{K^-}, \\
[\underline{\boldsymbol{v}}] &= \boldsymbol{v}^+ \otimes \boldsymbol{n}_{K^+} + \boldsymbol{v}^- \otimes \boldsymbol{n}_{K^-}, \\
[\underline{\boldsymbol{\tau}}] &= \underline{\boldsymbol{\tau}}^+ + \underline{\boldsymbol{\tau}}^-.
\end{aligned}
$$

If E is an edge of element K that lies on $\partial\Omega$, then the averages and jumps are defined by

$$
\{q\} = q \ , \quad \{\boldsymbol{v}\} = \boldsymbol{v} \ , \quad \{\underline{\boldsymbol{\tau}}\} = \underline{\boldsymbol{\tau}} \, ,
$$

and

$$
[q] = q\boldsymbol{n} \quad , \quad [\boldsymbol{v}] = \boldsymbol{v} \cdot \boldsymbol{n} \quad , \quad [\underline{\boldsymbol{v}}] = \boldsymbol{v} \otimes \boldsymbol{n} \quad , \quad [\underline{\boldsymbol{\tau}}] = \underline{\boldsymbol{\tau}}\boldsymbol{n} \quad ,
$$

where $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$. Now for vectors $\boldsymbol{v}$ and tensors $\boldsymbol{\tau}$, piecewise smooth on $\mathcal{T}_h$, we recall the following identity

$$
\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{v} \cdot \boldsymbol{\tau} \boldsymbol{n} \, \mathrm{ds} = \sum_{E \in \mathcal{E}_h} \int_E [\underline{\boldsymbol{v}}] : \{\underline{\boldsymbol{\tau}}\} \, \mathrm{ds} + \sum_{E \in \mathcal{E}_h \backslash \partial\Omega} \int_E \{\boldsymbol{v}\} \cdot [\underline{\boldsymbol{\tau}}] \, \mathrm{ds} \, , \tag{3.2}
$$

obtained by re-arranging terms.
We denote by $\mathcal{P}_k(K)$ the space of polynomial functions of degree at most $k \geq 0$ on K, and we finally recall or introduce the following standard trace and inverse inequalities:

$$
\begin{aligned}
\|v\|_E &\leq C \left[h_E^{-1/2} \|v\|_K + h_E^{1/2} \|\nabla v\|_K \right] \\
& \qquad \text{for all edge } E \subset \partial K, \text{ and all } v \in H^1(\mathcal{T}_h), \\
\|v_h\|_E &\leq C h_E^{-1/2} \|v_h\|_K \ \text{ for all edge } E \subset \partial K, \ \text{ and all } v_h \in \mathcal{P}_k(K), \\
\|\nabla v_h\|_K &\leq C h_K^{-1} \|v_h\|_K \ \text{ for all } v_h \in \mathcal{P}_k(K),
\end{aligned} \tag{3.3}
$$

where C is a positive constant independent of h_K and E is an edge of K. Here and henceforth $\| \cdot \|_K$ and $\| \cdot \|_E$ will denote respectively the L^2-norms on an element K and edge E.

3.2. Variational formulation: "continuous formulation". Before introducing the discontinuous Galerkin scheme, it is important when dealing with error estimates, to set out in clear terms the regularity of the weak solution of the continuous problem. The question of regularity of solutions to Stokes flow driven by the slip boundary condition of friction type is a much more subtle matter than the corresponding question for Stokes flow with the non-slip boundary condition, and complete results are not yet available. In what follows, we require that both the gradient of the velocity, and the pressure have a trace on line segments. This entails taking $(\boldsymbol{v}, p) \in H^2_{loc}(\Omega)^2 \times H^1_{loc}(\Omega)$. These assumptions are somewhat stronger than what is known in the literature [16] at least for Lipschitz domains. Now, to introduce the discontinuous Galerkin formulation, we need additional notations.

Let $m \geq 1$ be a natural number so that we consider the following infinite dimensional spaces:

$$\begin{array}{rcl} H^m(\mathcal{T}_h)^2 & = & \{\boldsymbol{v} \in L^2(\Omega)^2, \quad \boldsymbol{v}|_K \in H^m(K)^2, \quad \text{for all} \quad K \in \mathcal{T}_h\}, \\ V(\mathcal{T}_h) & = & H^2(\mathcal{T}_h)^2, \\ M(\mathcal{T}_h) & = & \{q \in L^2_0(\Omega), \quad q|_K \in H^1(K), \quad \text{for all} \quad K \in \mathcal{T}_h\}, \end{array} \tag{3.4}$$

endowed with the following norms

$$\begin{array}{rcl} \|\boldsymbol{v}\|^2_{V(\mathcal{T}_h)} & = & \displaystyle\sum_{K \in \mathcal{T}_h} \left[\|\nabla \boldsymbol{v}\|^2_K + h_K^2 |\boldsymbol{v}|^2_{2,K}\right] + \sum_{E \in \mathcal{E}_h^\ddagger} \frac{1}{h_E} \|[\underline{\boldsymbol{v}}]\|^2_E, \\ & \text{and} & \\ \|q\|^2_{M(\mathcal{T}_h)} & = & \displaystyle\sum_{K \in \mathcal{T}_h} \left[\|q\|^2_K + h_K^2 \|\nabla q\|^2_K\right]. \end{array} \tag{3.5}$$

Next, for $\boldsymbol{v}, \boldsymbol{w}$ in $V(\mathcal{T}_h)$, and q in $M(\mathcal{T}_h)$, we set

$$\begin{array}{rcl} j(\boldsymbol{v}) & = & \displaystyle\sum_{E \in \mathcal{E}_h^S} \int_E g|\boldsymbol{v}_{\boldsymbol{\tau}}| \, \mathrm{ds}, \\ a(\boldsymbol{v}, \boldsymbol{w}) & = & \displaystyle\sum_K \int_K \nabla \boldsymbol{v} : \nabla \boldsymbol{w} \mathrm{dx} - \sum_{E \in \mathcal{E}_h^\ddagger} \int_E (\{\nabla \boldsymbol{v}\} : [\underline{\boldsymbol{w}}] + \alpha \{\nabla \boldsymbol{w}\} : [\underline{\boldsymbol{v}}]) \, \mathrm{ds} \\ & & \displaystyle + \sum_{E \in \mathcal{E}_h^\ddagger} \frac{\beta}{h_E} \int_E [\underline{\boldsymbol{v}}] : [\underline{\boldsymbol{w}}] \mathrm{ds}, \\ b(\boldsymbol{v}, q) & = & \displaystyle -\sum_K \int_K q \, \mathrm{div} \, \boldsymbol{v} \mathrm{dx} + \sum_{E \in \mathcal{E}_h^\ddagger} \int_E \{q\}[\boldsymbol{v}] \mathrm{ds}, \end{array} \tag{3.6}$$

where $\alpha = \{\pm 1\}$ and β is a positive constant that will be made precise later. With these forms, we then consider the following problem:

$$\begin{cases} \text{Find} \quad (\boldsymbol{v}, p) \in V(\mathcal{T}_h) \times M(\mathcal{T}_h), \quad \text{such that} \\ \text{for all} \quad (\boldsymbol{w}, q) \in V(\mathcal{T}_h) \times M(\mathcal{T}_h), \\ a(\boldsymbol{v}, \boldsymbol{w} - \boldsymbol{v}) + b(\boldsymbol{w} - \boldsymbol{v}, p) + j(\boldsymbol{w}) - j(\boldsymbol{v}) \geq \ell(\boldsymbol{w} - \boldsymbol{v}), \\ b(\boldsymbol{v}, q) = 0. \end{cases} \tag{3.7}$$

Remark 3.1. One readily sees that the variational formulation (3.7) is equivalent to the following problem:

$$\begin{cases} \text{Find} \quad (\boldsymbol{v}, p) \in V(\mathcal{T}_h) \times M(\mathcal{T}_h), \quad \text{such that} \\ \text{for all} \quad (\boldsymbol{w}, q) \in V(\mathcal{T}_h) \times M(\mathcal{T}_h), \\ A(\boldsymbol{v}, p; \boldsymbol{w}, q) + j(\boldsymbol{w}) - j(\boldsymbol{v}) \geq \ell(\boldsymbol{w} - \boldsymbol{v}), \end{cases} \tag{3.8}$$

where $A(\boldsymbol{v}, p; \boldsymbol{w}, q) = a(\boldsymbol{v}, \boldsymbol{w}) + b(\boldsymbol{w}, p) - b(\boldsymbol{v}, q)$.

For the analysis of (3.8), we first recall the following crucial inequality obtained in [11]: for each real number $p \in [2, \infty)$, there is a constant $C(p)$ such that

$$\|\boldsymbol{v}\|_{L^p(\Omega)} \le C(p)\|\boldsymbol{v}\|_{V(\mathcal{T}_h)}, \quad \text{for all } \boldsymbol{v} \in V(\mathcal{T}_h). \tag{3.9}$$

With (3.9) in mind, Cauchy Shwarz and Holder's inequalities, one may readily deduce the following

LEMMA 3.1. *There are positive constants C_1, C_2 independent of the mesh size h such that for all $\boldsymbol{w}, \boldsymbol{v}$ in $V(\mathcal{T}_h)$ there hold*

$$\begin{aligned} j(\boldsymbol{v}) - j(\boldsymbol{w}) &\le C_1 \left[\sum_{E \in \mathcal{E}_h^S} \|g\|^2_{L^2(E)}\right]^{1/2} \|\boldsymbol{v} - \boldsymbol{w}\|_{V(\mathcal{T}_h)}, \\ \ell(\boldsymbol{w}) &\le C_2 \|\boldsymbol{w}\|_{V(\mathcal{T}_h)}. \end{aligned}$$

Now for all $\boldsymbol{w}$ and $\boldsymbol{v}$ in $V(\mathcal{T}_h)$, one deduces from $(3.3)_1$, that

$$\begin{aligned} &\left| \sum_{E \in \mathcal{E}_h^\ddagger} \int_E (\{\nabla \boldsymbol{v}\} : [\underline{\boldsymbol{w}}] + \alpha \{\nabla \boldsymbol{w}\} : [\underline{\boldsymbol{v}}]) \right| \\ &\le C_1 \left[\sum_{K \in \mathcal{T}_h} \left(\|\nabla \boldsymbol{v}\|_K^2 + h_K^2 |\boldsymbol{v}|_{2,K}^2\right)\right]^{1/2} \left[\sum_{E \in \mathcal{E}^\ddagger} h_E^{-1} \|[\underline{\boldsymbol{w}}]\|_E^2\right]^{1/2} \\ &+ C_2 \left[\sum_{K \in \mathcal{T}_h} \left(\|\nabla \boldsymbol{w}\|_K^2 + h_K^2 |\boldsymbol{w}|_{2,K}^2\right)\right]^{1/2} \left[\sum_{E \in \mathcal{E}^\ddagger} h_E^{-1} \|[\underline{\boldsymbol{v}}]\|_E^2\right]^{1/2} \end{aligned} \tag{3.10}$$

which, Holder's inequality leads to

LEMMA 3.2. *There is a constant C, independent of h, such that*

$$a(\boldsymbol{v}, \boldsymbol{w}) \le C \|\boldsymbol{v}\|_{V(\mathcal{T}_h)} \|\boldsymbol{w}\|_{V(\mathcal{T}_h)} \quad \textit{for all} \;\; \boldsymbol{v}, \boldsymbol{w} \in V(\mathcal{T}_h).$$

Similarly one obtains,

$$\left| \sum_{E \in \mathcal{E}_h^\ddagger} \int_E \{q\}[\boldsymbol{v}] \right| \le C_1 \left[\sum_{K \in \mathcal{T}_h} \|q\|_K^2 + h_K^2 |q|_{1,K}^2\right]^{1/2} \left[\sum_{E \in \mathcal{E}_h^\ddagger} h_E^{-1} \|[\boldsymbol{v}]\|_E^2\right]^{1/2}, \tag{3.11}$$

which with $(3.5)_2$ and (3.11) gives

LEMMA 3.3. *There is a constant C, independent of h, such that*

$$b(\boldsymbol{v}, q) \leq C\|\boldsymbol{v}\|_{V(\mathcal{T}_h)}\|q\|_{M(\mathcal{T}_h)} \quad \textit{for all} \quad \boldsymbol{v}, q \in V(\mathcal{T}_h) \times M(\mathcal{T}_h).$$

As far as the coercivity of the bilinear form $a(\cdot,\cdot)$ goes, there are two possibilities:

- For $\alpha = -1$ and $\boldsymbol{w} = \boldsymbol{v}$ in (3.6), one obtains

$$a(\boldsymbol{v}, \boldsymbol{v}) \quad = \quad \sum_K \|\nabla \boldsymbol{v}\|_K^2 + \beta \sum_{E \in \mathcal{E}_h^\ddagger} \frac{1}{h_E} \|[\underline{\boldsymbol{v}}]\|_E^2 = \min(1, \beta)\ \|\boldsymbol{v}\|_{V(\mathcal{T}_h)}^2.$$

- For $\alpha = 1$ and $\boldsymbol{w} = \boldsymbol{v}$ in (3.6), yields

$$\begin{aligned} a(\boldsymbol{v}, \boldsymbol{v}) = {} & \sum_K \|\nabla \boldsymbol{v}\|_K^2 - 2 \sum_{E \in \mathcal{E}_h^\ddagger} \int_E \{\nabla \boldsymbol{v}\} : [\underline{\boldsymbol{v}}] + \beta \sum_{E \in \mathcal{E}_h^\ddagger} h_E^{-1} \|[\underline{\boldsymbol{v}}]\|_E^2 \\ & \geq \sum_K \|\nabla \boldsymbol{v}\|_K^2 - C_1 \left[\sum_{K \in \mathcal{T}_h} \left(\|\nabla \boldsymbol{v}\|_K^2 + h_K^2 |\boldsymbol{v}|_{2,K}^2 \right) \right]^{1/2} \left[\sum_{E \in \mathcal{E}^\ddagger} h_E^{-1} \|[\underline{\boldsymbol{v}}]\|_E^2 \right]^{1/2} \\ & + \beta \sum_{E \in \mathcal{E}_h^\ddagger} h_E^{-1} \|[\underline{\boldsymbol{v}}]\|_E^2. \end{aligned}$$

So, by applying Young's inequality and choosing β appropriately, we can state the following

LEMMA 3.4. *There is a constant C, independent of h, such that*

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq C\|\boldsymbol{v}\|_{V(\mathcal{T}_h)}^2 \quad \textit{for all} \quad \boldsymbol{v} \in V(\mathcal{T}_h).$$

3.3. Finite element discretization: error estimates. In this paragraph, we discuss the solvability of the finite element scheme associated to (3.7). We also derive a priori error estimates in the energy norm of the error for the velocity and the mesh dependent norm of the error for the pressure.
In order to approximate $\boldsymbol{v}$ and p, we introduce two finite dimensional spaces $V_h \subset V(\mathcal{T}_h)$ and $M_h \subset M(\mathcal{T}_h)$, define as follows

$$\begin{aligned} V_h &= \{\boldsymbol{v}_h \in V(\mathcal{T}_h); \ \text{ for all } K \in \mathcal{T}_h, \ \ \boldsymbol{v}_h \in \mathcal{P}_k^2(K)\}, \\ M_h &= \{q_h \in M(\mathcal{T}_h); \ \text{ for all } K \in \mathcal{T}_h, \ \ q_h \in \mathcal{P}_{k-1}(K)\}. \end{aligned}$$

We endow V_h and M_h respectively with the usual norms

$$
\begin{aligned}
\|\boldsymbol{v}_h\|_{V_h}^2 &= \sum_{K\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h\|_K^2 + \sum_{E\in\mathcal{E}_h^\ddagger} h_E^{-1}\|[\underline{\boldsymbol{v}_h}]\|_E^2,\\
\|q_h\|_{M_h}^2 &= \sum_{K\in\mathcal{T}_h} \|q_h\|_K^2.
\end{aligned}
\tag{3.12}
$$

We note immediately that, using the inverse inequality $(3.3)_3$, we have that on V_h and M_h the DG norms (3.12) are equivalent to the norms (3.5) originally introduced in $V(\mathcal{T}_h)$ and $M(\mathcal{T}_h)$ respectively. In particular there are positive constants C_1, C_2 independent of the mesh size h such that

$$
\|\boldsymbol{v}_h\|_{V(\mathcal{T}_h)} \le C_1\|\boldsymbol{v}_h\|_{V_h} \quad \le \quad C_1\|\boldsymbol{v}_h\|_{V(\mathcal{T}_h)}, \tag{3.13}
$$

and

$$
\|q_h\|_{M(\mathcal{T}_h)} \le C_2\|q_h\|_{M_h} \quad \le \quad C_2\|q_h\|_{M(\mathcal{T}_h)}. \tag{3.14}
$$

The discrete problem is formulated as follows:

$$
\begin{cases}
\text{Find} \quad (\boldsymbol{v}_h, p_h) \in V_h \times M_h, \quad \text{such that}\\
\text{for all} \ \ (\boldsymbol{w}_h, q_h) \in V_h \times M_h,\\
a(\boldsymbol{v}_h, \boldsymbol{w}_h - \boldsymbol{v}_h) + b(\boldsymbol{w}_h - \boldsymbol{v}_h, p_h) + j(\boldsymbol{w}_h) - j(\boldsymbol{v}_h) \ge \ell(\boldsymbol{w}_h - \boldsymbol{v}_h),\\
b(\boldsymbol{v}_h, q_h) = 0.
\end{cases}
\tag{3.15}
$$

Remark 3.2. With the bilinear form $A(\cdot,\cdot)$ introduced earlier (see Remark 3.1), the variational formulation (3.15) can be re-written as

$$
\begin{cases}
\text{Find} \quad (\boldsymbol{v}_h, p_h) \in V_h \times M_h, \quad \text{such that}\\
\text{for all} \ \ (\boldsymbol{w}_h, q_h) \in V_h \times M_h,\\
A(\boldsymbol{v}_h, p_h; \boldsymbol{w}_h, q_h) + j(\boldsymbol{w}_h) - j(\boldsymbol{v}_h) \ge \ell(\boldsymbol{w}_h - \boldsymbol{v}_h).
\end{cases}
\tag{3.16}
$$

Based on (3.13), and (3.14), Lemmas 3.1, 3.2, 3.3 and 3.4, one obtains the following result

Proposition 3.1. *There are positive constants C_1, C_2, C_3, C_4 independent of the mesh size h such that for all $\boldsymbol{v}_h$ and $\boldsymbol{w}_h$ in V_h and q_h in M_h;*

$$
\begin{aligned}
j(\boldsymbol{v}_h) - j(\boldsymbol{w}_h) &\le C_1 \left[\sum_{E\in\mathcal{E}_h^S} \|g\|_{L^2(E)}^2\right]^{1/2} \|\boldsymbol{v}_h - \boldsymbol{w}_h\|_{V_h},\\
\ell(\boldsymbol{w}_h) &\le C_2\|\boldsymbol{w}_h\|_{V_h},\\
b(\boldsymbol{v}_h, q_h) &\le C_3\|\boldsymbol{v}_h\|_{V_h}\|q_h\|_{M_h},\\
a(\boldsymbol{v}_h, \boldsymbol{v}_h) &\ge C_4\|\boldsymbol{v}_h\|_{V_h}^2.
\end{aligned}
$$

Finally, to claim existence of DG solution $\boldsymbol{v}_h$ and p_h defined by (3.15), it is important to obtain the inf-sup condition involving the bilinear form $b(\cdot,\cdot)$ in the DG norm (3.12). That is, there is a constant C independent of h such that

$$\text{for all } q_h \in M_h, \quad C\|q\|_{M_h} \leq \sup_{\boldsymbol{v}_h\in V_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{V_h}}. \tag{3.17}$$

It should be noted that the compatibility condition (3.17) has been proved in [11] in the case where the continuous space of velocity $V = H_0^1(\Omega)^2$, and Ω is made of two sub non-overlapping domains. The situation here is slightly different, indeed $V = \{\boldsymbol{v} \in H^1(\Omega)^2 \ , \boldsymbol{v}|_\Gamma = 0 \ , \ \boldsymbol{v}\cdot\boldsymbol{n}|_S = 0\}$. But nevertheless, the proof in [11] can be adapted and will not be repeated here. Hence, the solvability of the variational problem (3.15) is obtained, and can be summarized as follows

Proposition 3.2. *The DG finite element formulation (3.15) has a unique solution $(\boldsymbol{v}_h, p_h)$ in $V_h \times M_h$.*

Having the existence of the solution of (3.15), our next task is to measure the quantities $\boldsymbol{v}-\boldsymbol{v}_h$, and $p-p_h$ in appropriate norms. For that purpose, we first put in place adequate instruments.
The pressure will be interpolated by the classical operator $\mathcal{I}_h \in \mathcal{L}(L_0^2(\Omega), M_h)$ (see [10]) given as follows. For any $K \in \mathcal{T}_h$

$$\int_K z_h(\mathcal{I}_h(q) - q) = 0, \quad \text{for all} \quad (q, z_h) \in L_0^2(\Omega) \times \mathcal{P}_{k-1}(K), \tag{3.18}$$

satisfying the following error estimate for $k \geq 1$ and $m = 0, 1$

$$\|q - \mathcal{I}_h(q)\|_{m,K} \leq Ch_K^{k-m}\|\nabla q\|_{k,K}, \text{ for all } \ q \in H^k(\Omega) \cap L_0^2(\Omega). \tag{3.19}$$

The velocity is interpolated by a Clément type operator $R_h \in \mathcal{L}(H^1(\Omega)^2, V_h)$ introduced by Girault and Scott [12] and satisfies for any $K \in \mathcal{T}_h$,

$$\begin{aligned}
b(R_h(\boldsymbol{w}) - \boldsymbol{w}, q_h) &= 0 \ \text{ for all } \ q_h \in M_h, \ \boldsymbol{w} \in V, & (3.20)\\
\|R_h(\boldsymbol{w}) - \boldsymbol{w}\|_{V(\mathcal{T}_h)} &\leq Ch^k|\boldsymbol{w}|_{k+1,\Omega} \quad \text{for all } \boldsymbol{w} \in H^{k+1}(\Omega)^2 \cap V, & (3.21)\\
\|R_h(\boldsymbol{w}) - \boldsymbol{w}\|_K &\leq Ch_K^{k+1}|\boldsymbol{w}|_{k+1,\triangle_K} \quad \text{for all } \boldsymbol{w} \in H^{k+1}(\Omega)^2, & (3.22)
\end{aligned}$$

where $\triangle_K$ is a suitable macro element containing K. Having these preliminaries in place, we are now in a position to state the *a priori* bound for the error committed.

Proposition 3.3. *Let $(\boldsymbol{f}, g) \in L^2(\Omega)^2 \times L^\infty(\Omega)$. Let $k = 1, 2$ or 3 be the degree of the polynomials in the definition of V_h and assume that the solution $(\boldsymbol{v}, p)$ of (1.1)–(1.5) belongs to $H^{k+1}(\Omega)^2 \times H^k(\Omega)$. Then the discontinuous Galerkin solution $(\boldsymbol{v}_h, p_h)$ satisfies the error estimate*

$$\begin{aligned}
\|\boldsymbol{v} - \boldsymbol{v}_h\|_{V(\mathcal{T}_h)} &\leq C\|\mathcal{I}_h(p) - p\|_{M(\mathcal{T}_h)} + C\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|_{V(\mathcal{T}_h)}^{1/2} \\
&+ C\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|_{V(\mathcal{T}_h)},
\end{aligned} \tag{3.23}$$

where C is a positive constant independent of h.

Proof. We start by decomposing the error in two parts, namely

$$\boldsymbol{e} = \boldsymbol{v} - \boldsymbol{v}_h = (\boldsymbol{v} - R_h(\boldsymbol{v})) + (R_h(\boldsymbol{v}) - \boldsymbol{v}_h)\,.$$

With the above decomposition of the error, it appears that we just need to estimate $R_h(\boldsymbol{v}) - \boldsymbol{v}_h$, as $\boldsymbol{v} - R_h(\boldsymbol{v})$ is given by (3.21) and (3.22).
Now, we use (3.13), the ellipticity of the bilinear form $a(\cdot,\cdot)$, and the linearity of the bilinear form $a(\cdot,\cdot)$ to obtain

$$\begin{aligned}
\|\boldsymbol{v}_h - R_h(\boldsymbol{v})\|^2_{V(\mathcal{T}_h)} &\leq C\|\boldsymbol{v}_h - R_h(\boldsymbol{v})\|^2_{V_h} \\
&\leq C\; a(\boldsymbol{v}_h - R_h(\boldsymbol{v}), \boldsymbol{v}_h - R_h(\boldsymbol{v})) \\
&= C\; [a(\boldsymbol{v}_h - \boldsymbol{v}, \boldsymbol{v}_h - R_h(\boldsymbol{v})) + a(\boldsymbol{v} - R_h(\boldsymbol{v}), \boldsymbol{v}_h - R_h(\boldsymbol{v}))] \\
&= C\; (T_1 + T_2), \qquad (3.24)
\end{aligned}$$

with

$$T_1 = a(\boldsymbol{v}_h - \boldsymbol{v}, \boldsymbol{v}_h - R_h(\boldsymbol{v})), \quad T_2 = a(\boldsymbol{v} - R_h(\boldsymbol{v}), \boldsymbol{v}_h - R_h(\boldsymbol{v})).$$

Thanks to Lemma 3.2, one has

$$T_2 \leq C\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|_{V(\mathcal{T}_h)}\|\boldsymbol{v}_h - R_h(\boldsymbol{v})\|_{V(\mathcal{T}_h)}. \qquad (3.25)$$

To proceed with the bound of T_1, we re-write it as follows

$$T_1 = a(\boldsymbol{v}_h, \boldsymbol{v}_h - R_h(\boldsymbol{v})) - a(\boldsymbol{v}, \boldsymbol{v}_h - R_h(\boldsymbol{v})).$$

We next express $a(\boldsymbol{v}_h, \boldsymbol{v}_h - R_h(\boldsymbol{v}))$ and $a(\boldsymbol{v}, \boldsymbol{v}_h - R_h(\boldsymbol{v}))$ using (3.15) and (3.7). We first set $\boldsymbol{w}_h = R_h(\boldsymbol{v})$ in (3.15), then

$$a(\boldsymbol{v}_h, \boldsymbol{v}_h - R_h(\boldsymbol{v})) \leq b(R_h(\boldsymbol{v}) - \boldsymbol{v}_h, p_h) + j(R_h(\boldsymbol{v})) - j(\boldsymbol{v}_h) - \ell(R_h(\boldsymbol{v}) - \boldsymbol{v}_h).$$

Hence T_1 is bounded as follows

$$T_1 \leq b(R_h(\boldsymbol{v}) - \boldsymbol{v}_h, p_h) + j(R_h(\boldsymbol{v})) - j(\boldsymbol{v}_h) - \ell(R_h(\boldsymbol{v}) - \boldsymbol{v}_h) - a(\boldsymbol{v}, \boldsymbol{v}_h - R_h(\boldsymbol{v})). \qquad (3.26)$$

Next, we set successively $\boldsymbol{w}$ equal $\boldsymbol{v}_h$ and $\boldsymbol{v} - R_h(\boldsymbol{v})$ in (3.7), one obtains after adding the resulting equations

$$-a(\boldsymbol{v}, \boldsymbol{v}_h - R_h(\boldsymbol{v})) \leq b(\boldsymbol{v}_h - R_h(\boldsymbol{v}), p) + j(\boldsymbol{v}_h) - 2j(\boldsymbol{v}) + j(2\boldsymbol{v} - R_h(\boldsymbol{v})) - \ell(\boldsymbol{v}_h - R_h(\boldsymbol{v})). \qquad (3.27)$$

Returning to (3.26) with (3.27) one obtains

$$\begin{aligned}
T_1 &\leq b(R_h(\boldsymbol{v}) - \boldsymbol{v}_h, p_h - p) + 4j(R_h(\boldsymbol{v}) - \boldsymbol{v}) \\
&\leq b(R_h(\boldsymbol{v}) - \boldsymbol{v}_h, p_h - \mathcal{I}_h(p)) + b(R_h(\boldsymbol{v}) - \boldsymbol{v}_h, \mathcal{I}_h(p) - p) \qquad (3.28) \\
&\quad +4j(R_h(\boldsymbol{v}) - \boldsymbol{v}),
\end{aligned}$$

where the triangle inequality has been used for the terms involving $j(\cdot)$. Now taking into account the weak incompressibility condition in (3.15) and (3.7), one gets

$$b(\boldsymbol{v}_h - \boldsymbol{v}, q_h) = 0 \quad \text{for all } q_h \in M_h$$

which by the linearity of $b(\cdot,\cdot)$ and (3.20) yields

$$b(R_h(\boldsymbol{v}) - \boldsymbol{v}_h, q_h) = b(R_h(\boldsymbol{v}) - \boldsymbol{v}, q_h) = 0 \quad \text{for all } q_h \in M_h. \tag{3.29}$$

So, (3.28) becomes

$$T_1 \quad \leq \quad b(R_h(\boldsymbol{v}) - \boldsymbol{v}_h, \mathcal{I}_h(p) - p) + 4j(R_h(\boldsymbol{v}) - \boldsymbol{v}), \tag{3.30}$$

which by Lemma 3.1 and Lemma 3.3 gives

$$T_1 \quad \leq \quad C\|R_h(\boldsymbol{v}) - \boldsymbol{v}_h\|_{V(\mathcal{T}_h)}\|\mathcal{I}_h(p) - p\|_{M(\mathcal{T}_h)} + C\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|_{V(\mathcal{T}_h)}. \tag{3.31}$$

Now, returning to (3.24) with (3.25) and (3.31) and Young's inequality, one obtains

$$\begin{aligned} \|\boldsymbol{v}_h - R_h(\boldsymbol{v})\|_{V(\mathcal{T}_h)} \quad &\leq \quad C\|\mathcal{I}_h(p) - p\|_{M(\mathcal{T}_h)} + C\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|^{1/2}_{V(\mathcal{T}_h)} \\ &+ \quad C\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|_{V(\mathcal{T}_h)}. \end{aligned} \tag{3.32}$$

The result announced in Proposition 3.3 then follows from (3.32) and the triangle inequality. □

Remark 3.3. (a) Once more, it is apparent that the error estimate depends solely upon the regularity of the solution $(\boldsymbol{v}, p)$.

(b) The error estimate obtained in Proposition 3.3 is typical for variational inequalities of second kind in the sense that the error is dominated by the contribution appearing on the boundary, that is $\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|^{1/2}_{V(\mathcal{T}_h)}$.

(c) One observes that for DG mixed finite element approximations, the error estimates on the unknown variables cannot necessarily be obtained simultaneously unlike the classical mixed finite element method (see [10]). Hence, the DG mixed method can be regarded as "decoupling" approach when it comes to finding error estimates.

(d) From the interpolation error estimates (3.19)—(3.22), and assuming that the exact solution $(\boldsymbol{v}, p) \in H^{k+1}(\Omega)^2 \times H^k(\Omega)$, then Proposition 3.3 gives

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{V(\mathcal{T}_h)} \leq Ch^{k/2}(|\boldsymbol{v}|_{k+1} + |p|_k).$$

(e) Proposition 3.3 does not yield optimal estimates even if the solution $(\boldsymbol{v}, p)$ is in $H^{k+1}(\Omega)^2 \times H^k(\Omega)$. But an optimal rate of convergence can be achieved if one assumes the following regularity of the solution $(\boldsymbol{v}, p) \in H^{k+1}(\overline{\Omega})^2 \times H^k(\Omega)$. We refer the reader to [21], where a similar analysis has been done. Of course, the situation here is slightly more involved because we are dealing with a saddle point problem, but it is conjectured that the arguments presented in [21] carry over to our problem.

As far as the error estimate about the pressure is concerned, we have the following

PROPOSITION 3.4. *Under the assumption of proposition 3.3, then we have the following a priori error estimate for the pressure*

$$\|p - p_h\|_{M(\mathcal{T}_h)} \leq C\|p - \mathcal{I}_h(p)\|_{M(\mathcal{T}_h)} + C\|\boldsymbol{v} - \boldsymbol{v}_h\|_{V(\mathcal{T}_h)}, \tag{3.33}$$

where C is a positive constant independent of h.

Proof. We first let $\widetilde{V}_h = V_h \cap \{\boldsymbol{w}_h|_S = 0\}$.
For $\boldsymbol{w} = \boldsymbol{v} \pm \boldsymbol{w}_h$ with $\boldsymbol{w}_h \in \widetilde{V}_h$ in (3.7) one obtains (after utilization of the triangle inequality and the fact that $\boldsymbol{w}_h|_S = 0$)

$$a(\boldsymbol{v}, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p) \geq \ell(\boldsymbol{w}_h) \text{ for all } \boldsymbol{w}_h \in \widetilde{V}_h, \tag{3.34}$$

$$-a(\boldsymbol{v}, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p) \geq -\ell(\boldsymbol{w}_h) \text{ for all } \boldsymbol{w}_h \in \widetilde{V}_h. \tag{3.35}$$

Putting together (3.34) and (3.35) yields

$$a(\boldsymbol{v}, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p) = \ell(\boldsymbol{w}_h) \text{ for all } \boldsymbol{w}_h \in \widetilde{V}_h. \tag{3.36}$$

Repeating the same steps with the DG finite element approximations (3.15), one gets

$$a(\boldsymbol{v}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h) = \ell(\boldsymbol{w}_h) \text{ for all } \boldsymbol{w}_h \in \widetilde{V}_h. \tag{3.37}$$

Subtracting (3.37) from (3.36), we obtain

$$b(\boldsymbol{w}_h, p_h - p) = a(\boldsymbol{v}_h - \boldsymbol{v}, \boldsymbol{w}_h) \text{ for all } \boldsymbol{w}_h \in \widetilde{V}_h. \tag{3.38}$$

Now, from the compatibility condition (3.17), (3.14), (3.13) and (3.38), Lemma 3.2, and Lemma 3.3, one gets for $q_h \in M_h$

$$\begin{aligned} C\|p_h - q_h\|_{M(\mathcal{T}_h)} &\leq \sup_{\boldsymbol{w}_h \in \widetilde{V}_h} \frac{b(\boldsymbol{w}_h, p_h - q_h)}{\|\boldsymbol{w}_h\|_{V(\mathcal{T}_h)}} \\ &= \sup_{\boldsymbol{w}_h \in \widetilde{V}_h} \frac{b(\boldsymbol{w}_h, p_h - p) + b(\boldsymbol{w}_h, p - q_h)}{\|\boldsymbol{w}_h\|_{V(\mathcal{T}_h)}} \\ &= \sup_{\boldsymbol{w}_h \in \widetilde{V}_h} \frac{a(\boldsymbol{v}_h - \boldsymbol{v}, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p - q_h)}{\|\boldsymbol{w}_h\|_{V(\mathcal{T}_h)}} \\ &\leq C_1\|\boldsymbol{v}_h - \boldsymbol{v}\|_{V(\mathcal{T}_h)} + C_2\|p - q_h\|_{M(\mathcal{T}_h)}. \end{aligned} \tag{3.39}$$

Hence by the triangle inequality $\|p - p_h\|_{M(\mathcal{T}_h)} \leq \|p - q_h\|_{M(\mathcal{T}_h)} + \|q_h - p_h\|_{M(\mathcal{T}_h)}$ and taking $q_h = \mathcal{I}_h(p)$ one obtains the desired inequality. □

4. Conclusion. In this work, we have derived *a priori* error estimates for a discontinuous Galerkin approximation of the steady incompressible Stokes driven by a slip boundary condition of friction type in two dimensions. We have considered symmetric and non-symmetric discontinuous approximations and established convergence of the approximations by using the energy norm for both the velocity and pressure. Our analysis is based on a suitable extension of Babuska-Brezzi's theory for mixed problems. It is observed that the convergence rate depends on the regularity of the solution, while the *a priori* error estimate is controlled by the expression on the boundary S. To our knowledge, this is the first study of discontinuous Galerkin approximations for Stokes problem driven a slip boundary condition of friction type. Numerical experiments are under investigation.

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