

# SHEAR FLOWS OF A NEW CLASS OF POWER-LAW FLUIDS\*

CHRISTIAAN LE ROUX, Pretoria,  
KUMBAKONAM R. RAJAGOPAL, College Station

*Abstract.* We consider the flow of a class of incompressible fluids which are constitutively defined by the symmetric part of the velocity gradient being a function, which can be non-monotone, of the deviator of the stress tensor. These models are generalizations of the stress power-law models introduced and studied by *J. Málek, V. Průša, K. R. Rajagopal*: Generalizations of the Navier-Stokes fluid from a new perspective. *Int. J. Eng. Sci.* 48 (2010), 1907–1924. We discuss a potential application of the new models and then consider some simple boundary-value problems, namely steady planar Couette and Poiseuille flows with no-slip and slip boundary conditions. We show that these problems can have more than one solution and that the multiplicity of the solutions depends on the values of the model parameters as well as the choice of boundary conditions.

*Keywords:* non-Newtonian fluid, Couette flow, Poiseuille flow, slip boundary condition

## 1. INTRODUCTION

Recently, Rajagopal [11], [12], [13] discussed a class of implicit models for describing the response of fluids that include the class of incompressible fluids with pressure dependent viscosity. While the classical rate type viscoelastic fluid models due to Burgers [3] and Oldroyd [9], [10] are implicit models<sup>1</sup>, they cannot describe fluids that have pressure dependent viscosity. Moreover, when the rate dependent terms are absent these models do not yet have an implicit structure. The very general

---

\*K. R. Rajagopal thanks the National Science Foundation for support of his work.

<sup>1</sup>The seminal viscoelastic fluid model due to Maxwell [7] is not an implicit model. It is not an explicit expression for the Cauchy stress as a function of the velocity gradient; it is however an explicit expression for the symmetric part of the velocity gradient in terms of the Cauchy stress and its time derivative.

class of fluids introduced by Noll [8], namely simple fluids, consists of explicit models wherein the Cauchy stress tensor is determined by the history of the density and the deformation in the case of compressible simple fluids, and is determined within a spherical part of the history of the deformation gradient in the incompressible case. The latter class of models cannot describe fluids with pressure dependent viscosity<sup>2</sup>. These models also cannot describe the response of the class of materials that are referred to as “Bingham fluids”<sup>3</sup>.

Let us consider the class of fluid models that are usually referred to as generalized Stokesian fluids. In such fluids, the Cauchy stress tensor  $\mathbb{T}$  is a function of the density and the symmetric part of the velocity gradient, i.e.

$$(1.1) \quad \mathbb{T} = \mathbb{F}(\varrho, \mathbb{D}),$$

where  $\varrho$  is the density and  $\mathbb{D} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ , where  $\mathbf{v}$  is the velocity. The classical compressible and incompressible Navier-Stokes fluids are given by the constitutive relations

$$(1.2) \quad \mathbb{T} = -p(\varrho)\mathbb{1} + \lambda(\varrho)(\text{tr } \mathbb{D})\mathbb{1} + 2\mu(\varrho)\mathbb{D}$$

and

$$(1.3) \quad \mathbb{T} = -\hat{p}\mathbb{1} + 2\hat{\mu}\mathbb{D},$$

respectively, where  $p(\varrho)$  is the thermodynamic pressure,  $\lambda(\varrho)$  is the bulk viscosity,  $\mu(\varrho)$  is the shear viscosity,  $\mathbb{1}$  is the identity tensor,  $\text{tr } \mathbb{D}$  is the trace of  $\mathbb{D}$ ,  $\hat{p}$  is the indeterminate part of the stress due to the constraint of incompressibility, and  $\hat{\mu}$  is the viscosity. Relation (1.3) can be written as  $\mathbb{D} = (2\hat{\mu})^{-1}(\mathbb{T} + \hat{p}\mathbb{1})$ . Moreover, relation (1.2) can be rewritten as

$$(1.4) \quad \mathbb{D} = \frac{2\mu(\varrho)p(\varrho) - \lambda(\varrho)(\text{tr } \mathbb{T})}{2\mu(\varrho)(3\lambda(\varrho) + 2\mu(\varrho))} \mathbb{1} + \frac{1}{2\mu(\varrho)} \mathbb{T}$$

if  $3\lambda(\varrho) + 2\mu(\varrho) \neq 0$ .<sup>4</sup> Thus, in these cases,  $\mathbb{D}$  can be expressed explicitly in terms of  $\varrho$  and  $\mathbb{T}$ :

$$(1.5) \quad \mathbb{D} = \mathbb{H}(\varrho, \mathbb{T}).$$

---

<sup>2</sup> In a general simple fluid the material moduli cannot depend on the Lagrange multiplier associated with a constraint.

<sup>3</sup> If by a fluid we mean a body that cannot resist shear stress, then what is referred to as a “Bingham fluid” is not a fluid.

<sup>4</sup> If  $3\lambda(\varrho) + 2\mu(\varrho) = 0$ , which is the Stokes assumption, then the thermodynamic pressure is the mean normal stress, i.e.  $p(\varrho) = -\frac{1}{3} \text{tr } \mathbb{T}$ , and  $\mathbb{D} - \frac{1}{3}(\text{tr } \mathbb{D})\mathbb{1} = (2\mu(\varrho))^{-1}(\mathbb{T} - \frac{1}{3}(\text{tr } \mathbb{T})\mathbb{1})$ . So if the motion is isochoric, i.e.  $\text{tr } \mathbb{D} = 0$ , we obtain a relation of the form (1.5).

In general, relation (1.1) cannot be inverted to obtain an expression of the form (1.5). Similarly, if we are to start with a class of models of the general form (1.5), if this relation is invertible, we immediately recover models that belong to the classical Stokesian fluid. However, there is a whole swathe of non-invertible models that are potential candidates to describe interesting phenomena that cannot be described by classical Stokesian models. It is precisely this class of models that cannot be inverted that forms the basis for this study.

Málek, Průša and Rajagopal [6] have considered a class of fluid models wherein the symmetric part of the velocity gradient has a power-law relationship to the Cauchy stress, which they refer to as *stress power-law models*. They show that this class contains models that are not equivalent to any classical power-law model, that is, no classical power-law can exhibit the type of response exhibited by members of the class of stress power-law fluids. For example, one finds that if the power-law index of the stress power-law lies within a certain range then one can have the response depicted in Fig. 1. Thus, the relationship is such that the shear stress cannot be expressed as a function of the shear rate, but the shear rate can be expressed as a function of the shear stress. The question that immediately confronts the modeler is whether such models have any application. The answer is yes. In fact, even more complicated models than the models considered by Málek et al. [6] might have interesting applications.

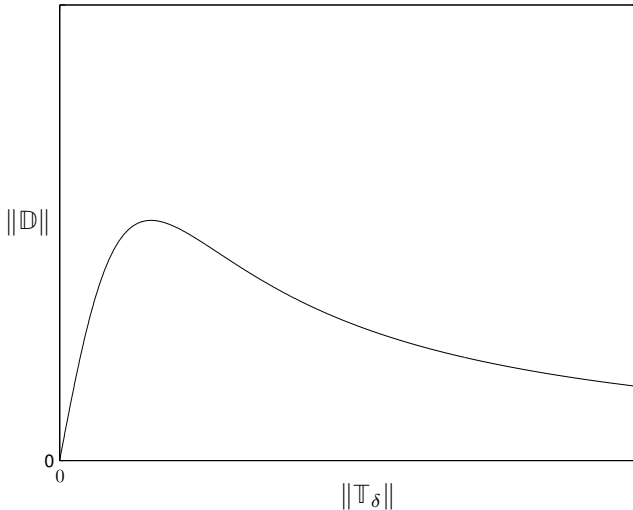


Figure 1. Example of a non-monotone stress power-law model.

Let us consider the one-dimensional response depicted in Fig. 2. We first note that it cannot be described by the classical Stokesian fluid model or the class of

fluid models studied by Málek et al. [6]. Not only is the shear rate a non-monotone function of the shear stress, this function is increasing near the origin and at large values of the shear rate. The question is whether such a response can describe a sensible physical problem. We try to answer this question in the affirmative within the context of an interesting phenomenon.

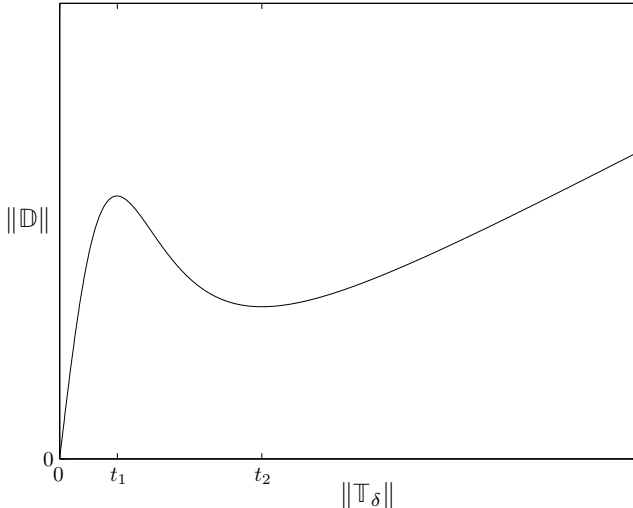


Figure 2. A mechanical model of the coagulation and lysis of blood.

Blood is invariably modeled as a single-component fluid when it flows in reasonably large blood vessels. It is in reality a very complex mixture of a fluid (plasma), red and white blood cells, platelets, numerous proteins, ions, etc. In large vessels this complex mixture can be approximated reasonably well as a fluid. However, in capillaries and arterioles it cannot even be described as a continuum. A myriad of biochemical reactions keep the blood in a delicate state of balance. Platelets are “activated” by prolonged exposure to high shear stresses. This can be approximated by the platelets being “activated” when the shear stress reaches a critical value ( $t_1$  in Fig. 2). Then a cascade of biochemical reactions takes place that lead to coagulation of the blood (see Zarnitsina et al. [14], [15], Lawson and Rajagopal [5], Anand et al. [1], [2] and references therein for a detailed discussion of the same). The viscosity of the coagulated blood is far greater than the viscosity of normal blood and causes a reduction in the shear rate of the blood. As the coagulation<sup>5</sup> proceeds, the generalized viscosity of the blood increases. During coagulation, red blood cell-

---

<sup>5</sup> Coagulation involves enzymatic reactions of plasma zymogens, anionic phospholipids and calcium ions that result in the formation of thrombin. The platelet aggregate and the fibrin mesh constitute the blood clot.

rouleau aggregates form and this, in addition to the platelet aggregates and the fibrin mesh, leads to the increased viscosity. As the shear stress increases further, we have “lysis”: the rouleau structure disaggregates and the fibrin mesh is dismantled. Of course, it is not merely the shear stress that is the cause of lysis; there are a plethora of biochemical reactions at play. We can however incorporate the effect of the biochemical reactions and the disaggregation of the red blood cell rouleaus within a mechanical framework that effectively leads to the blood shear-thinning, thereby leading to increased shear rates as the shear stress increases further, once the shear stress reaches a critical value ( $t_2$  in Fig. 2). Thus, Fig. 2 can describe in an approximate fashion the coagulation and lysis of blood; the consequences of the biochemical reactions are replaced by an equivalent mechanical response.

We show that for the class of models considered wherein the shear rate depends non-monotonically on the shear stress as depicted in Fig. 2, even simple flows such as those between parallel plates, the counterparts of classical Couette and Poiseuille flows, show interesting new features. In the case of Couette flow, when the no-slip condition is enforced at the plates, though one might find a velocity field of the classical form, such a velocity field can have associated with it non-unique stress fields. That is, the same flow can be engendered by different stress fields. When the free slip condition is applied at one plate and either the no-slip or Navier-slip condition at the other plate, the problem has a unique solution. However, if the Navier slip condition is applied at both plates then the problem can have one, two or three solutions (velocity fields), depending on the values of the model parameters, the velocities of the plates and the shear forces applied to the fluid at the plates. The corresponding Poiseuille problems have similar non-uniqueness properties.

The outline of the paper is as follows. In the next section we develop a model to describe the one-dimensional shear rate versus shear stress response depicted in Fig. 2 and its generalization to three dimensions. In Section 3 we study several simple but important boundary value problems for such models, namely steady planar Couette, Poiseuille and Couette-Poiseuille flows with no-slip, partial slip (Navier slip) and traction (free slip) boundary conditions.

## 2. CONSTITUTIVE RELATION

Unlike the usual procedure of providing constitutive equations for the stress in terms of kinematical quantities, we shall express the symmetric part of the velocity gradient  $\mathbb{D}$  as a function of the deviator  $\mathbb{T}_\delta$  of the Cauchy stress tensor  $\mathbb{T}$ . We shall consider the specific constitutive representation

$$(2.1) \quad \mathbb{D} = [\alpha(1 + \beta\|\mathbb{T}_\delta\|^2)^n + \gamma] \mathbb{T}_\delta,$$

where  $\alpha$  and  $\beta$  are positive numbers,  $\gamma$  is a nonnegative number,  $n$  is a real number,

$$(2.2) \quad \mathbb{D} := \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \quad \mathbb{T}_\delta := \mathbb{T} - \frac{1}{3}(\text{tr } \mathbb{T})\mathbb{I},$$

$$\|\mathbb{T}_\delta\|^2 = \text{tr}(\mathbb{T}_\delta(\mathbb{T}_\delta)^T) = (T_\delta)_{ij}(T_\delta)_{ij},$$

$\mathbf{v}$  is the velocity,  $\mathbb{T}$  is the Cauchy stress tensor,  $\text{tr } \mathbb{T}$  is the trace of  $\mathbb{T}$ , and  $\mathbb{I}$  is the identity tensor. The constants  $\alpha$  and  $\gamma$  have the same dimensions as the reciprocal of viscosity, and  $\beta$  has the same dimensions as  $\|\mathbb{T}_\delta\|^{-2}$ . Observe that equation (2.1) automatically satisfies the constraint of incompressibility:  $\text{div } \mathbf{v} = \text{tr } \mathbb{D} = 0$ .

When  $n = 0$ , relation (2.1) reduces to the classical Navier-Stokes model of an incompressible fluid (with viscosity  $\mu = (2(\alpha + \gamma))^{-1}$ ). When  $\gamma = 0$ , relation (2.1) reduces to the stress power-law model considered by Málek et al. [6]. They discuss qualitative issues concerning the stress power-law models versus the standard power-law models in detail. The critical distinction is that the stress power-law model can allow for a relation in which the shear rate is a non-monotone function of the shear stress and the shear stress is therefore not a function of the shear rate. This possibility is at the heart of the phenomena that we wish to describe.

We shall henceforth assume that  $\gamma > 0$  and  $n < -1/2$ , i.e.  $2n + 1 < 0$ . In our analysis of equation (2.1) we shall often refer to the following result.

**Lemma 2.1.** *Let  $n < -1/2$  and  $a, b, c > 0$ , let  $f(t) = [a(1 + bt^2)^n + c]t$ ,  $t \geq 0$ , and let*

$$(2.3) \quad d_n = 2 \left( \frac{2n - 2}{2n + 1} \right)^{n-1}.$$

*Then the following hold:*

- (a)  $f(t) - ct \rightarrow 0^+$  as  $t \rightarrow \infty$ .
- (b) If  $d_n \leq c/a$  then  $f(t)$  is strictly increasing.
- (c) If  $c/a < d_n$  then there exist numbers  $0 < t_1 < t_2$  such that  $f(t)$  is strictly increasing on  $[0, t_1]$  and  $[t_2, \infty)$  and strictly decreasing on  $[t_1, t_2]$ . In this case the equation  $f(t) = g$  has
  - (i) one solution if  $0 \leq g < f(t_2)$  or  $f(t_1) < g$ ;
  - (ii) two distinct solutions if  $g$  equals  $f(t_1)$  or  $f(t_2)$ ;
  - (iii) three distinct solutions if  $f(t_2) < g < f(t_1)$ .

**Proof.** First,  $f(0) = 0$ ,  $f(t) > 0$  for all  $t > 0$ , and  $f(t) - ct = h(t) \rightarrow 0^+$  as  $t \rightarrow \infty$ , where  $h(t) := a(1 + bt^2)^n t$ . Furthermore,

$$h'(t) = a(1 + (2n + 1)bt^2)(1 + bt^2)^{n-1} = a \left( 1 + \frac{2nbt^2}{1 + bt^2} \right) (1 + bt^2)^n.$$

Thus,  $f'(0) = a + c > 0$  and  $f'(t) \rightarrow c^-$  as  $t \rightarrow \infty$ . Moreover,  $h(t)$  has a global maximum at  $t = t_0 := (-(2n+1)b)^{-1/2}$ , where the sign of  $h'(t)$  changes from positive to negative. Furthermore, since

$$f''(t) = 2abnt(3 + (2n + 1)bt^2)(1 + bt^2)^{n-2},$$

the graph of  $f$  has an inflection point and  $f'(t)$  has a global minimum at  $t = \sqrt{3}t_0$ , where the sign of  $f''(t)$  changes from negative to positive. Now,  $f'(\sqrt{3}t_0) = c - ad_n$ . So if  $0 < c/a < d_n$  then it follows from the intermediate value theorem that there exist numbers  $t_1 \in (t_0, \sqrt{3}t_0)$  and  $t_2 > \sqrt{3}t_0$  such that  $f'(t_1) = f'(t_2) = 0$ . Moreover, since  $f'(t)$  is strictly decreasing on  $(0, \sqrt{3}t_0)$  and strictly increasing on  $(\sqrt{3}t_0, \infty)$ , the points  $t_1$  and  $t_2$  are unique. Hence, in this case,  $f(t)$  is a non-monotone function that is strictly increasing on  $[0, t_1]$  and  $[t_2, \infty)$  and strictly decreasing on  $[t_1, t_2]$ . On the other hand, if  $d_n \leq c/a$  then  $f(t)$  is strictly increasing since  $f'(t) > 0$  for all  $t \in (0, \sqrt{3}t_0) \cup (\sqrt{3}t_0, \infty)$ . Lastly, assertions (i)–(iii) in part (c) follow from the intermediate value theorem and the properties of  $f$  established above.  $\square$

**Remark 2.1.**

- (a) Since  $f$  is odd, analogous statements hold for  $t \leq 0$ .
- (b) Formula (2.3) can be written as  $d_n = 2([1 + m^{-1}]^{m+1})^{-3/2}$  with  $m = -(2n+1)$ . Thus,  $d_n$  converges monotonely from below to  $2e^{-3/2} \doteq 0.4463$  as  $n \rightarrow -\infty$ . For example,  $d_{-1} = 0.125$ ,  $d_{-2} = 0.25$ ,  $d_{-3} \doteq 0.3052$ ,  $d_{-10} \doteq 0.3987$ , and  $d_{-100} \doteq 0.4413$ .

Now, equation (2.1) is  $\|\mathbb{D}\| = f(\|\mathbb{T}_\delta\|)$ , where  $f$  is the function in Lemma 2.1 with  $a = \alpha$ ,  $b = \beta$  and  $c = \gamma$ . Thus, Lemma 2.1 shows that if  $n < -1/2$  and  $0 < \gamma/\alpha < d_n$ , this model has the non-monotone behaviour illustrated in Fig. 2. Figs. 3–5 show the effects of varying the model parameters  $\beta$ ,  $n$ , and  $\gamma$ . Note that all the curves in Fig. 3 (as well as the solid curves in Figs. 4 and 5) are monotone increasing since they represent borderline cases with  $n = -2$  and  $\gamma/\alpha = 0.25 = d_{-2}$ . The straight lines in Figs. 3 and 4 correspond to the limit cases  $\beta = 0$  and  $n = 0$ , respectively, where equation (2.1) reduces to the Navier-Stokes model. The dotted curve in Fig. 5 shows the limit case  $\gamma = 0$ , where equation (2.1) reduces to the stress power-law model considered in [6].

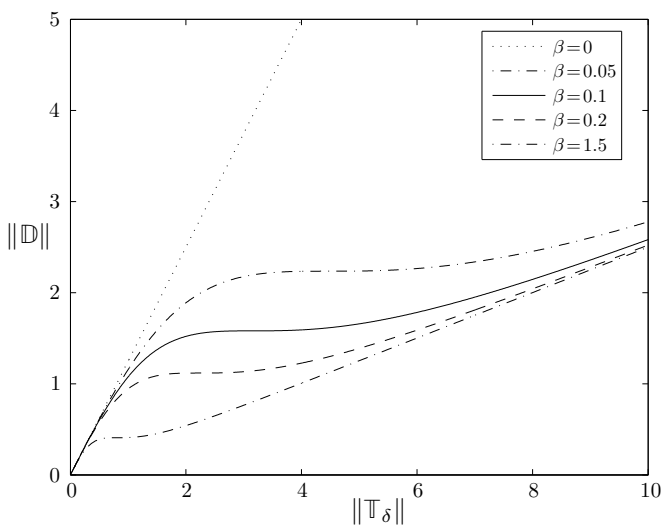


Figure 3.  $\|\mathbb{D}\| = f(\|\mathbb{T}_\delta\|)$  for some values of  $\beta$  ( $\alpha = 1$ ,  $\gamma = 0.25$ ,  $n = -2$ ).

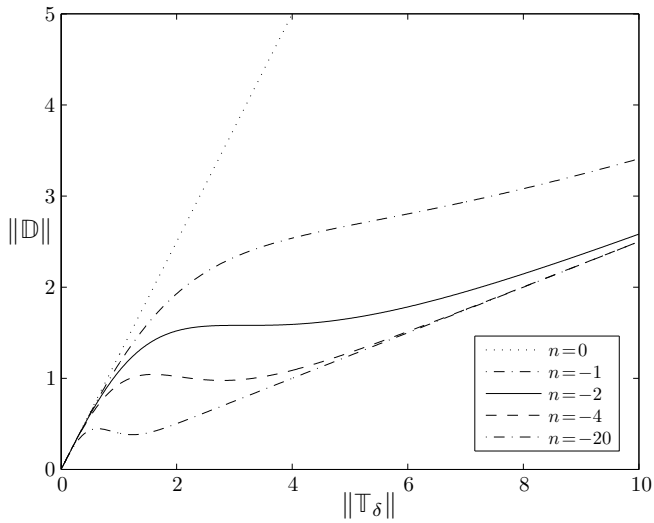


Figure 4.  $\|\mathbb{D}\| = f(\|\mathbb{T}_\delta\|)$  for some values of  $n$  ( $\alpha = 1$ ,  $\beta = 0.1$ ,  $\gamma = 0.25$ ).



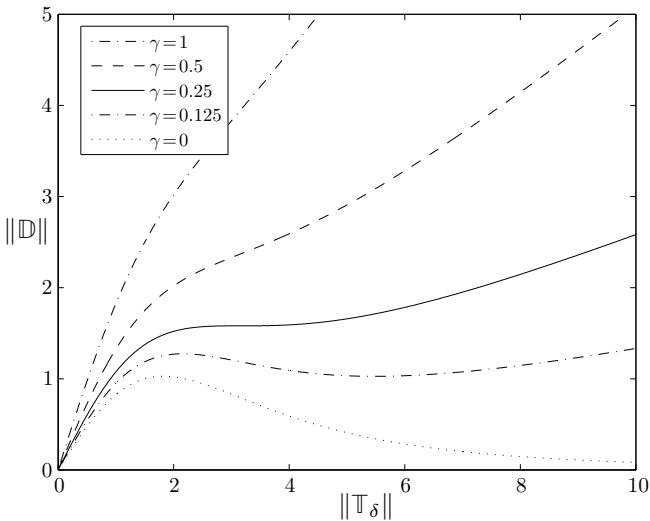


Figure 5.  $\|\mathbb{D}\| = f(\|\mathbb{T}_\delta\|)$  for some values of  $\gamma$  ( $\alpha = 1$ ,  $\beta = 0.1$ ,  $n = -2$ ).

### 3. BOUNDARY VALUE PROBLEMS

Let us consider flows of an incompressible fluid modeled by equation (2.1). In the classical approach where the stress is given in terms of the velocity gradient, we substitute the expression for the stress into the balance of linear momentum

$$(3.1) \quad \varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b},$$

where  $\varrho$  is the density,  $d/dt$  is the material time derivative and  $\mathbf{b}$  is the external body force per unit mass, to obtain a partial differential equation in  $\mathbf{v}$  and, due to the incompressibility of the fluid, the pressure  $p$ . We also have to meet the balance of mass, which reduces due to the constraint of incompressibility to

$$(3.2) \quad \operatorname{div} \mathbf{v} = 0.$$

In contrast, the constitutive relation (2.1) does not yield an expression for the stress to substitute into the balance of linear momentum. We now have to solve the constitutive equation (2.1) and the balance of linear momentum (3.1) simultaneously (equation (3.2) is satisfied automatically). This is a system of nine first-order partial differential equations for the components of  $\mathbf{v}$  and  $\mathbb{T}$ , whereas the classical approach leads to a system of three second-order (or higher-order) differential equations, (3.1), and one first-order differential equation, (3.2), for  $p$  and the components of  $\mathbf{v}$ .

We shall find it convenient to non-dimensionalize the constitutive relation (2.1) and the balance of linear momentum (3.1). Let  $L$  be a characteristic length and  $V$  a characteristic speed (the values of  $L$  and  $V$  will depend on the problem under consideration). Then we define dimensionless variables

$$(3.3) \quad \begin{aligned} \mathbf{x}^* &= \frac{1}{L} \mathbf{x}, & t^* &= \frac{V}{L} t, & \mathbf{v}^* &= \frac{1}{V} \mathbf{v}, & \mathbb{D}^* &= \frac{L}{V} \mathbb{D}, \\ \mathbb{T}^* &= \frac{\alpha L}{V} \mathbb{T}, & \mathbb{T}_\delta^* &= \frac{\alpha L}{V} \mathbb{T}_\delta = (\mathbb{T}^*)_\delta, \end{aligned}$$

and dimensionless constants and body force

$$(3.4) \quad \mathcal{R}_1 = \alpha \varrho V L, \quad \mathcal{R}_2 = \frac{\beta V^2}{\alpha^2 L^2}, \quad \mathcal{R}_3 = \frac{\gamma}{\alpha}, \quad \mathbf{b}^* = \frac{L}{V^2} \mathbf{b}.$$

Since  $\alpha$  has the dimensions of the reciprocal of viscosity,  $\mathcal{R}_1$  is an analogue of the Reynolds number. (We can replace  $\alpha$  by  $\alpha + \gamma$  in (3.3)–(3.4), but this would not make a fundamental difference to what follows.) The dimensionless forms of the constitutive relation and the balance of linear momentum are

$$(3.5) \quad \mathbb{D}^* = [(1 + \mathcal{R}_2 \|\mathbb{T}_\delta^*\|^2)^n + \mathcal{R}_3] \mathbb{T}_\delta^*,$$

$$(3.6) \quad \frac{d\mathbf{v}^*}{dt^*} = \frac{1}{\mathcal{R}_1} \operatorname{div}^* \mathbb{T}^* + \mathbf{b}^*.$$

We shall henceforth only work with these dimensionless variables and equations. For the sake of simplicity we shall omit the asterisks.

1. Let us consider a steady planar flow  $\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y$ , where  $\mathbf{e}_x$  and  $\mathbf{e}_y$  are the unit vectors in the  $x$  and  $y$  coordinate directions, respectively, wherein the deviator of the stress takes the special form

$$(3.7) \quad \mathbb{T} - \frac{1}{3}(\operatorname{tr} \mathbb{T}) \mathbb{I} = \mathbb{T}_\delta = T_{xy}(\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x), \quad T_{xy} = T = T(y).$$

Then it follows from the constitutive equation (3.5) that

$$(3.8) \quad \mathbb{D} = [(1 + 2\mathcal{R}_2 T^2)^n + \mathcal{R}_3] T(\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x).$$

This implies that the velocity field has the form

$$(3.9) \quad \mathbf{v} = v_x(y) \mathbf{e}_x + (Ax + B) \mathbf{e}_y,$$

where  $A$  and  $B$  are constants and

$$(3.10) \quad \frac{dv_x}{dy} + A = 2[(1 + 2\mathcal{R}_2 T^2)^n + \mathcal{R}_3] T.$$

2. Suppose in addition that we consider only flows in which  $A = B = 0$ , i.e.  $\mathbf{v} = v_x(y)\mathbf{e}_x$ . Then  $(\nabla\mathbf{v})\mathbf{v} = 0$ . Thus, if the body force is also zero, it follows from the balance of linear momentum (3.6) and equation (3.7) that

$$(3.11) \quad \frac{1}{\mathcal{R}_1} \frac{\partial}{\partial x} \left( \frac{1}{3} \text{tr } \mathbb{T} \right) + T'(y) = \frac{1}{\mathcal{R}_1} \frac{\partial}{\partial y} \left( \frac{1}{3} \text{tr } \mathbb{T} \right) = \frac{1}{\mathcal{R}_1} \frac{\partial}{\partial z} \left( \frac{1}{3} \text{tr } \mathbb{T} \right) = 0.$$

This implies that

$$(3.12) \quad T(y) = Cy + E, \quad -\frac{1}{3} \frac{\partial}{\partial x} (\text{tr } \mathbb{T}) = C$$

for some (dimensionless) constants  $C$  and  $E$ . We usually refer to the mean normal stress as the pressure; thus  $C$  is the pressure gradient along the  $x$ -axis (the flow direction). By substituting  $T(y)$  from (3.12) into equation (3.10) we obtain

$$(3.13) \quad \frac{dv_x}{dy} = 2 \left[ (1 + 2\mathcal{R}_2(Cy + E)^2)^n + \mathcal{R}_3 \right] (Cy + E).$$

3. We shall consider planar Couette, Poiseuille and Couette-Poiseuille flows between two infinite parallel plates at  $y = 1$  and  $y = -1$ . We assume that these boundaries are impermeable, so that

$$(3.14) \quad \mathbf{v} \cdot \mathbf{n} = 0,$$

where  $\mathbf{n}$  denotes the outward unit normal vector. For velocity fields of the form (3.9), this implies that  $v_y = Ax + B = 0$  for all  $x$ . Thus  $A = B = 0$ , as we assumed earlier.

Furthermore, we shall impose boundary conditions of the form

$$(3.15) \quad \lambda K([\mathbb{T}\mathbf{n}]_\tau - \boldsymbol{\sigma}) + (1 - \lambda)(\mathbf{v}_\tau - \mathbf{w}) = 0,$$

where  $\lambda$  is a constant in  $[0, 1]$ ,  $K$  is a positive constant,  $\mathbf{v}_\tau = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ ,  $[\mathbb{T}\mathbf{n}]_\tau = [\mathbb{T}_\delta\mathbf{n}]_\tau$  is the tangential traction, and  $\boldsymbol{\sigma}$  and  $\mathbf{w}$  are given vector fields such that  $\boldsymbol{\sigma} \cdot \mathbf{n} = 0$  and  $\mathbf{w} \cdot \mathbf{n} = 0$  everywhere on the boundary. Condition (3.15) includes a number of standard boundary conditions as special cases. If  $\lambda = 0$  and  $\mathbf{w}$  is the velocity of the boundary, (3.15) is the no-slip boundary condition  $\mathbf{v}_\tau = \mathbf{w}$ . If  $0 < \lambda < 1$ ,  $\boldsymbol{\sigma} = 0$  and  $\mathbf{w}$  is the velocity of the boundary, (3.15) is the Navier partial slip condition. If  $\lambda = 1$ , (3.15) is the (tangential) traction boundary condition  $[\mathbb{T}\mathbf{n}]_\tau = \boldsymbol{\sigma}$ , which becomes the condition of free slip (frictionless slip) when  $\boldsymbol{\sigma} = 0$ . We shall call this boundary condition “free slip” even when  $\boldsymbol{\sigma} \neq 0$  in order to avoid possible confusion with the traction boundary condition  $\mathbb{T}\mathbf{n} = \boldsymbol{\sigma}$ .

To write (3.15) in dimensionless form, we define  $\mathbf{v}^*$  and  $\mathbb{T}^*$  as in (3.3) and define

$$(3.16) \quad \mathcal{K} = \frac{K}{\alpha L}, \quad \boldsymbol{\sigma}^* = \frac{\alpha L}{V} \boldsymbol{\sigma}, \quad \mathbf{w}^* = \frac{1}{V} \mathbf{w}.$$

If we omit the asterisks, we obtain a boundary condition of the same form as (3.15) with  $\mathcal{K}$  in place of  $K$ . By applying this boundary condition with  $\lambda = \lambda_1$ ,  $\boldsymbol{\sigma} = \sigma \mathbf{e}_x$  and  $\mathbf{w} = w_1 \mathbf{e}_x$  at  $y = 1$ , and  $\lambda = \lambda_{-1}$ ,  $\boldsymbol{\sigma} = \sigma_{-1} \mathbf{e}_x$  and  $\mathbf{w} = w_{-1} \mathbf{e}_x$  at  $y = -1$ , we obtain

$$(3.17) \quad \lambda_1 \mathcal{K}(C + E - \sigma_1) + (1 - \lambda_1)(v_x(1) - w_1) = 0,$$

$$(3.18) \quad \lambda_{-1} \mathcal{K}(C - E - \sigma_{-1}) + (1 - \lambda_{-1})(v_x(-1) - w_{-1}) = 0.$$

We shall now consider equation (3.13) with these boundary conditions.

**Remark 3.1.** It is important to keep in mind that we only consider solutions of a very special kind (fully developed unidirectional steady flows with stress fields of the form (3.7)). The original boundary-value problem (equation (3.1) with boundary conditions of the form (3.15) at both plates; equation (3.2) is satisfied automatically) might have additional solutions irrespective of whether the corresponding boundary-value problem (3.13), (3.17)–(3.18) has one or more solutions.

### 3.1. Planar Couette flow

Assume that the pressure gradient  $C$  is zero. Then the general solution of equation (3.13) is

$$(3.19) \quad v_x(y) = 2[(1 + 2\mathcal{R}_2 E^2)^n + \mathcal{R}_3] E y + F$$

for some constant  $F$ .

**3.1.1. No slip.** When  $\lambda_1 = \lambda_{-1} = 0$ , boundary conditions (3.17) and (3.18) reduce to  $v_x(1) = w_1$  and  $v_x(-1) = w_{-1}$ , respectively. This yields the solution

$$(3.20) \quad v_x(y) = \frac{1}{2}(w_1 - w_{-1})y + \frac{1}{2}(w_1 + w_{-1}) = \frac{1}{2}w_1(1 + y) + \frac{1}{2}w_{-1}(1 - y).$$

The velocity profile is independent of the material constants and is therefore the same as for the Navier-Stokes model and the stress power-law models mentioned earlier. However, the stress that engenders the flow is not necessarily unique; the derivation of (3.20) does not require finding  $E$  because

$$(3.21) \quad 4[(1 + 2\mathcal{R}_2 E^2)^n + \mathcal{R}_3] E = w_1 - w_{-1}.$$

It follows from Lemma 2.1 (with  $a = 4$ ,  $b = 2\mathcal{R}_2$ ,  $c = 4\mathcal{R}_3$ ) that if  $d_n \leq \mathcal{R}_3$  then this equation has a unique solution, and if  $\mathcal{R}_3 < d_n$  then it has either one, two or three solutions, depending on the value of  $w_1 - w_{-1}$ . (The corresponding equation for the Navier-Stokes model has a unique solution. The equation for the stress-power-law model with  $\mathcal{R}_3 = 0$  has either no, one or two solutions.)

**3.1.2. Navier slip.** Let  $\lambda_1, \lambda_{-1} \in [0, 1)$  with  $\lambda_1 > 0$  or  $\lambda_{-1} > 0$ . Substitution of formula (3.19) into boundary conditions (3.17)–(3.18) yields the equations

$$(3.22) \quad \lambda_1 \mathcal{K}(E - \sigma_1) + (1 - \lambda_{-1})(2[(1 + 2\mathcal{R}_2 E^2)^n + \mathcal{R}_3]E + F - w_1) = 0,$$

$$(3.23) \quad -\lambda_{-1} \mathcal{K}(E + \sigma_{-1}) + (1 - \lambda_{-1})(-2[(1 + 2\mathcal{R}_2 E^2)^n + \mathcal{R}_3]E + F - w_{-1}) = 0.$$

Elimination of  $F$  in (3.22)–(3.23) yields

$$(3.24) \quad \left[ 4(1 + 2\mathcal{R}_2 E^2)^n + 4\mathcal{R}_3 + \frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} + \frac{\lambda_{-1} \mathcal{K}}{1 - \lambda_{-1}} \right] E \\ = \frac{\lambda_1 \mathcal{K} \sigma_1}{1 - \lambda_1} - \frac{\lambda_{-1} \mathcal{K} \sigma_{-1}}{1 - \lambda_{-1}} + w_1 - w_{-1}.$$

If the right-hand side of this equation is zero, it has the unique solution  $E = 0$ . In this case,

$$(3.25) \quad v_x(y) = F = w_1 + \frac{\lambda_1 \mathcal{K} \sigma_1}{1 - \lambda_1}.$$

In general, it follows from Lemma 2.1 that equation (3.24) has a unique solution if

$$(3.26) \quad d_n \leq \mathcal{R}_3 + \frac{\mathcal{K}}{4} \left( \frac{\lambda_1}{1 - \lambda_1} + \frac{\lambda_{-1}}{1 - \lambda_{-1}} \right).$$

If this inequality is not satisfied, equation (3.24) has either one, two or three solutions, depending on the value of its right-hand side. Note that this is the case even if the no-slip condition is applied at one of the boundaries, i.e. if either  $\lambda_1 = 0$  or  $\lambda_{-1} = 0$ . Each solution  $E$  corresponds to a distinct stress function  $T(y) \equiv E$  and a distinct velocity profile because equations (3.19) and (3.24) imply that

$$(3.27) \quad v_x(y) = \frac{1}{2} \left[ \frac{\lambda_1 \mathcal{K} \sigma_1}{1 - \lambda_1} - \frac{\lambda_{-1} \mathcal{K} \sigma_{-1}}{1 - \lambda_{-1}} \right. \\ \left. - \left( \frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} + \frac{\lambda_{-1} \mathcal{K}}{1 - \lambda_{-1}} \right) E + w_1 - w_{-1} \right] y + F.$$

For each solution  $E$  of equation (3.24), the corresponding value of  $F$  is determined by equation (3.22) or (3.23).

**Remark 3.2.** Elimination of the non-linear term in equations (3.22)–(3.23) yields the equation

$$(3.28) \quad F = \frac{1}{2} \left[ \frac{\lambda_1 \mathcal{K} \sigma_1}{1 - \lambda_1} + \frac{\lambda_{-1} \mathcal{K} \sigma_{-1}}{1 - \lambda_{-1}} + \left( \frac{\lambda_{-1} \mathcal{K}}{1 - \lambda_{-1}} - \frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} \right) E + w_1 + w_{-1} \right].$$

Note that the value of  $F$  is independent of  $E$  when  $\lambda_1 = \lambda_{-1}$ . From equations (3.27) and (3.28) we get

$$(3.29) \quad v_x(y) = \frac{1}{2} \left[ \frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} (\sigma_1 - E) + w_1 \right] (1 + y) + \frac{1}{2} \left[ \frac{\lambda_{-1} \mathcal{K}}{1 - \lambda_{-1}} (\sigma_{-1} + E) + w_{-1} \right] (1 - y),$$

which shows that the velocity profile is in effect an affine function of  $E$  (as is the stress, by equations (3.7) and (3.12)). Of course,  $E$  depends nonlinearly on the data  $(\sigma_1, w_1, \sigma_{-1}$  and  $w_{-1})$  via equation (3.24).

**3.1.3. Free slip.** Here we consider the situation when  $\lambda_1 = 1$  or  $\lambda_{-1} = 1$ .

(a) If  $\lambda_1 = 1$ , boundary condition (3.17) reduces to  $E = \sigma_1$ . Thus, if  $0 \leq \lambda_{-1} < 1$ , we can solve for  $F$  in (3.23) and get

$$(3.30) \quad v_x(y) = 2 \left[ (1 + 2\mathcal{R}_2 \sigma^2)^n + \mathcal{R}_3 \right] \sigma_1 (1 + y) + \frac{\lambda_{-1} \mathcal{K}}{1 - \lambda_{-1}} (\sigma_1 + \sigma_{-1}) + w_{-1}.$$

(b) If  $\lambda_{-1} = 1$  and  $0 \leq \lambda_1 < 1$ , it follows as in (a) that  $E = -\sigma_{-1}$  and

$$(3.31) \quad v_x(y) = 2 \left[ (1 + 2\mathcal{R}_2 \sigma_{-1}^2)^n + \mathcal{R}_3 \right] \sigma_{-1} (1 - y) + \frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} (\sigma_1 + \sigma_{-1}) + w_1.$$

(c) If  $\lambda_1 = \lambda_{-1} = 1$ , boundary conditions (3.17) and (3.18) reduce to  $E = \sigma_1$  and  $E = -\sigma_{-1}$ , respectively. Hence there is no solution (of the form chosen for the velocity) if  $\sigma_1 \neq -\sigma_{-1}$ . (The original boundary-value problem with partial differential equation (3.1) might yet have one or more solutions.) If  $\sigma_1 = -\sigma_{-1}$ , there are infinitely many solutions:  $v_x$  is as in formula (3.19) with  $E = \sigma_1$  and arbitrary  $F$ .

**Example 3.1.** Let  $\mathcal{R}_2 = 0.05$ ,  $\mathcal{R}_3 = 0.125$ ,  $n = -2$ ,  $\mathcal{K} = 1$ ,  $\lambda_{-1} = 0$  (no slip at  $y = -1$ ),  $\sigma_1 = \sigma_{-1} = 0$ ,  $w_1 = 4.8$  and  $w_{-1} = 0$ .

(a) If  $\lambda_1 = 0$  (no slip at  $y = 1$ ) then  $\mathcal{R}_3 < d_{-2} = 0.25$  and therefore equation (3.21) has either one, two or three solutions. In this instance there are three solutions:  $E_1 \doteq 1.5448$ ,  $E_2 \doteq 3.0488$  and  $E_3 \doteq 8.6336$ . Hence, there are three possible

stress functions  $T(y)$ , namely  $T(y) \equiv E_i$ ,  $i = 1, 2, 3$ , and each of these engenders the velocity field  $v_x(y) = 2.4(1 + y)$  (by (3.20)).

- (b) If  $0 < \lambda_1 < \frac{1}{3}$  (“small” Navier slip at  $y = 1$ ) then inequality (3.26) does not hold and equation (3.24) has either one, two or three solutions. For example, if  $\lambda_1 = 0.25$  then there is only one solution,  $E_4 \doteq 1.2584$ , and thus  $T(y) \equiv E_4$  and  $v_x(y) = (2.4 - E_4/6)(1 + y)$  (by (3.29)).
- (c) If  $\lambda_1 \geq \frac{1}{3}$  (“large” Navier slip at  $y = 1$ ) then inequality (3.26) holds and equation (3.24) has a unique solution. For example, if  $\lambda_1 = 0.5$  then the solution is  $E_5 \doteq 0.9984$ . Thus  $T(y) \equiv E_5$  and  $v_x(y) = (2.4 - E_5/2)(1 + y)$ . Similarly, if  $\lambda_1 = 0.75$  then  $E = E_6 \doteq 0.6701$ ,  $T(y) \equiv E_6$  and  $v_x(y) = (2.4 - 3E_6/2)(1 + y)$ .
- (d) If  $\lambda_1 = 1$  (free slip at  $y = 1$ ) then  $T(y) \equiv 0$  and  $v_x(y) \equiv 0$  (by (3.30)).

The velocity profiles in (a)–(c) are shown in Fig. 6.

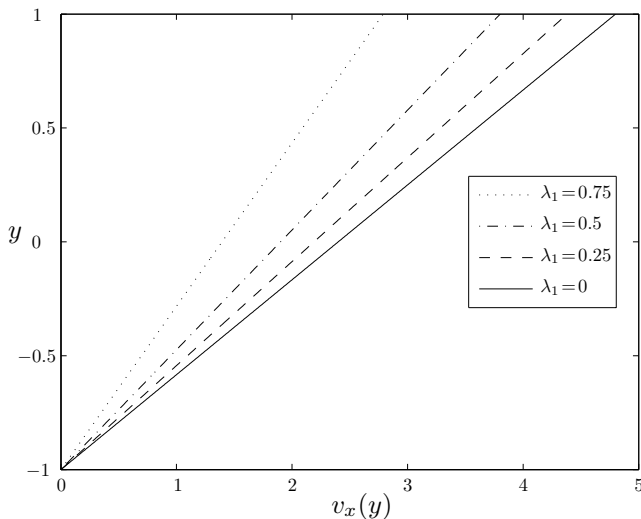


Figure 6. Velocity profiles in Example 3.1.

**Example 3.2.** As in Example 3.1 (b), let  $\mathcal{R}_2 = 0.05$ ,  $\mathcal{R}_3 = 0.125$ ,  $n = -2$ ,  $\mathcal{K} = 1$ ,  $\lambda_{-1} = 0$  (no slip at  $y = -1$ ),  $\lambda_1 = 0.25$  (Navier slip at  $y = 1$ ),  $\sigma_1 = \sigma_{-1} = 0$  and  $w_{-1} = 0$ . Then equation (3.24) has either one, two or three solutions, depending on the value of  $w_1$ . For each solution  $E$ , the corresponding stress function is  $T(y) \equiv E$  and the velocity function is  $v_x(y) = (w_1 - E/6)(1 + y)$ .

- (a) If  $w_1 = 4.8$ , there is only one solution, which is given in Example 3.1 (b). Similarly, if  $w_1 = 1.6$  then  $E \doteq 0.3373$ , and if  $w_1 = 3.2$  then  $E \doteq 0.7192$ .
- (b) If  $w_1 = 5.8$ , equation (3.24) has three solutions, namely  $E_7 \doteq 2.1119$ ,  $E_8 \doteq 3.1473$  and  $E_9 \doteq 5.0030$ . Hence, there are three possible stress functions and each engenders a distinct velocity profile.

(c) If  $w_1 = 6.4$ , equation (3.24) has only one solution,  $E \doteq 6.5607$ . Similarly, if  $w_1 = 8.0$  then  $E \doteq 9.0917$ , and if  $w_1 = 9.6$  then  $E \doteq 11.2290$ .

The velocity profiles corresponding to the above cases are shown in Fig. 7. Note the positions of the three solutions with  $w_1 = 5.8$  amongst the other solutions. In all cases except  $E = E_7$  and  $E = E_9$ , the slip velocity  $v_x(1)$  increases as  $w_1$  increases.

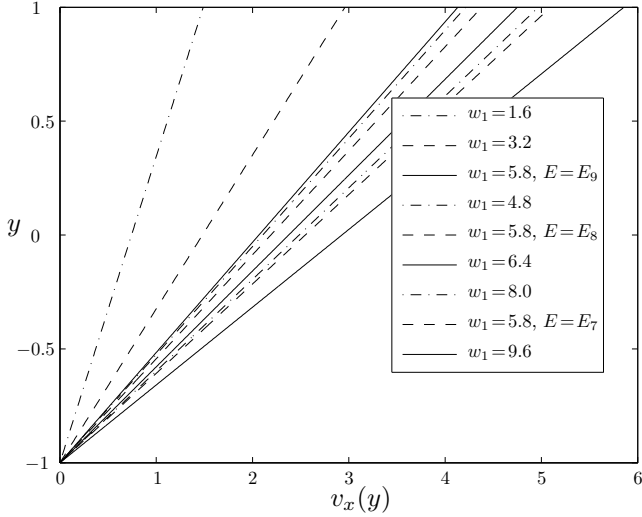


Figure 7. Velocity profiles in Example 3.2.

### 3.2. Planar Poiseuille flow

Here we consider the situation when the pressure gradient  $C$  is negative and the boundary plates do not move, i.e.  $w_1 = w_{-1} = 0$ . If  $n \neq -1$  the solution of equation (3.13) is

$$(3.32) \quad v_x(y) = \frac{1}{2(n+1)CR_2} (1 + 2R_2(Cy + E)^2)^{n+1} + R_3(Cy^2 + 2Ey) + F$$

for some constant  $F$ . If  $n = -1$  the solution is

$$(3.33) \quad v_x(y) = \frac{1}{2CR_2} \ln(1 + 2R_2(Cy + E)^2) + R_3(Cy^2 + 2Ey) + F.$$

**3.2.1. No slip.** When  $\lambda_1 = \lambda_{-1} = 0$ , the boundary conditions are  $v_x(1) = 0$  and  $v_x(-1) = 0$ . If  $n \neq -1$  this yields the equations

$$(3.34) \quad \frac{1}{2(n+1)CR_2} (1 + 2R_2(C + E)^2)^{n+1} + R_3(C + 2E) + F = 0,$$

$$(3.35) \quad \frac{1}{2(n+1)CR_2} (1 + 2R_2(E - C)^2)^{n+1} + R_3(C - 2E) + F = 0.$$



By subtracting (3.35) from (3.34) we get the equation

$$(3.36) \quad H_n(E) + 4R_3E = 0,$$

where

$$(3.37) \quad H_n(E) := \frac{1}{2C} (I_n(E + C) - I_n(E - C)),$$

$$(3.38) \quad I_n(E) := \frac{1}{(n+1)R_2} (1 + 2R_2E^2)^{n+1}.$$

Similarly, if  $n = -1$  we obtain the equation

$$(3.39) \quad H_{-1}(E) + 4R_3E = 0,$$

where  $H_{-1}$  is defined as in (3.37) with

$$(3.40) \quad I_{-1}(E) := \frac{1}{R_2} \ln(1 + 2R_2E^2).$$

If  $n < -1$  then  $n + 1 < 0$  and the function with values  $1 + 2R_2x^{n+1}$ ,  $x > 0$ , is strictly decreasing. For  $E < 0$ ,  $CE > 0$  and thus  $(C + E)^2 > (C - E)^2$ . Hence  $H_n(E) + 4R_3E < H_n(E) < 0$  if  $E < 0$ . Similarly,  $H_n(E) + 4R_3E > 0$  if  $E > 0$ . This shows that  $E = 0$  is the only solution of equation (3.36). If  $-1 < n < -\frac{1}{2}$  then  $n + 1 > 0$  and it follows by a similar argument that  $E = 0$  is the only solution of equation (3.36). Hence, for all  $n < -\frac{1}{2}$ ,  $n \neq -1$ , we have the unique solution

$$(3.41) \quad v_x(y) = \frac{1}{2(n+1)CR_2} [(1 + 2R_2C^2y^2)^{n+1} - (1 + 2R_2C^2)^{n+1}] + CR_3(y^2 - 1).$$

If  $n = -1$  it follows in a similar fashion that  $E = 0$  and

$$(3.42) \quad v_x(y) = \frac{1}{2CR_2} [\ln(1 + 2R_2C^2y^2) - \ln(1 + 2R_2C^2)] + CR_3(y^2 - 1).$$

**Remark 3.3.** Equation (3.13) with  $E = 0$  can be written as  $v'_x(y) = -f(y)$ , where  $f$  is the function in Lemma 2.1 with  $a = -2C$ ,  $b = 2R_2C^2$  and  $c = -2CR_3$ . It follows from the proof of Lemma 2.1 that if  $R_3 < d_n$  then  $f'(y)$  changes sign at  $y = t_1$  and at  $y = t_2$  for some numbers  $t_1 \in (t_0, \sqrt{3}t_0)$  and  $t_2 > \sqrt{3}t_0$ , where  $t_0 = (-2(2n+1)R_2C^2)^{-1/2}$ . So if  $R_3 < d_n$  and  $3 \leq -2(2n+1)R_2C^2$  then  $\sqrt{3}t_0 \leq 1$  and the graph of  $v_x(y)$  has inflection points at  $y = \pm t_1 \in (-1, 1)$ . Málek et al. [6] mention that the existence of inflection points in the velocity profiles of certain flows of perfect fluids is related to the question of the stability of those flows (it is a necessary condition for instability; see [4]). It is not clear whether that also applies to the present setting. However, it is important to determine the stability/instability of the special flows considered here in order to evaluate their physical significance.

**Example 3.3.** Consider the velocity field (3.41) with  $\mathcal{R}_2 = 0.05$ ,  $\mathcal{R}_3 = 0.125$ , and  $n = -2$ . The sufficient conditions for the existence of inflection points given in Remark 3.3 now reduce to  $C \leq -\sqrt{10}$ . Fig. 8 shows the velocity profiles for some values of the pressure gradient  $C$ . Note the inflection points in the profiles with  $C = -4, -8, -16$ . From the profiles with  $C = -4$  and  $C = -8$  we also deduce that the volume flow rate per unit depth,  $\int_{-1}^1 v_x(y) dy$ , is a non-monotone function of  $C$ .

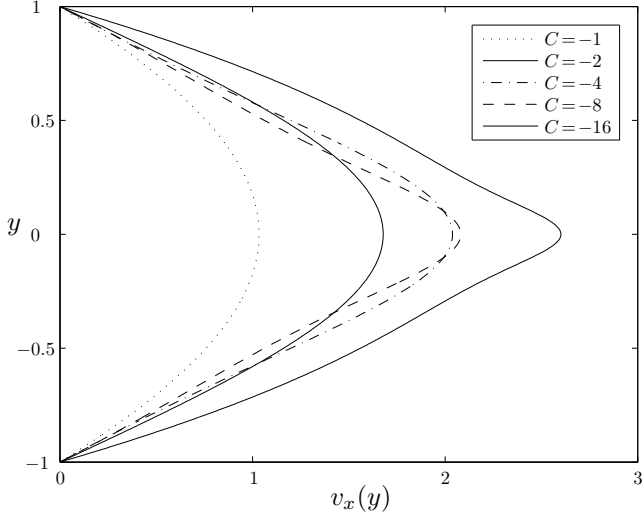


Figure 8. Velocity profiles in Example 3.3.

**3.2.2. Navier slip.** Let  $\lambda_1, \lambda_{-1} \in [0, 1)$  with  $\lambda_1 > 0$  or  $\lambda_{-1} > 0$ . Substitution of (3.32) in boundary conditions (3.17)–(3.18) yields

$$(3.43) \quad (1 - \lambda_1) \left[ \frac{1}{2(n+1)CR_2} (1 + 2R_2(C+E)^2)^{n+1} + R_3(C+2E) + F \right] + \lambda_1 \mathcal{K}(C+E - \sigma_1) = 0,$$

$$(3.44) \quad (1 - \lambda_{-1}) \left[ \frac{1}{2(n+1)CR_2} (1 + 2R_2(C-E)^2)^{n+1} + R_3(C-2E) + F \right] + \lambda_{-1} \mathcal{K}(C-E - \sigma_{-1}) = 0.$$

By eliminating  $F$  from (3.43)–(3.44) we get the equation

$$(3.45) \quad H_n(E) + LE = G,$$

where  $H_n$  is defined as in (3.37) and

$$(3.46) \quad L := 4R_3 + \frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} + \frac{\lambda_{-1} \mathcal{K}}{1 - \lambda_{-1}},$$

$$(3.47) \quad G := \frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} (\sigma_1 - C) + \frac{\lambda_{-1} \mathcal{K}}{1 - \lambda_{-1}} (\sigma_{-1} - C).$$

Similarly, for  $n = -1$  we obtain the equation

$$(3.48) \quad H_{-1}(E) + LE = G.$$

For each solution  $E$  of equation (3.45), the corresponding value of  $F$  is determined by (3.43) or (3.44) and  $v_x(y)$  is given by (3.32); a similar statement holds for  $n = -1$ . We cannot apply Lemma 2.1 directly to equation (3.45) or (3.48) but we shall show that the solvability properties of these equations are analogous to those of equation (3.24), which can be written as

$$(3.49) \quad h(E) + LE = G_*,$$

where  $h(E) = 4(1 + 2R_2E^2)^n E$  is the function in the proof of Lemma 2.1 with  $a = 4$  and  $b = 2R_2$ ,  $L$  is as in (3.46) and  $G_*$  denotes the right-hand side of equation (3.24). In the proof of Lemma 3.1 below we show that the graph of  $H_n$  has essentially the same form as the graph of  $h$ .

If  $G = 0$ , equation (3.45) or (3.48) is of the same form as equation (3.36) or (3.39), respectively, and has therefore the unique solution  $E = 0$ . In this case,  $v_x(y)$  is equal to the no-slip solution (3.41) or (3.42) plus the constant

$$\frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} (\sigma_1 - C).$$

In general, the situation is as follows.

**Lemma 3.1.** *For each  $n < -\frac{1}{2}$  there is a unique positive number  $L_n$ , which depends on  $n$ ,  $C$ , and  $R_2$ , such that the following hold:*

- (a) *If  $L \geq L_n$  then equation (3.45) (or (3.48)) has a unique solution.*
- (b) *If  $0 < L < L_n$  then equation (3.45) (or (3.48)) has either one, two or three solutions, depending on the value of  $G$ .*

**Proof.** We start by establishing some properties of the function  $H_n$ ,  $n < -\frac{1}{2}$ .

1.  $H_n$  is an odd function by virtue of its definition. Thus,  $H'_n$  is even and  $H''_n$  is odd, and it suffices to consider  $E \geq 0$ .

2. If  $n < -1$  then  $n + 1 < 0$  and the function with values  $(1 + 2R_2x)^{n+1}$ ,  $x \geq 0$ , is strictly decreasing. If  $-1 < n < -\frac{1}{2}$  then  $n + 1 > 0$  and the function  $(1 + 2R_2x)^{n+1}$ ,  $x \geq 0$ , is strictly increasing. For  $n = -1$ , the function  $\ln(1 + 2R_2x)$ ,  $x \geq 0$ , is strictly increasing. Further, if  $E > 0$  then  $CE < 0$  and so  $(C + E)^2 < (C - E)^2$ . Thus, in every case,  $H_n(E) > 0$  for all  $E > 0$ .

3. By differentiating  $H_n(E)$  we see that for all  $n < -\frac{1}{2}$ ,

$$(3.50) \quad H'_n(E) = \frac{1}{2C} (h(E+C) - h(E-C)) = \frac{1}{2C} (h(E+C) + h(C-E)),$$

where  $h$  is the function in equation (3.49). (Thus,  $H'_n(E)$  can be viewed as a central difference approximation of  $h'(E)$ .)

4. By (3.50),  $H'_n(0) = -h(-C)/C > 0$  and  $H'_n(-C) = -h(-2C)/(2C) > 0$ . In addition, if  $0 < E < -C$  then  $E+C < 0 < E-C$ , so that  $h(E+C) < 0 < h(E-C)$  and  $H'_n(E) > 0$ . Thus,  $H'_n(E) > 0$  for all  $E \in [0, -C]$ .

Furthermore, let  $t_0$  be as in the proof of Lemma 2.1 with  $a$  and  $b$  as in equation (3.49), i.e.

$$t_0 = (-2(2n+1)R_2)^{-1/2}.$$

If  $-C < t_0 + C$  and  $E \in (-C, t_0 + C]$ , then  $0 < E+C < E-C \leq t_0$ . Thus, since  $h$  is strictly increasing on  $[0, t_0]$ ,  $h(E+C) < h(E-C)$  and  $H'_n(E) > 0$ . Hence,

$$(3.51) \quad H'_n(E) > 0 \quad \text{for all } E \in [0, \max\{t_0 + C, -C\}].$$

5. By equation (3.50),  $H''_n(E) = (h'(E+C) - h'(E-C))/(2C)$ . Now, if  $E \in [-C, t_0 - C)$  then  $E+C \in [0, t_0)$  and  $h'(E+C) > 0$ . Also, if  $E = t_0 - C$  then  $h'(E+C) = h'(t_0) = 0$ . Further, if  $E \in (t_0 + C, t_0 - C]$  then  $E-C \in (t_0, t_0 - 2C]$  and  $h'(E-C) < 0$ . Also, if  $E = t_0 + C$  then  $h'(E-C) = h'(t_0) = 0$ . It follows that  $H''_n(E) < 0$  for all  $E \in [\max\{t_0 + C, -C\}, t_0 - C]$ .

In addition, if  $-C < t_0 + C$  and  $E \in [-C, t_0 + C)$ , then  $0 \leq E+C < E-C < t_0$ . Thus, since  $h'$  is strictly decreasing on  $[0, t_0]$ ,  $h'(E+C) > h'(E-C)$  and  $H''_n(E) < 0$ . Hence,

$$(3.52) \quad H''_n(E) < 0 \quad \text{for all } E \in [-C, t_0 - C].$$

6. If  $E \geq t_0 - C$  then  $t_0 \leq E+C < E-C$  and  $h(E+C) > h(E-C)$  since  $h$  is strictly decreasing on  $[t_0, \infty)$ . Thus,

$$(3.53) \quad H'_n(E) < 0 \quad \text{for all } E \geq t_0 - C.$$

7. By inequalities (3.51) and (3.53),  $H'_n(t_0 - C) < 0 < H'_n(\max\{t_0 + C, -C\})$ . Hence, by the intermediate value theorem,  $H'_n(E_0) = 0$  for some  $E_0 \in (\max\{t_0 + C, -C\}, t_0 - C)$ . Moreover, by virtue of inequalities (3.51)–(3.53), there is no other point  $E \geq 0$  such that  $H'_n(E) = 0$ . This implies that  $H'_n(E) > 0$  for all  $E \in [0, E_0)$  and  $H'_n(E) < 0$  for all  $E > E_0$ . Thus  $H_n(E)$  attains its global maximum at  $E = E_0$ .

8. Since  $h(t) \rightarrow 0^+$  as  $t \rightarrow \infty$ , it follows from equation (3.50) and inequality (3.53) that  $H'_n(E) \rightarrow 0^-$  as  $E \rightarrow \infty$ . Moreover, by applying the mean value theorem to the definition of  $H_n$  (see (3.37)), we find that  $H_n(E) = I'_n(\xi) = h(\xi)$  for some  $\xi \in (E + C, E - C)$ . Thus,  $H_n(E) \rightarrow 0^+$  as  $E \rightarrow \infty$ .

The properties derived in paragraphs 1–8 show that the graph of  $H_n$  is qualitatively the same as that of  $h$  in the proof of Lemma 2.1. We do not have formulas for  $E_0$  (the analogue of  $t_0$ ) or the global minimum of  $H'_n(E)$  (the analogue of  $-ad_n$ ) but in the paragraphs below we identify an interval that contains the point  $E_3$  (the analogue of  $\sqrt{3}t_0$ ) where this minimum is attained.

9. If  $t_0 - C < \sqrt{3}t_0 + C$  and  $E \in (t_0 - C, \sqrt{3}t_0 + C]$ , then  $0 < t_0 < E + C < E - C \leq \sqrt{3}t_0$ . Thus, since  $h'$  is strictly decreasing on  $[0, \sqrt{3}t_0]$ ,  $h'(E + C) > h'(E - C)$  and  $H''_n(E) < 0$ . Thus, in view of inequality (3.52),

$$(3.54) \quad H''_n(E) < 0 \quad \text{for all } E \in [-C, \max\{\sqrt{3}t_0 + C, t_0 - C\}].$$

If  $E \geq \sqrt{3}t_0 - C$  then  $\sqrt{3}t_0 \leq E + C < E - C$  and  $h'(E + C) < h'(E - C)$  since  $h'$  is strictly increasing on  $[\sqrt{3}t_0, \infty)$ . Hence,

$$(3.55) \quad H''_n(E) > 0 \quad \text{for all } E \geq \sqrt{3}t_0 - C.$$

By inequalities (3.54)–(3.55),  $H''_n(\max\{\sqrt{3}t_0 + C, t_0 - C\}) < 0 < H''_n(\sqrt{3}t_0 - C)$ . Hence, by the intermediate value theorem,  $H''_n(E_3) = 0$  for some  $E_3 \in (\max\{\sqrt{3}t_0 + C, t_0 - C\}, \sqrt{3}t_0 - C)$ .

10. Since  $h'$  is strictly decreasing on  $[0, \sqrt{3}t_0]$  and strictly increasing on  $[\sqrt{3}t_0, \infty)$ ,  $h'(\cdot + C)$  is strictly decreasing on  $[-C, \sqrt{3}t_0 - C]$  and  $h'(\cdot - C)$  is strictly increasing on  $[\sqrt{3}t_0 + C, \infty)$ . Therefore  $H''_n$  is strictly increasing on  $[\max\{\sqrt{3}t_0 + C, -C\}, \sqrt{3}t_0 - C]$  and so, in view of inequalities (3.54) and (3.55), there is no other point  $E \geq -C$  such that  $H''_n(E) = 0$ . This implies that  $H''_n(E) < 0$  for all  $E \in [-C, E_3)$  and  $H''_n(E) > 0$  for all  $E > E_3$ . Hence, since  $H'_n(E) > 0$  on  $[0, E_0]$ , which contains  $[0, -C]$  (see paragraph 7 above), it follows that  $H'_n(E)$ ,  $E \geq 0$ , attains its global minimum at  $E = E_3$  and at no other point.

We can now apply the arguments in the proof of Lemma 2.1 to the function  $\hat{f}(E) := H_n(E) + LE$  (instead of  $t_1$  and  $t_2$  we have analogous points  $E_1 \in (E_0, E_3)$  and  $E_2 > E_3$ ) to deduce the assertions of the present lemma with  $L_n = -H'_n(E_3)$ .  $\square$

**3.2.3. Free slip.** Here we consider the situation when  $\lambda_1 = 1$  or  $\lambda_{-1} = 1$ .

(a) If  $\lambda_1 = 1$ , boundary condition (3.17) reduces to  $E = \sigma - C$ . Thus, if  $0 \leq \lambda_{-1} < 1$ , we can solve for  $F$  in equation (3.18), which becomes equation (3.44).

Thus, for  $n \neq 1$  we obtain

$$(3.56) \quad v_x(y) = \frac{1}{2(n+1)CR_2} [(1 + 2R_2(C(y-1) + \sigma_1)^2)^{n+1} - (1 + 2R_2(\sigma - 2C)^2)^{n+1}] + R_3(C(y-1)^2 + 2\sigma_1(y+1) - 4C) + \frac{\lambda_{-1}\mathcal{K}}{1 - \lambda_{-1}}(\sigma_1 + \sigma_{-1} - 2C)$$

and  $T(y) = C(y-1) + \sigma_1$ . Note that  $[\mathbb{T}\mathbf{n}]_t = \mathbb{T}\mathbf{n} = \sigma_1\mathbf{e}_x$  at the upper plate. Similarly, for  $n = -1$  we obtain  $T(y) = C(y-1) + \sigma_1$  and

$$(3.57) \quad v_x(y) = \frac{1}{2CR_2} [\ln(1 + 2R_2(C(y-1) + \sigma)^2) - \ln(1 + 2R_2(\sigma_1 - 2C)^2)] + R_3(C(y-1)^2 + 2\sigma_1(y+1) - 4C) + \frac{\lambda_{-1}\mathcal{K}}{1 - \lambda_{-1}}(\sigma_1 + \sigma_{-1} - 2C).$$

(b) If  $\lambda_{-1} = 1$ , boundary condition (3.18) reduces to  $E = C - \sigma_{-1}$ . Thus, if  $0 \leq \lambda < 1$ , we can solve for  $F$  in equation (3.17), which becomes equation (3.43). Thus, for  $n \neq -1$  we obtain

$$(3.58) \quad v_x(y) = \frac{1}{2(n+1)CR_2} [(1 + 2R_2(C(y+1) - \sigma_{-1})^2)^{n+1} - (1 + 2R_2(\sigma_{-1} - 2C)^2)^{n+1}] + R_3(C(y+1)^2 + 2\sigma_{-1}(1-y) - 4C) + \frac{\lambda_1\mathcal{K}}{1 - \lambda_1}(\sigma_1 + \sigma_{-1} - 2C)$$

and for  $n = -1$  we obtain

$$(3.59) \quad v_x(y) = \frac{1}{2CR_2} [\ln(1 + 2R_2(C(y+1) - \sigma_{-1})^2) - \ln(1 + 2R_2(\sigma_{-1} - 2C)^2)] + R_3(C(y+1)^2 + 2\sigma_{-1}(1-y) - 4C) + \frac{\lambda_1\mathcal{K}}{1 - \lambda_1}(\sigma_1 + \sigma_{-1} - 2C).$$

Here  $T(y) = C(y+1) - \sigma_{-1}$  and so  $[\mathbb{T}\mathbf{n}]_t = \mathbb{T}\mathbf{n} = \sigma_{-1}\mathbf{e}_x$  at the lower plate.

(c) If  $\lambda = \lambda_{-1} = 1$ , boundary conditions (3.17) and (3.18) reduce to  $E = \sigma_1 - C$  and  $E = C - \sigma_{-1}$ , respectively. Hence there is no solution if  $\sigma_1 + \sigma_{-1} \neq 2C$ . If  $\sigma_1 + \sigma_{-1} = 2C$ , there are infinitely many solutions:  $v_x$  is as in formula (3.32) or (3.33) with  $E = (\sigma_1 - \sigma_{-1})/2$  and arbitrary  $F$ .

### 3.3. Planar Couette-Poiseuille flow

Suppose that the pressure gradient  $C$  is negative and that at least one of the plates is moving, i.e.  $w_1 \neq 0$  or  $w_{-1} \neq 0$ . Then  $v_x$  is as in formula (3.32) or (3.33), and for  $n \neq -1$  boundary conditions (3.17)–(3.18) become

$$(3.60) \quad (1 - \lambda_1) \left[ \frac{1}{2(n+1)CR_2} (1 + 2R_2(C + E)^2)^{n+1} + R_3(C + 2E) + F - w_1 \right] \\ + \lambda_1 \mathcal{K}(C + E - \sigma_1) = 0,$$

$$(3.61) \quad (1 - \lambda_{-1}) \left[ \frac{1}{2(n+1)CR_2} (1 + 2R_2(C - E)^2)^{n+1} \right. \\ \left. + R_3(C - 2E) + F - w_{-1} \right] + \lambda_{-1} \mathcal{K}(C - E - \sigma_{-1}) = 0.$$

**3.3.1. No slip.** When  $\lambda_1 = \lambda_{-1} = 0$ , the boundary conditions are  $v_x(1) = w_1$  and  $v_x(-1) = w_{-1}$ , which means that the expressions in square brackets in equations (3.60)–(3.61) are equal to zero. By proceeding as in Section 3.2.1 we obtain the equation

$$(3.62) \quad H_n(E) + 4R_3E = w_1 - w_{-1}.$$

If  $w_1 = w_{-1}$  then it follows as in Section 3.2.1 that  $E = 0$ . In this case,  $v_x$  is given by formula (3.41) or (3.42) with the term  $w_1$  added to the right-hand side.

If  $w_1 \neq w_{-1}$ , the situation is the same as in Section 3.2.2 but with  $L := 4R_3$  and  $G := w_1 - w_{-1}$ . For each solution  $E$  of equation (3.62),  $F$  is determined by one of the boundary conditions and  $v_x$  is given by formula (3.32) or (3.33).

**3.3.2. Navier slip.** In this case the situation is essentially the same as in Section 3.2.2. We obtain equation (3.45) or (3.48) with  $L$  as in definition (3.46) and

$$G := \frac{\lambda_1 \mathcal{K}}{1 - \lambda_1} (\sigma_1 - C) + \frac{\lambda_{-1} \mathcal{K}}{1 - \lambda_{-1}} (\sigma_{-1} - C) + w_1 - w_{-1}.$$

**3.3.3. Free slip.** In this case, once again, the situation is essentially the same as in Section 3.2.3. The difference is that we must add  $w_{-1}$  to the right-hand sides of formulas (3.56)–(3.57) and add  $w_1$  to the right-hand sides of formulas (3.58)–(3.59).

In this paper we studied simple flows of a class of fluid models which is constitutively defined by specifying the symmetric part of the velocity gradient in terms of the deviatoric part of the stress, the relation being non-invertible for some values of the material parameters. Thus, these models are not classical Stokesian fluid models. In a simple shear flow, the class of models considered here allow the shear rate to depend non-monotonically on the shear stress. As discussed in the introduction, such models could be useful if one would like to capture the phenomenon of the coagulation and lysis of blood without modeling the detail of the bio-chemical reactions that take place.

We studied problems with different types of boundary conditions, namely the no-slip, Navier slip and free slip (or tangential traction) boundary conditions. Even within the context of simple one-dimensional flows such as planar Couette and Poiseuille flows, we find that intriguing new solutions are possible. For example, in the problem of planar Couette flow with no-slip boundary conditions, the same flow field can be associated with up to three distinct stress fields. Another example: for some values of the material parameters, the boundary-value problem for planar Poiseuille flow with Navier slip boundary conditions can have either one, two or three distinct solutions. The solutions that we establish are explicit exact solutions and thus the multiplicity that we find is not due to some numerical artifice. Exact solutions are useful in that they not only provide elegant solutions to specific problems, they can be used to test numerical schemes that are developed to solve problems in more complicated or three-dimensional flow domains. Moreover, although our analysis deals with a specific family of power-law type models, the main arguments can be applied to other non-monotone constitutive relations of the type depicted in Fig. 2. We therefore expect that the corresponding flow problems for such fluid models will have similar non-uniqueness properties.

When one has non-unique solutions, some of the solutions might be unstable. There has been no work concerning the stability of flows of the fluids considered here and this would be a very worthwhile undertaking. It would also be worthwhile to study flows of such fluids in more complex geometries.

#### *References*

- [1] *M. Anand, K. Rajagopal, K. R. Rajagopal*: A model incorporating some of the mechanical and biomechanical factors underlying clot formation and dissolution in flowing blood. *J. Theor. Med.* 5 (2003), 183–218.
- [2] *M. Anand, K. Rajagopal, K. R. Rajagopal*: A model for the formation and lysis of blood clots. *Pathophysiology of Homeostasis and Thrombosis* 34 (2005), 109–120.



- [3] *J. M. Burgers*: Mechanical considerations—model systems—phenomenological theories of relaxation and viscosity. In: First Report on Viscosity and Plasticity (*J. M. Burgers*, ed.). Nordemann Publishing Company, New York, 1935.
- [4] *P. G. Drazin, W. H. Reid*: Hydrodynamic Stability (2nd ed.). Cambridge University Press, 2004.
- [5] *J. Lawson, K. Rajagopal*: Regulation of hemostatic system function by biochemical and mechanical factors. In: Modeling of Biological Materials (*F. Mollica, L. Preziosi, K. R. Rajagopal*, eds.). Birkhäuser, Boston, 2007, pp. 179–210.
- [6] *J. Málek, V. Průša, K. R. Rajagopal*: Generalizations of the Navier-Stokes fluid from a new perspective. *Int. J. Eng. Sci.* *48* (2010), 1907–1924.
- [7] *J. C. Maxwell*: On the dynamical theory of gases. *Philos. Trans. R. Soc. Lond. Ser. A* *157* (1866), 26–78.
- [8] *W. A. Noll*: A new mathematical theory of simple materials. *Arch. Ration. Mech. Anal.* *48* (1972), 1–50.
- [9] *J. G. Oldroyd*: On the formulation of rheological equations of state. *Proc. R. Soc. Lond. Ser. A* *200* (1950), 523–541.
- [10] *J. G. Oldroyd*: Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids. *Proc. R. Soc. Lond. Ser. A* *245* (1958), 278–297.
- [11] *K. R. Rajagopal*: On implicit constitutive theories. *Appl. Math.* *48* (2003), 279–319.
- [12] *K. R. Rajagopal*: On implicit constitutive theories for fluids. *J. Fluid. Mech.* *550* (2006), 243–249.
- [13] *K. R. Rajagopal, A. R. Srinivasa*: On the thermodynamics of fluids defined by implicit constitutive relations. *Z. Angew. Math. Phys.* *59* (2008), 715–729.
- [14] *V. I. Zarnitsina, A. V. Pokhilko, F. I. Ataullakhanov*: A mathematical model for spatio-temporal dynamics of intrinsic pathway of blood coagulation. I. The model description. *Thromb. Res.* *84* (1996), 225–236.
- [15] *V. I. Zarnitsina, A. V. Pokhilko, F. I. Ataullakhanov*: A mathematical model for spatio-temporal dynamics of intrinsic pathway of blood coagulation. II. Results. *Thromb. Res.* *84* (1996), 333–344.

*Authors' addresses:* *C. Le Roux*, Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa, e-mail: [ianleroux@yahoo.com](mailto:ianleroux@yahoo.com); *K. R. Rajagopal*, Department of Mechanical Engineering, Texas A&M University, College Station, TX-77845, USA, e-mail: [krajagopal@tamu.edu](mailto:krajagopal@tamu.edu).