Riesz theory and Fredholm determinants in Banach algebras

by

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To my daughter Karabo

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Notations

$\mathcal{A}$	A Banach algebra
$ran(x)$	The rank of an element x of $\mathcal{A}$
$\mathcal{F}$	The ideal of finite rank elements of $\mathcal{A}$
$\mathcal{F}_n$	The set of rank n elements of $\mathcal{A}$
$\mathcal{N}$	The ideal of nuclear elements of $\mathcal{A}$
$\mathcal{K}$	The norm closure of $\mathcal{F}$
$\delta_n(a)$	The nth approximation number of $a \in \mathcal{A}$
$\mathcal{C}_1$	Elements of $\mathcal{A}$ with summable approximation numbers
$\mathcal{F}(\mathcal{A})$	The set of Fredholm elements of $\mathcal{A}$
$\mathcal{R}(\mathcal{A})$	The set of Riesz elements of $\mathcal{A}$
X	A Banach space
X^*	The dual of a Banach space X
$B(X)$	Bounded linear operators on X
$F(X)$	Finite rank operators on a Banach space X
$\mathcal{C}_1(X)$	Operators on X with summable approximation numbers

Introduction

Fredholm determinant theory of integral operators was introduced by I. Fredholm at the turn of the centuary [Fre03], by studying the properties of the eigenvalues of these operators and their connections with the analyticity of the determinant. This theory sparked the development of functional analysis, in particular a comprehensive study of the spectral theory of compact operators. In fact a few years later after Fredholm published his 1903 paper, E. Schmidt and F. Riesz developed an approach to functional analysis which was determinant free, marking the beginning of abstract operator theory [Sch07, Rie18].

After contributions of J. Schauder [Sch30] and S. Banach [Ban32] there was a gap of about two decades until in the fifties when A.F. Ruston, T. Leżański and A. Grothendieck almost simultaneously introduced determinants for nuclear operators on Banach spaces [Rus51, Lez53, Gro56]. The works of the latter were complicated due to the problem of the approximation property, hence, later on attention was given to the Hilbert space case.

The counterexample of P. Enflo [Enf73] suggested that Fredholm determinants could reasonably be investigated only in cases where the Banach space has the approximation property. In this case the Fredholm determinant is well defined on the ideal of nuclear operators with all the desired properties.

During the eighties, Pietsch [Pie87] gave an axiomatic approach to the theory of trace and determinant defined on certain operator ideals by investigating the relationship between traces, determinants and eigenvalues. This monograph provides a beautiful interplay between Riesz theory and the theory of traces and determinants in operator ideals. It is therefore natural and important to establish whether such a theory can be extended to general Banach algebras. The aim of this thesis is on one hand to obtain a complete Riesz decomposition theorem for Riesz elements in a semiprime Banach algebra and on the other hand to extend the existing theory of traces and determinants to a more general setting of Banach algebras.

Already in 1978, J. Puhl [Puh78] introduced the notion of a trace on the socle of a semiprime Banach algebra. In the case where the Banach algebra consists of bounded linear operators on some Banach space, the socle coincides with the ideal of finite rank operators. Puhl showed that as in the classical case, the trace of an element of the socle has a spectral representation known as the Lidskij trace formula. Moreover if the algebra has some approximation property, the trace can be extended uniquely to the ideal of nuclear elements of the algebra (although not necessarily spectral).

To our knowledge, B. Aupetit and du T. Mouton [AdTM96] were the first to introduce a Fredholm determinant for elements of the socle of a semisimple Banach algebra. If one uses similar axioms for a functional to be a trace on the socle as Pietsch did, it follows directly that such a functional satisfies the Lidskij trace formula, hence justifying the definition of Aupetit and Mouton of their Fredholm determinant. In their approach, Aupetit and Mouton made use of the theory of analytic multifunctions on the socle. However, as the classical case shows, the spectral determinant does not extend continuously to the ideal of nuclear elements as a spectral determinant.

Chapter one contains some notations, definitions and standard facts that will be used throughout the thesis. We will introduce the rank notion as defined by P. Nylen and L. Rodman [NR90] in order to define multiplicity of isolated spectral points of elements of Banach algebras. Some of the standard facts will be stated without their proofs, however, corresponding references will be included. We also use our rank notion to give a characterization of the essential spectral radius of elements of $\mathcal{A}$. This characterization turns out to settle a conjecture of P. Nylen and L. Rodman [NR90], Conjecture 5.11 in the affirmative [BS98].

In Chapter 2 we use our rank notion to establish a complete Riesz decomposition theorem for Riesz elements of a semiprime Banach algebra. For a comprehensive study of these elements we refer the reader to [BMSW82]. The main characteristics of Riesz elements seem to be the fact that they are precisely those elements for which the spectral points are isolated with finite multiplicities [BMSW82], Corollary R.2.5.

In Chapter 3 we provide an alternative formula for the determinant on the ideal $\mathcal{F}$ of finite rank elements of the algebra (as the one suggested by Aupetit and Mouton) emanating from Plemelj's type formulas. We will show that our determinant extends continuously to the ideal of nuclear elements $\mathcal{N}$ of $\mathcal{A}$, provided $\mathcal{A}$ has some approximation property, and that the determinant possesses all the desired properties. The approach we follow complements the recent work of I. Gohberg, S. Goldberg and N. Krupnik [GGK96].

We remark that earlier this year, B. Carl and C. Schiebold [CS99] established the importance of Fredholm determinants in solving some nonlinear equations in soliton physics such as the Korteweg-de Vries and sine-Gordon equations.

Chapter 1

On the essential spectrum

Throughout the thesis $\mathcal{A}$ will denote a unital Banach algebra over the complex field $\mathbb{C}$ unless otherwise stated in some chapters. For an element $a \in \mathcal{A}$, $\sigma(a)$ denotes the **spectrum** of a and $r(a)$ the **spectral radius** of a.

We adopt the rank notion of P. Nylen and L. Rodman [NR90], Definition 2.1. We say $x \in \mathcal{A}$ is of **rank one** if for every $y \in \mathcal{A}$ there is a complex number λ_y such that $xyx = \lambda_y x$. We say $x \in \mathcal{A}$ is of **rank** n for some integer n if x can be expressed as a sum of n rank one elements of $\mathcal{A}$ but cannot be expressed as a sum of less than n rank one elments of $\mathcal{A}$. We denote the set of finite rank elements of $\mathcal{A}$ by $\mathcal{F}$ and the norm closure of $\mathcal{F}$ by $\mathcal{K}$. We denote by $\mathcal{F}_n$, the component of $\mathcal{F}$ consisting of rank n elements only. We will establish some properties of this rank notion in Proposition 1.1.1 where we will prove that $\mathcal{F}$ is a two-sided ideal of $\mathcal{A}$.

For an element $a \in A$, $\sigma_K(a)$ and $r_K(a)$ will respectively denote the **essential spectrum** of a and the **essential spectral radius** of a, that is, the spectrum and spectral radius of $a+\mathcal{K}$ in the quotient Banach algebra $\mathcal{A}/\mathcal{K}$. Clearly $\sigma_K(a) \subset \sigma(a)$.

We say $x \in \mathcal{A}$ is **Fredholm** if $x+\mathcal{K}$ is invertible in $\mathcal{A}/\mathcal{K}$ and denote the set of the Fredholm elements of $\mathcal{A}$ by $\mathcal{F}(\mathcal{A})$. We say an element $x \in \mathcal{A}$ is **Riesz** if the essential spectrum of x, $\sigma_K(x)$, relative to $\mathcal{K}$ consists of zero only. The set of Riesz elements of $\mathcal{A}$ will be denoted by $\mathcal{R}(\mathcal{A})$.

A complex number λ is called a **Fredholm point** of $x \in \mathcal{A}$ if $\lambda - x \in \mathcal{F}(\mathcal{A})$ and λ is called a **Riesz point** of $x \in \mathcal{A}$ if $\lambda - x$ is invertible or λ is a Fredholm point of x which is an isolated point of $\sigma(x)$.

As in the classical theory of bounded operators on a Banach space the notion of multiplicity is essential. We say $\lambda \in \sigma(a)$ has **finite multiplicity** if λ is an isolated spectral point such that the corresponding **Riesz idempotent** $e_\lambda := \frac{1}{2\pi i}\int_{|\mu-\lambda|=\epsilon}(\mu - a)^{-1}d\mu$ is of finite rank. The multiplicity of λ is defined to be the rank of e_λ. Spectral points of finite multiplicity will often be denoted by f.m. spectral points. We denote the rank of e_λ by m_λ if it is finite.

1.1 Finite rank elements

We will assume the definition of rank given in the opening of the chapter and establish some facts about the rank.

Proposition 1.1.1. *([NR90], Proposition 2.2). Let $a,b \in \mathcal{A}$. The rank function has the following properties:*

(a) $rank(a+b) \leq rank(a) + rank(b)$,

(b) if $rank(a) = 1$, then for every $b \in \mathcal{A}$ either $ab = 0$ or $rank(ab) = 1$. Moreover $rank(ba) = 1$ unless $ba = 0$,

(c) $rank(ab) \leq min\{rank(a), rank(b)\}$,

(d) the set $\mathcal{F}$ is a two-sided ideal in $\mathcal{A}$, and

(e) the subalgebra $\mathcal{A}(a_1, \ldots, a_s)$ of $\mathcal{A}$, generated by 1 and a finite number of finite rank elements $a_1, \ldots, a_s$ of $\mathcal{A}$, is finite dimensional, as a vector space over $\mathbb{C}$.

Proof. (a) For the case where at least one of a and b is of infinite rank the result follows trivially. Hence, suppose both rank(a) and rank(b) are finite, that is rank$(a) = n_1$ and rank$(b) = n_2$. So, $a = a_1 + \ldots + a_{n_1}$, where rank$(a_i) = 1$, $(i = 1, \ldots, n_1)$. Also $b = b_1 + \ldots + b_{n_2}$, where rank$(b_j) = 1$, $(j = 1, \ldots, n_2)$.

Then $a + b = a_1 + \ldots + a_{n_1} + b_1 + \ldots + b_{n_2}$, from which we deduce that $\text{rank}(a + b) \leq n_1 + n_2 = \text{rank}(a) + \text{rank}(b)$.

(b) Since $\text{rank}(a) = 1$, for every $c \in \mathcal{A}$ there is a $\lambda \in \mathbb{C}$ such that $aca = \lambda a$. Arbitrarily choose $d \in \mathcal{A}$ and assume $ab \neq 0$. Then $abdab = (\lambda_0)ab$, for some $\lambda_0 \in \mathbb{C}$. That is $(ab)d(ab) = \lambda_0(ab)$ which shows that $\text{rank}(ab) = 1$. By a similar argument, it can be shown that $\text{rank}(ba) = 1$ unless $ba = 0$.

(c) If both $\text{rank}(a)$ and $\text{rank}(b)$ are infinite we then have nothing to prove. Suppose $\text{rank}(a) = n \leq \text{rank}(b)$. So, $ab = (a_1 + \ldots + a_n)b$ with $\text{rank}(a_i) = 1$, $(i = 1, \ldots, n)$. That is, $ab = a_1 b + \ldots + a_n b$. Thus, from (a) $\text{rank}(ab) \leq \text{rank}(a_1 b) + \ldots + \text{rank}(a_n b)$. By (b) $\text{rank}(ab) \leq n = \min\{\text{rank}(a), \text{rank}(b)\}$. The analogous argument works for the case where $\text{rank}(b) = n \leq \text{rank}(a)$.

(d) Let $a, b \in \mathcal{F}$ and $d \in \mathcal{A}$. We apply (a) and (c) to get the following: $\text{rank}(\alpha a + \beta b) \leq \text{rank}(\alpha a) + \text{rank}(\beta b) \leq \text{rank}(a) + \text{rank}(b)$, $(\alpha, \beta \in \mathbb{C})$. That is, $\text{rank}(\alpha a + \beta b) < \infty$. Hence, $\mathcal{F}$ is a subspace of $\mathcal{A}$.

Also from part (b), $\text{rank}(db) \leq \min\{\text{rank}(d), \text{rank}(b)\} \leq \text{rank}(b) < \infty$. Thus, $db \in \mathcal{F}$ and by a similar argument, $bd \in \mathcal{F}$. Whence, $\mathcal{F}$ is a two-sided ideal of $\mathcal{A}$.

(e) Since any finite rank element of $\mathcal{A}$ is a sum of a finite number of rank one elements of $\mathcal{A}$, we assume that $\text{rank}(a_i) = 1$, $(i = 1, \ldots, s)$.

The algebra $\mathcal{A}(a_1, a_2, \ldots, a_s)$ consists of all polynomials in $1, a_1, a_2, \ldots, a_s$. Let $a_0 = 1$. For all $1 \leq i \leq s$ and $0 \leq k \leq s$ it follows that $a_i a_k a_i = \lambda_{ik} a_i$.

Thus all powers of finite products of elements of $\mathcal{A}(a_1, a_2, \ldots, a_s)$ will reduce to scalar multiples of products of distinct elements from $\{a_0, a_1, \ldots, a_s\}$. Hence $\mathcal{A}(a_1, a_2, \ldots, a_s)$ will be spanned by all possible finite products of distinct elements from $\{a_0, a_1, a_2, \ldots, a_s\}$ which can only be finite in number.

Therefore $\dim(\mathcal{A}(a_1, a_2, \ldots, a_s)) < \infty$ and the theorem is established.

□

Corollary 1.1.1. *([NR90], Corollary 2.3). Every $x \in \mathcal{A}$ with rank$(x) < \infty$ is algebraic. That is, there is a non-zero polynomial $P(t)$ such that $P(x) = 0$. In particular, the spectrum $\sigma(x)$ is a finite set.*

Proof. The subalgebra $\mathcal{A}(x)$, generated by the identity 1 and x is finite dimensional, which follows from Proposition 1.1.1 (e). Since $x^n \in \mathcal{A}(x)$, $(n = 1, 2, \ldots)$, the x^n's cannot all be linearly independent over $\mathbb{C}$. This says that scalars $\lambda_1, \ldots, \lambda_k$ exist, not all zero, such that

$$\lambda_k x^{n_k} + \ldots + \lambda_1 x^{n_1} = 0.$$

Whence,

$$P(t) = \lambda_k t^{n_k} + \ldots + \lambda_1 t^{n_1}$$

is the required polynomial for which $P(x) = 0$.

Lastly, by the Spectral Mapping Theorem, it follows that $\sigma(P(x)) = P(\sigma(x))$. But,

$$\begin{aligned} \sigma(P(x)) &= \sigma(0) \\ &= \{0\}. \end{aligned}$$

So, $P(\sigma(x)) = \{0\}$. Since $P(t)$ has a finite number of zeros, it follows that $\sigma(x)$ is a finite set and we are done. □

The following lemma will be needed to show that if $\mathcal{A} = B(X)$, then the rank notion coincides with the classical notion of finite dimensional range.

Lemma 1.1.1. *Let $T \in B(X)$, with X a Banach space and dim($T(X)$) $= n < \infty$. Then, T has a representation of the form*

$$Tx = f_1(x)y_1 + \ldots + f_n(x)y_n,$$

where $\{y_1, \ldots, y_n\}$ and $\{f_1, \ldots, f_n\}$ are sets in X and X^* respectively.

Proof. There is an independent set $\{y_1, \ldots, y_n\}$ in Y, such that span$\{y_1, \ldots, y_n\} = T(X)$. Then for each $x \in X$,

$$Tx = \sum_{i=1}^{n} f_i(x)y_i. \tag{1.1}$$

Since this representation is unique the coefficients $f_i(x)$ are clearly seen to define linear functionals on X. Since $T(X)$ has a finite dimension, all norms on $T(X)$ are equivalent, hence there exists a constant $K > 0$ such that

$$\sum_{i=1}^{n} |f_i(x)| \leq K\|\sum_{i=1}^{n} f_i(x)y_i\| \text{ for any } x \in X.$$

Hence

$$\begin{aligned}\sum_{i=1}^{n} |f_i(x)| &\leq K\|Tx\|,\\ &\leq K\|T\|\|x\|.\end{aligned}$$

This shows that all the f_i's are bounded. □

The $y_1, \ldots, y_n$ in representation (1.1) are chosen to be linearly independent. We could arrange it so that $f_1, \ldots, f_n$ are also linearly independent but we will not need this fact for our purpose.

We next prove a theorem that confirms that if $\mathcal{A} = B(X)$, where X is a Banach space, the rank notion applied to an elment x of $\mathcal{A}$ boils down to the rank of x as an operator, that is, the dimension of the range of x.

Theorem 1.1.1. *([NR90], Theorem 2.4). Let $\mathcal{A} = B(X)$, where X is a Banach space. Let $T \in \mathcal{A}$. Then, $rank(T) = n$ if and only if $dim(T(X)) = n$.*

Proof. By the representation in Lemma 1.1.1 it is clear that any operator $T \in B(X)$ with $\dim(T(X)) = n$ can be written as a sum of n operators each with one-dimensional range. Hence it suffices to prove the theorem for $n = 1$.

Now suppose $\text{rank}(T) = 1$. Since $T^2 = \lambda T$ for some scalar λ, we can assume, if $\lambda \neq 0$, by rescaling, that $T^2 = T$. The rescaling can be done by considering $T_0 = \frac{T}{\lambda}$, if $\lambda \neq 0$. So that, $T_0^2 = \frac{T^2}{\lambda^2} = \frac{\lambda T}{\lambda^2} = \frac{T}{\lambda} = T_0$. Hence either $T^2 = T$ or $T^2 = 0$.

Case 1: Suppose $T^2 = T$. That is, T is a projection. Thus, X is a direct sum of T-invariant subspaces X_0 and X_1 such that $Ta = a$, for all $a \in X_1$ and $Tb = 0$, for all $b \in X_0$.

We now assume $\dim(X_1) > 1$. So, there are $a_1, a_2 \in X_1$, which are linearly independent. Let X_2 be a direct complement of $\text{span}\{a_1\}$ in X_1. Clearly $a_2 \in X_2$. Define $S \in \mathcal{A}$ by $Sb = 0, \quad b \in X_0 \oplus X_2$ and $Sa_1 = a_1$. So,

$$\begin{aligned} TSTa_1 &= TSa_1 \\ &= Ta_1 \\ &= a_1. \end{aligned}$$

Also,

$$\begin{aligned} TSTa_2 &= TSa_2 \\ &= T0 \\ &= 0. \end{aligned}$$

So, there is no $\lambda \in \mathbb{C}$ such that $TST = \lambda T$, which contradicts the assumption that $\text{rank}(T) = 1$. So, we must have

$$\begin{aligned} \dim(T(X)) &= \dim(X_1) \\ &= 1 \\ &= \text{rank}(T). \end{aligned}$$

Case 2: Suppose $T^2 = 0$. We also carry out the argument by contradiction. We suppose that $\dim(T(X)) > 1$. Let $a_1 = Tc_1$ and $a_2 = Tc_2$ be linearly independent for some c_1 and $c_2 \in X$. It then follows from the linearity of T that c_1 and c_2 are also linearly independent. Since $T^2 = 0$, it follows that

$$\{a_1, a_2, c_1, c_2\}$$

is linearly independent, because

$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 c_1 + \lambda_4 c_2 = 0$$

implies that

$$T(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 c_1 + \lambda_4 c_2) = 0,$$

which implies that

$$\lambda_1 T a_1 + \lambda_2 T a_2 + \lambda_3 T c_1 + \lambda_4 T c_2 = 0.$$

Hence

$$\lambda_1 T^2 c_1 + \lambda_2 T^2 c_2 + \lambda_3 T c_1 + \lambda_4 T c_2 = 0.$$

So,

$$\lambda_3 T c_1 + \lambda_4 T c_2 = 0, \text{ because } T^2 = 0.$$

That is,

$$\lambda_3 a_1 + \lambda_4 a_2 = 0,$$

which means that

$$\lambda_3 = \lambda_4 = 0, \text{ because } a_1 \text{ and } a_2 \text{ are linearly independent.}$$

Since

$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 c_1 + \lambda_4 c_2 = 0,$$

it follows that

$$\lambda_1 a_1 + \lambda_2 a_2 = 0,$$

Therefore,

$$\lambda_1 = \lambda_2 = 0, \text{ because } a_1 \text{and } a_2 \text{ are linearly independent.}$$

We then have

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.$$

We now let

$$M = \text{span}\{a_1, a_2, c_1, c_2\}.$$

Clearly, M is T-invariant. Restricting T to the subspace M, we can represent T by the following matrix:

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Consider

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly

$$TST = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, there is no scalar $\lambda \in \mathbb{C}$ such that $TST = \lambda T$. This contradicts the assumption that $\text{rank}(T) = 1$. We therefore have

$$\dim(T(X)) = 1 = \text{rank}(T).$$

For the converse, assume that $\dim(T(X)) = 1$. So, by using Lemma 1.1.1 it follows that

$$Ta = \psi(a)b, \ (a \in X), \ b \text{ a fixed vector and } \psi \text{ fixed in } X^*. \tag{1.2}$$

Note that from equation (1.2) we have

$$b = \frac{Ta}{\psi(a)}, \ (a \neq 0).$$

So, for any $S \in B(X)$, we get

$$TSb = \psi(Sb)b.$$

Therefore,

$$\frac{TSTa}{\psi(a)} = \psi(Sb)b.$$

That is,

$$TSTa = \psi(Sb)\psi(a)b.$$

That is,

$$\begin{aligned} TSTa &= \frac{Ta}{\psi(a)}\psi(a)\psi(Sb) \\ &= \psi(Sb)Ta. \end{aligned}$$

Therefore,

$$TSTa = \psi(Sb)Ta, \text{ for all } a \in X.$$

So,

$$TST = \lambda T \text{ for all } S \in B(X), \text{ where } \lambda = \psi(Sb).$$

Whence, $\text{rank}(T) = 1$. □

Remark 1.1.1. Let $\mathcal{A} = M_n(\mathbb{C})$, the algebra of all n by n matrices over $\mathbb{C}$. The space $\mathcal{A}$ with the usual matrix multiplication, can be viewed as the algebra $B(\mathbb{C}^n)$ with multiplication being the composition of operators and the classical matrix rank of an n by n matrix T is exactly the dimension of $T(\mathbb{C}^n)$ which by Theorem 1.1.5 equals $\text{rank}(T)$.

Let us consider for a moment the case where $\mathcal{A}$ is a semiprime Banach algebra. Recall that $\mathcal{A}$ is called semiprime if $uxu = 0$ for all $x \in \mathcal{A}$ then $u = 0$.

The example given in Theorem 1.1.1 is semiprime. If K is a compact Hausdorff space then it is easy to see that the algebra $C(K)$ of continuous functions on K is semiprime. In fact if $\mathcal{A}$ is any C^*-algebra then $\mathcal{A}$ is semiprime. To see this suppose $aba = 0$ for all $b \in \mathcal{A}$ which implies that $(ab)^2 = 0$ for every $b \in \mathcal{A}$. Thus if we choose $b = a^*$ we have $(aa^*)^2 = 0$ and since aa^* is positive, it clearly follows that $aa^* = 0$. Hence $0 = \|aa^*\| = \|a\|^2$, which implies that $a = 0$.

Remark 1.1.2. In the case where $\mathcal{A}$ is semiprime it follows from [Puh78], Remark 1.2.5 that $\mathcal{A}$ contains minimal left ideals. Moreover $\mathcal{F}$ coincides with the socle of $\mathcal{A}$, that is the ideal generated by the set of minimal idempotents of $\mathcal{A}$. For more details on minimal idempotents see [Ric60]

1.2 Nuclear elements

An element u of $\mathcal{A}$ is called **nuclear** if there exist rank one elements $u_i \in \mathcal{A}$ such that $u = \sum_{i=1}^{\infty} u_i$ and $\sum_{i=1}^{\infty} \|u_i\| < \infty$. We denote by $\mathcal{N}$ the class of nuclear elements of $\mathcal{A}$. For $u \in \mathcal{N}$, we let $\nu(u) = \inf(\sum_{i=1}^{\infty} \|u_i\|)$ where the infimum is taken over all such possible representations of u. In fact in the case where $\mathcal{A}$ is semiprime it was shown by Puhl that $\mathcal{N}$ is complete in the norm ν

[Puh78]. This definition is motivated by the definition of nulear operators on a Banach space X. We establish the completeness of $\mathcal{N}$ in the norm ν for any Banach algebra which is not necessarily semiprime.

Proposition 1.2.1. *$\mathcal{N}$ is a two-sided ideal of $\mathcal{A}$ with $\mathcal{F} \subset \mathcal{N}$. Moreover ν is a norm on $\mathcal{N}$ satisfying $\nu(xuy) \leq ||x||\nu(u)||y||$ for any $x, y \in \mathcal{A}$ and $u \in \mathcal{N}$.*

Proof. That $\mathcal{N}$ is a two-sided ideal of $\mathcal{A}$ and $\mathcal{F} \subset \mathcal{N}$ is clear. We will only show that ν is a norm on $\mathcal{N}$ satisfying the inequality $\nu(xuy) \leq ||x||\nu(u)||y||$ for any $x, y \in \mathcal{A}$ and $u \in \mathcal{N}$.

By definition $\nu(u) < \infty$ for any nuclear element u. Let $u = \sum_{i=1}^{\infty} u_i$ be any nuclear representation such that $\nu(u) = 0$. It then follows from $||u|| \leq \sum_{i=1}^{\infty} ||u_i||$ that $||u|| \leq \nu(u) = 0$. Therefore $u = 0$. Conversely suppose $u = \sum_{i=1}^{\infty} u_i = 0$. It follows trivially that $\nu(u) = 0$.

That $\nu(\lambda u) = |\lambda|\nu(u)$ for all $\lambda \in \mathbb{C}$ is also clear from the definition of ν. To establish the triangle inequality, let $u, w \in \mathcal{N}$. So, $\nu(u + w) \leq \sum_{i=1}^{\infty} ||u_i|| + \sum_{i=1}^{\infty} ||w_i||$ for all nuclear representations of u and w. It then follows that $\nu(u+w) \leq \nu(u) + \nu(w)$. That $\nu(xuy) \leq ||x||\nu(u)||y||$ follows from the fact that $\mathcal{N}$ is a two-sided ideal and $||.||$ is an algebra norm on $\mathcal{A}$. □

Proposition 1.2.2. *$\mathcal{N}$ is complete with respect to ν.*

Proof. Let $\{w_n\}$ be any Cauchy sequence in $\mathcal{N}$. Since $\|w_n - w_m\| \leq \nu(w_n - w_m)$, it follows that $\{w_n\}$ is a Cauchy sequence in $\mathcal{A}$ with respect to the norm $\|.\|$. Thus there is a $w \in \mathcal{A}$ such that $\lim_{n\to\infty} \|w - w_n\| = 0$. We choose an increasing sequence of positive integers n_k such that $\nu(w_n - w_m) < \frac{1}{2^{k+2}}$, $n, m \geq n_k$. We consider the subsequence $\{w_{n_k}\}$ of $\{w_n\}$. Clearly $\nu(w_{n_{k+1}} - w_{n_k}) < \frac{1}{2^{k+2}}$ for each k. For each k we choose a nuclear representation $\sum_{i=1}^{\infty} w_i^{(k)}$ of $w_{n_{k+1}} - w_{n_k}$ satisfying

$$\sum_{i=1}^{\infty} \|w_i^{(k)}\| < \frac{1}{2^{k+2}}.$$

For all l we can express $w_{n_{k+l}} - w_{n_k}$ as follows:

$$w_{n_{k+l}} - w_{n_k} = [w_{n_{k+l}} - w_{n_{k+l-1}}] + [w_{n_{k+l-1}} - w_{n_{k+l-2}}] + \ldots + [w_{n_{k+1}} - w_{n_k}].$$

Letting l go to infinity in the $\mathcal{A}$ norm $\|.\|$, we get $w - w_{n_k} = \sum_{j=k}^{\infty} \sum_{i=1}^{\infty} w_i^{(j)}$ with $\sum_{j=k}^{\infty} \sum_{i=1}^{\infty} \|w_i^{(j)}\| < \sum_{j=k}^{\infty} \frac{1}{2^{j+2}} = \frac{1}{2^{k+1}}$. Thus $w - w_{n_k} \in \mathcal{N}$ and hence $w \in \mathcal{N}$. Also we have $\nu(w - w_{n_k}) \leq \frac{1}{2^{k+1}} \to 0$. Therefore $w_n \to w$ in the nuclear norm ν. □

Proposition 1.2.3. *The ideal $\mathcal{F}$ is ν-dense in $\mathcal{N}$.*

Proof. Suppose $u \in \mathcal{N}$ has a representation $u = \sum_{i=1}^{\infty} u_i$. Then

$$\nu(u - \sum_{i=1}^{n-1} u_i) = \nu(\sum_{i=n}^{\infty} u_i) \leq \sum_{i=n}^{\infty} \|u_i\| \to 0.$$

Thus $\mathcal{F}$ is ν-dense in $\mathcal{N}$. □

Remark 1.2.1. If $\mathcal{A} = B(X)$ for some Banach space X,then $u \in \mathcal{N}$ if and only if u is a nuclear operator. This follows from the fact that $u \in \mathcal{A}$ is of rank one if and only if u is a rank one operator (See Theorem 1.1.1).

1.3 A characterization of the essential spectrum

In 1967 T. Yamamoto [Yam67] proved a theorem on the asymptotic behaviour of the singular values of an $n \times n$ matrix over $\mathbb{C}$. For a number of years this classical result was a subject of extension to $B(X)$. In 1990, P. Nylen and L. Rodman [NR90] introduced the notion of spectral radius property in Banach algebras in order to generalize this classical result. In [BS98], the author and A. Ströh proved a conjecture of P. Nylen and L. Rodman [NR90], Conjecture 5.11 in the affirmative, namely that any unital Banach algebra has the spectral radius property. In fact we showed that a slightly more general spectral property holds. We showed that for every element which has spectral points which are not of finite multiplicity, the essential spectral radius is the supremum of the set of absolute values of the spectral points that are not of finite multiplicitity.

The spectral radius property is defined in [NR90] in terms of a spectral point sequence which corresponds to each element of the Banach algebra $\mathcal{A}$.

We outline the construction of the spectral point sequence as in [NR90]. For each $a \in \mathcal{A}$ we consider $S_1 = \{\lambda : |\lambda| = r(a)\} \bigcap \sigma(a)$. If S_1 consists of f.m. spectral points, we set $\mu_i(a) = r(a), \quad (i = 1, 2, \ldots, n_1)$ where n_1 is the sum of the multiplicities of the points of S_1. If S_1 contains some point which is not of f.m. we set $\mu_i(a) = r(a), \quad (i = 1, 2, \ldots)$. In the former case, we proceed by considering $S_2 = \{\lambda : |\lambda| = r(a_1)\} \bigcap \sigma(a_1)$, where $a_1 = (1 - e_{S_1})a(1 - e_{S_1})$ and e_{S_1} is the Riesz idempotent corresponding to S_1. We view a_1 as an element of the Banach algebra $(1 - e_{S_1})\mathcal{A}(1 - e_{S_1})$ with unit $1 - e_{S_1}$. If S_2 consists of f.m. spectral points, we set $\mu_{n_1+j}(a) = r(a_1), \quad (j = 1, 2, \ldots, n_2)$ where n_2 is the sum of the multiplicities of the points of S_2. If S_2 contains some point which is not of f.m. we set $\mu_{n_1+j}(a) = r(a_1), \quad (j = 1, 2, \ldots)$. Continuing this process we obtain a nonincreasing sequence of positive numbers which can either be infinite or finite (we have the latter case if $\sigma(a)$ consists only of f.m. spectral points). We let $n(a)$ denote the length of the sequence. If the sequence is infinite, we denote its limit by $\mu(a)$. We will show that any unital Banach algebra has the following spectral property which was first introduced in [NR90]:

For every $a \in \mathcal{A}$ *for which the spectral point sequence* $\{\mu_n(a)\}_{n=1}^{\infty}$ *is of infinite length and* $\mu(a) = \mu_m(a)$ *for some integer* m, *we have that* $r_K(a) = \mu(a)$.

The proof is based on a well-established Fredholm theory in a Banach algebra relative to an inessential ideal [BMSW82].

We will need the following facts in the sequel:

Theorem 1.3.1. *([BMSW82], Lemma R.2.3) If* λ *is a Riesz point of* $\sigma(x)$, *then it is an isolated point of* $\sigma(x)$ *and* $e_\lambda \in \mathcal{K}$.

Theorem 1.3.2. *([BMSW82], Theorem R.2.4) Let* $\mathcal{A}$ *be a unital Banach algebra and* $x \in \mathcal{A}$. *Then every Fredholm point of* x *which is a boundary point of* $\sigma(x)$ *is a Riesz point of* x.

We recall that each element of $\mathcal{F}$ has a finite spectrum. By [BMSW82], Theorem R.2.6 it follows that $\mathcal{F}$ and hence $\mathcal{K}$ are inessential ideals of $\mathcal{A}$. As an important corollary of the punctured neighbourhood theorem [BMSW82], Theorem F.3.9, Barnes, Murphy, Smyth and West obtained very useful results, [BMSW82], Theorem R.2.7 and Theorem 1.3.1 above, on the structure of the spectrum of an element in a Banach algebra. Note that Theorem R.2.7 was stated for a semisimple Banach algebra, but it holds without the hypothesis of semisimplicity. We will indicate a short proof of the following special case that will be needed:

Proposition 1.3.1. *([BS98], Proposition 2.1) Let $a \in A$ and $\Omega = \{\lambda \in \sigma(a) : |\lambda| > r_K(a)\}$. Then every point λ of Ω is an isolated point and $e_\lambda \in \mathcal{K}$.*

Proof. We first show that if $\lambda \in \Omega$, then λ is a boundary point of $\sigma(a)$. If this is not the case we can find a neighbourhood of λ which consists entirely of spectral points and does not intersect $\sigma_K(a)$. Let

$$t_0 = \sup\{t \geq 0 : (1+\epsilon)\lambda \in \sigma(a) \text{ for all } 0 \leq \epsilon \leq t\}.$$

Then $(1+t_0)\lambda$ is in the boundary of $\sigma(a)$ and it is not isolated. On the other hand, since $(1+t_0)|\lambda| > r_K(a)$, $(1+t_0)\lambda - a$ is Fredholm relative to $\mathcal{K}$. It follows directly from Theorem 1.3.2 that $(1+t_0)\lambda$ is isolated, which yields a contradiction. Hence λ is a boundary point of $\sigma(a)$. Applying Theorem 1.3.2 again, this time to λ instead of $(1+t_0)\lambda$, it follows that λ is an isolated spectral point. That $e_\lambda \in \mathcal{K}$ follows from Theorem 1.3.1. □

From this proposition we immediately obtain the well-known characterization of Riesz elements in a general Banach algebra $\mathcal{A}$ [BMSW82], Corollary R.2.5.

Corollary 1.3.1. *An element $a \in \mathcal{A}$ is Riesz if and only if all the nonzero spectral points are isolated and are of finite multiplicities.*

Lemma 1.3.1. *If I is any two-sided ideal in $\mathcal{A}$, then I and ideal $\overline{I}$ which is the norm closure of I have the same sets of idempotents.*

Proof. Let p be an idempotent in $\overline{I}$. Then $p\overline{I}p$ is a Banach algebra with identity p, algebraic structure inherited from $\mathcal{A}$ and norm $||.||_p$ defined by

$$||pap||_p = \sup\{||(pap)(pbp)|| : b \in \overline{I},\ ||pbp|| = 1\}.$$

Moreover, since I is norm dense in $\overline{I}$, it follows that $\overline{pIp} = p\overline{I}p$. Hence pIp is a dense two-sided ideal in the unital Banach algebra $p\overline{I}p$, which only holds if $pIp = p\overline{I}p$. Hence $p = ppp \in p\overline{I}p = pIp \subset I$. □

Note that if $a \in \mathcal{A}$ for which the **spectral point sequence** $\{\mu_n(a)\}_{n=1}^{\infty}$ is infinite and its limit $\mu(a)$ is attained by μ_m for some m, then $\sigma(a)$ must contain a point which is not of f.m.

Theorem 1.3.3. *([BS98], Theorem 2.3) Let $a \in \mathcal{A}$ be such that $\sigma(a)$ contains a point that is not of finite multiplicity. Then $r_K(a) = \sup\{|\lambda| : \lambda \in \sigma(a)$ is not a spectral point of finite multiplicity$\}$.*

Proof. Choose $\lambda \in \sigma(a)$ such that $r_K(a) < |\lambda|$. It follows from Proposition 1.3.1 that $e_\lambda \in \dot{\mathcal{K}}$. From Lemma 1.3.1 we have that $e_\lambda \in \mathcal{F}$, hence λ is a spectral point of finite multiplicity. Thus,

$$\sup\{|\lambda| : \lambda \in \sigma(a) \text{ is not a spectral point of finite multiplicity}\} \leq r_K(a).$$

Now suppose that

$$\sup\{|\lambda| : \lambda \in \sigma(a) \text{ is not a spectral point of finite multiplicity}\} < r_K(a).$$

Then $\{\lambda \in \sigma(a) : |\lambda| = r_K(a)\}$ consists of a finite number of f.m. spectral points. We can find a $\lambda_0 \in \sigma(a)$ such that $|\lambda_0| < r_K(a)$ and $\Lambda = \{\lambda : |\lambda| > |\lambda_0|\}$ consists of a finite number of f.m. spectral points. Let e_Λ denote the Riesz idempotent corresponding to the set Λ, then the rank of e_Λ is finite and e_Λ commutes with a. Then,

$$\begin{aligned}
|\lambda_0| &= \lim_{n\to\infty} \|[(1-e_\Lambda)a(1-e_\Lambda)]^n\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \|a^n - a^n e_\Lambda\|^{\frac{1}{n}} \\
&\geq \lim_{n\to\infty} (\inf_{k\in K} \|a^n - k\|)^{\frac{1}{n}} \\
&= r_K(a),
\end{aligned}$$

which yields a contradiction. □

As a direct corollary of Theorem 1.3.2 we affirmatively answer [NR90] Conjecture 5.11.

Corollary 1.3.2. *([BS98], Corollary 2.4) Every unital Banach algebra has the spectral radius property.*

The goal in [NR90] was to obtain a Yamamoto type theorem in the general framework of Banach algebras. The situation is the following: Let $a \in \mathcal{A}$ and

$$\delta_j(a) = \inf\{\|a - b\| : b \in A, rank(b) < j\}.$$

For each j, $\delta_j(a)$ is called the j^{th} approximation number of a. Clearly the sequence $\{\delta_j(a)\}_j$ is non-increasing and satisfies some properties of the classical approximation numbers of operators on a Banach space [Pie87]. In the case where $\mathcal{A}$ is the algebra of $n \times n$ matrices, $\delta_j(a)$ is exactly the j^{th} singular value for the matrix a.

Let $a \in \mathcal{A}$ and $\{\mu_n(a)\}_n$ be the corresponding spectral point sequence. The generalized Yamamoto's Theorem in Banach algebras states that for every $n \leq n(a)$ one has

$$\lim_{m\to\infty} (\delta_n(a^m))^{\frac{1}{m}} = \mu_n(a). \tag{1.3}$$

From [NR90] Theorem 4.2 this asymptotic relation holds for all $a \in \mathcal{A}$ for which either $n(a) < \infty$ or $n(a) = \infty$ and $\mu_n(a) > \mu(a)$. In the case where there exists an n such that $\mu_n(a) = \mu(a)$, the spectral radius property was needed to obtain (1.3) [NR90], Theorem 5.1. As a consequence of Corollary 1.3.2, (1.3) holds in general.

Chapter 2

Riesz decomposition results

In this chapter we will assume that $\mathcal{A}$ is a semiprime Banach algebra unless otherwise stated. We extend decomposition results of Barnes [Bar68] for elements of the closure of the socle of a semi-simple Banach algebra to Riesz elements in a semiprime Banach algebra $\mathcal{A}$ and provide a complete Riesz decomposition theorem for Riesz elements. We refer the reader to [Pie87], Theorem 3.2.14 for the well-known Riesz decomposition theorem in $B(X)$. In Section 2.3 we show that these results can be obtained by an extensive use of spectral analysis and C*-algebra techniques if $\mathcal{A}$ is a C*-algebra.

Call an element h in $\mathcal{A}$ **hermitian** if its numerical range is contained in $\mathbb{R}$ (i.e for any norm one linear functional ϕ on $\mathcal{A}$ satisfying $\phi(1) = 1$ we have

$\phi(h) \in \mathbb{R}$). An element x of $\mathcal{A}$ is called **regular** if there exists a $b \in \mathcal{A}$ such that $xbx = x$. It is clear from this definition that xb and bx are idempotents. The element b is called a **generalized inverse** of x. When such a b exists, it is not necessarily unique, however it is shown in [Rak88] that there is only one such a b such that bx and xb are hermitian idempotents. Following the classical notation we call the unique b the **Moore-Penrose inverse** of x.

Remark 2.0.1. If $\mathcal{A}$ is a C^*-algebra then an element $x \in \mathcal{A}$ is hermitian if and only if x is self-adjoint [Zhu93], Theorem 13.9. In this case the existence of a Moore-Penrose inverse b implies that both xb and bx are self-adjoint idempotents. We refer the reader to [HM93] for a comprehensive study of Moore-Penrose inverses in C^*-algebras.

For a subset $\mathcal{B}$ of $\mathcal{A}$ we let $\mathrm{L}[\mathcal{B}] = \{a \in \mathcal{A} : ab = 0 \text{ for } b \in \mathcal{B}\}$ and call it the left annihilator of $\mathcal{B}$ in $\mathcal{A}$. The right annihilator of $\mathcal{B}$ in $\mathcal{A}$ denoted by $\mathrm{R}[\mathcal{B}]$ is defined analogously.

For $x \in \mathcal{A}$, L_x and R_x will denote the **left and the right** multiplication operator respectively. Clearly $Ker(L_x) = \mathrm{R}[\{x\}]$ and $Ker(R_x) = \mathrm{L}[\{x\}]$ where for a linear operator T we use $Ker(T)$ to denote the kernel of T.

If $x \in \mathcal{A}$ we say an idempotent p in $\mathcal{A}$ is a left **Barnes idempotent** for x in $\mathcal{A}$ if $x\mathcal{A} = (1-p)\mathcal{A}$ and we say an idempotent q in $\mathcal{A}$ is a **right**

Barnes idempotent for x if $\mathcal{A}x = \mathcal{A}(1-q)$. As noted in [BMSW82], p. 26

- the Barnes idempotents, if they exist, are not necessarily unique;
- if p is a left Barnes idempotent for x in $\mathcal{A}$, then $Ker(R_x) = \mathcal{A}p$;
- if q is a right Barnes idempotent for x in $\mathcal{A}$, then $Ker(L_x) = q\mathcal{A}$.

It then follows from this remark that R$[\mathcal{A}x] = q\mathcal{A}$ and L$[x\mathcal{A}] = \mathcal{A}p$.

2.1 A Riesz decomposition theorem

Recall that for a linear operator T on a Banach space X and for any nonnegative integer n we clearly have the following inclusions: $Ker(T^n) \subseteq Ker(T^{n+1})$ and $T^{n+1}(X) \subseteq T^n(X)$. Moreover if there is an m such that $Ker(T^m) = Ker(T^{m+1})$ and $T^m(X) = T^{m+1}(X)$, then for any nonnegative integer n we will have $Ker(T^m) = Ker(T^{m+n})$ and $T^m(X) = T^{m+n}(X)$. The smallest nonnegative integer m satisfying $Ker(T^m) = Ker(T^{m+1})$ is called the ascent of T and the smallest nonnegative integer n satisfying $T^n(X) = T^{n+1}(X)$ is called the descent of T. If there is no such an integer we define the ascent or descent to be infinite.

Remark 2.1.1. It is a well-known fact from operator theory that if a linear operator T on a Banach space X has finite ascent and descent, then they are equal and $X = T^m(X) \oplus Ker(T^m)$ where m is their common value. See for instance [Pal94], Lemma 8.5.2.

For $x \in \mathcal{A}$ the quantities $\alpha_l(x)$ and $\delta_l(x)$ will denote the **ascent** and **descent** of L_x respectively whereas $\alpha_r(x)$ and $\delta_r(x)$ will denote the ascent and descent of R_x respectively.

In this section we show that for any Riesz element a of a semiprime Banach algebra $\mathcal{A}$ and any nonzero complex number λ it follows that $k := \alpha_l(\lambda - a) = \delta_l(\lambda - a) = \alpha_r(\lambda - a) = \delta_r(\lambda - a) < \infty$. We use this to deduce the following Riesz decomposition result: $\mathrm{R}[\mathcal{A}(\lambda - a)^k] \oplus \mathcal{A}(\lambda - a)^k = \mathcal{A}$ and analogously $\mathrm{L}[(\lambda - a)^k\mathcal{A}] \oplus (\lambda - a)^k\mathcal{A} = \mathcal{A}$. These results were proved by Barnes [Bar68] for elements of the closure of the socle of a semisimple algebra.

Theorem 2.1.1. *([BMSW82], Theorem F.1.10)Let $\mathcal{A}$ be a unital semiprime algebra and $x \in \mathcal{A}$. Then x is left (right) invertible modulo $\mathcal{K}$ if and only if x has a right (left) Barnes idempotent in $\mathcal{K}$.*

Lemma 2.1.1. *Let x be a regular element of $\mathcal{A}$ and $c \in \mathcal{A}$ such that $xcx = x$, then $p = 1 - xc$ and $q = 1 - cx$ are left and right Barnes idempotent for x*

respectively.

Proof. $\mathcal{A}(1-q) = \mathcal{A}cx \subseteq \mathcal{A}x$ and $\mathcal{A}x = \mathcal{A}xcx = \mathcal{A}x(1-q) \subseteq \mathcal{A}(1-q)$. Therefore $\mathcal{A}x = \mathcal{A}(1-q)$. By a similar argument $x\mathcal{A} = (1-p)\mathcal{A}$. □

A powerful result of Atkinson characterizes Fredholm operators (those operators which are invertible modulo the compact operators) as those which have closed ranges and finite dimensional kernels and cokernels. The following result provides a similar characterization in a Banach algebra context.

Proposition 2.1.1. *Let $x \in \mathcal{A}$. Then x is left (right) invertible modulo $\mathcal{K}$ if and only if 1) x is regular and 2) the left and right hermitian Barnes idempotents $1-xc$ and $1-cx$ respectively are in $\mathcal{F}$ where c is the Moore-Penrose inverse of x.*

Proof. Suppose x is regular with c its Moore-Penrose inverse such that $1-xc$ and $1-cx$ are in $\mathcal{F}$. Since by Lemma 2.1.1 the elements $1-xc$ and $1-cx$ are Barnes idempotents for x, it follows from Theorem 2.1.1 that x is both right and left invertible modulo $\mathcal{K}$.

Conversely, suppose x is left (right) invertible modulo $\mathcal{K}$. By Theorem 2.1.1 x has a left (right) Barnes idempotents in $\mathcal{K}$, say p and q respectively. That is $x\mathcal{A} = (1-p)\mathcal{A}$ and $\mathcal{A}x = \mathcal{A}(1-q)$. So $xy = 1-p$ and $zx = 1-q$

for some y and z in $\mathcal{A}$. That is, $xyx = (1-p)x$ and $xzx = x(1-q)$. Since $px = 0$, it follows that $xyx = x$, whence, x is regular. □

As a corollary of Proposition 2.1.1 we generalize (1.1) of [Bar68] p. 496 on elements of the closure of the socle of $\mathcal{A}$ to Riesz elements of $\mathcal{A}$.

Corollary 2.1.1. *Let $a \in \mathcal{R}(\mathcal{A})$ and λ a nonzero scalar. Then $L_{(\lambda-a)^m}$ is injective if and only if $R_{(\lambda-a)^m}$ is surjective for any nonnegative integer m.*

Proof. Suppose $L_{(\lambda-a)^m}$ is injective. That is $ker(L_{(\lambda-a)^m}) = \{0\}$. Let n_m is the right Barnes idempotent for $(\lambda - a)^m$. It follows that $\mathcal{A}(\lambda - a)^m = \mathcal{A}(1 - n_m)$ and $Ker(L_{(\lambda-a)^m}) = n_m\mathcal{A}$. Hence $n_m \in Ker(L_{(\lambda-a)^m})$ and so $n_m = 0$. Therefore $\mathcal{A}(\lambda - a)^m = \mathcal{A}$. That is $R_{(\lambda-a)^m}$ is surjective.

Conversely, suppose $R_{(\lambda-a)^m}$ is surjective. That is $\mathcal{A}(\lambda - a)^m = \mathcal{A}$. Since $\mathcal{A}(\lambda - a)^m = \mathcal{A}(1 - n_m)$, it follows that $\mathcal{A} = \mathcal{A}(1 - n_m)$, that is $\mathcal{A}n_m = \{0\}$. That is $Ker(L_{(\lambda-a)^m}) = \{0\}$. □

Recall that a right(left) ideal $\mathcal{I}$ of $\mathcal{A}$ is said to be of finite order if and only if $\mathcal{I}$ can be expressed as a sum of a finite number of minimal right(left) ideals of $\mathcal{A}$. We define the order of $\mathcal{I}$ to be the smallest number of minimal right(left) ideals whose sum is $\mathcal{I}$. Conventionally we consider the zero ideal to be of finite order.

We will need the following well-known result of Barnes [Bar68]:

Theorem 2.1.2. *([Bar68], Theorem 2.2) Let $\mathcal{I}$ be a nonzero right ideal of finite order m. Then $\mathcal{I} = p\mathcal{A}$ where $p \in \mathcal{A}$ is a rank m idempotent.*

Lemma 2.1.2. *([Tay66], Lemma 3.4) Let $T \in B(X)$ for some Banach space X. 1) If $Ker(T) \cap T^n(X) = \{0\}$ for some nonnegative integer n then $\alpha(T) \leq n$ and 2) If $\alpha(T) \leq n$ then $Ker(T^k) \cap T^n(X) = \{0\}$ for $k = 1, 2, \ldots$.*

Proof. Suppose $Ker(T) \cap T^n(X) = \{0\}$. Let $x \in Ker(T^{n+1})$. Then $T^{n+1}x = T(T^n x) = 0$, so that $T^n x \in Ker(T) \cap T^n(X) = \{0\}$, and so $T^n x = 0$, that is $x \in Ker(T^n)$.

Conversely, suppose $\alpha(T) \leq n$ and let $x \in Ker(T^k) \cap T^n(X)$, where k is a positive integer. Then $x = T^n y$ for some y and $T^k x = 0$, so that $y \in Ker(T^{n+k})$. However $Ker(T^{n+k}) = Ker(T^n)$ and thus $T^n y = 0$. Therefore $Ker(T^k) \cap T^n(X) = \{0\}$. □

Proposition 2.1.2. *Let $\mathcal{A}$ be semiprime Banach algebra and let $a \in \mathcal{R}(\mathcal{A})$. Then, for any non-zero scalar λ and a nonnegative integer k, $\mathrm{R}[\mathcal{A}(\lambda - a)^k]$ is of finite order and $\alpha_l(\lambda - a) < \infty$. Similarly $\mathrm{L}[(\lambda - a)^k \mathcal{A}]$ is of finite order and $\alpha_r(\lambda - a) < \infty$.*

Proof. Let c_k be the Moore-Penrose inverse of $(\lambda - a)^k$. Since $(\lambda - a)^k$ is invertible modulo $\mathcal{F}$, it follows from Proposition 2.1.1 that the right Barnes idempotent n_k for $(\lambda - a)^k$ corresponding to c_k is in $\mathcal{F}$, that is it is of finite rank. Since R$[\mathcal{A}(\lambda - a)^k]= n_k\mathcal{A}$, it follows from Theorem 2.1.2 that R$[\mathcal{A}(\lambda - a)^k]$ is of finite order. We next show that $\alpha_l(\lambda - a)< \infty$. For each $k \in \mathbb{N}$ we let $R_k = n_1\mathcal{A}\cap(\lambda - a)^k\mathcal{A}$. Suppose $\alpha_l(\lambda - a)= \infty$ from which it follows from Lemma 2.1.2 that $R_k \neq \{0\}$. Since the sequence $\{R_k\}$ is decreasing and R$[\mathcal{A}(\lambda - a)]$ is of finite order, $\{R_k\}$ should become constant. Recall that R_k's are of finite order, hence there is a minimal idempotent $e \in R_k$ for all k. That is

$$e \in \mathrm{R}[\mathcal{A}(\lambda - a)] \cap (\lambda - a)^k\mathcal{A} \text{ for all } k. \tag{2.1}$$

Since $a \in \mathcal{R}(\mathcal{A})$, $\lim_{n\to\infty} |||a^n + \mathcal{K}|||^{\frac{1}{n}} = 0$ where $|||.|||$ denotes a norm in $\mathcal{A}/\mathcal{K}$. We thus can find a $q \in \mathbb{N}$ and an $f \in \mathcal{F}$ such that $||a^q - f|| < \frac{|\lambda|^q}{4}$. Consider $T = L^e_a$, $R = L^e_f$ and $S = T^q - R = L^e_{a^q-f}$ where for $x \in \mathcal{A}$ L^e_x denotes the restriction of L_x to $\mathcal{A}e$ (cf. [Pal94], p.669). Since L^e_d is contractive for all $d \in \mathcal{A}$, we have that $||S|| < \frac{|\lambda|^q}{4}$. We will show that the ascent of $\lambda - L^e_a$ is finite which due to Lemma 2.1.2 will contradict (2.1). Suppose the ascent of $\lambda - L^e_a$ is infinite. By Riesz lemma we can find $x_k \in ker((\lambda - T)^{k+1})$ with $||x_k|| = 1$

and $\|x_k - x\| \geq \frac{3}{4}$ for all $x \in ker((\lambda - T)^k)$. Let $y_k = \lambda^{-q}Rx_k$. That is,

$$\begin{aligned} y_k &= \lambda^{-q}(T^q x_k - Sx_k) \\ &= x_k - \lambda^{-1}(\lambda - T)\sum_{j=0}^{q-1}(\lambda^{-1}T)^j x_k - \lambda^{-q}Sx_k. \end{aligned}$$

Since $\lambda^{-1}(\lambda - T)\sum_{j=0}^{q-1}(\lambda^{-1}T)^j x_k \in ker((\lambda - T)^k)$, it follows that for any $k > l$,

$$y_k - y_l = x_k - x - \lambda^{-q}Sx_k + \lambda^{-q}Sx_l, \quad \text{for some } x \in ker((\lambda - T)^k).$$

Thus,

$$\begin{aligned} \|y_k - y_l\| &\geq \|x_k - x\| - \|\lambda^{-q}Sx_k\| - \|\lambda^{-q}Sx_l\| \\ &\geq \frac{3}{4} - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{4}. \end{aligned}$$

But this cannot be because R is a finite rank operator, hence compact. Therefore the ascent of $\lambda - L_a^e$ has to be finite which contradicts (2.1). Thus the assumption that $\alpha_r(\lambda - a) = \infty$ is violated. The other part of the proposition follows analogously. □

Remark 2.1.2. 1) For $a \in \mathcal{R}(\mathcal{A})$ we adopt the notation in the proof of Proposition 2.1.2 by letting n_k be the hermitian right Barnes idempotent for $(\lambda - a)^k$ corresponding to the Moore-Penrose inverse c_k of $(\lambda - a)^k$. Similarly $1 - r_k$

will denote the hermitian left Barnes idempotent of $(\lambda - a)^k$. See Proposition 2.1.1 for details.

2) For $a \in \mathcal{F}(\mathcal{A})$ we will denote by n_a the hermitian right Barnes idempotent for a corresponding to the Moore-Penrose inverse of a and by $1 - r_a$ we denote the hermitian left Barnes idempotent corresponding to the Moore-Penrose inverse of a.

From these two remarks it is clear that if $a \in \mathcal{R}(\mathcal{A})$ then $n_k = n_{(\lambda-a)^k}$ and $r_k = r_{(\lambda-a)^k}$, $\lambda \neq 0$.

Lemma 2.1.3. *Let $a \in \mathcal{R}(\mathcal{A})$ and λ a nonzero complex number. If $R_{(\lambda-a)}$ has a finite ascent, say $\alpha_r(\lambda - a) = m$, then $L[((\lambda - a)^m - n_m)\mathcal{A}] = \{0\}$. Moreover $\mathcal{A} = \mathcal{A}((\lambda - a)^m - n_m)$.*

Proof. Let $b \in \mathrm{L}[((\lambda - a)^m - n_m)\mathcal{A}]$. That is $b(\lambda - a)^m - bn_m = 0$, which implies that $b(\lambda - a)^{2m} = bn_m(\lambda - a)^m = 0$. Since $\alpha_r(\lambda - a) = m$, it follows that $b(\lambda - a)^m = 0$. This implies that $b \in \mathrm{L}[(\lambda - a)^m\mathcal{A}] = An_m$. So $b = bn_m = b(\lambda - a)^m = 0$. Therefore $\mathrm{L}[((\lambda - a)^m - n_m)\mathcal{A}] = \{0\}$. For the second part of the lemma, since $\{0\} = \mathrm{L}[((\lambda - a)^m - n_m)\mathcal{A}] = Ker(R_{((\lambda-a)^m-n_m)})$, it follows that $R_{((\lambda-a)^m-n_m)}$ is injective. Since $R_{((\lambda-a)^m-n_m)} = R_{(\lambda-u)}$ for some $u \in \mathcal{R}(\mathcal{A})$, it follows by Corollary 2.1.1 that $\mathcal{A} = \mathcal{A}((\lambda - a)^m - n_m)$. □

Proposition 2.1.3. *Let $a \in \mathcal{R}(\mathcal{A})$ and λ a nonzero complex number. Then $\alpha_l(\lambda - a) = \delta_l(\lambda - a) = \alpha_r(\lambda - a) = \delta_r(\lambda - a) < \infty$.*

Proof. Let $\alpha_l(\lambda - a) = m$ and $b \in \mathcal{A}(\lambda - a)^m$, that is $b = c(\lambda - a)^m$ for some $c \in \mathcal{A}$. Since by Lemma 2.1.3 we have $\mathcal{A} = \mathcal{A}((\lambda - a)^m - n_m)$, it follows that $c = d((\lambda - a)^m - n_m)$ for some $d \in \mathcal{A}$. So

$$\begin{aligned} b &= d((\lambda - a)^m - n_m)(\lambda - a)^m \text{ for some } d \in \mathcal{A} \\ &= d(\lambda - a)^{2m} - dn_m(\lambda - a)^m \\ &= d(\lambda - a)^{2m} \text{ because } dn_m(\lambda - a)^m = 0. \end{aligned}$$

So $b \in \mathcal{A}(\lambda - a)^{2m}$. Therefore $\delta_r(\lambda - a) \leq \alpha_l(\lambda - a)$. Analogously $\delta_l(\lambda - a) \leq \alpha_r(\lambda - a)$. Since $\alpha_l(\lambda - a)$, $\delta_l(\lambda - a)$, $\alpha_r(\lambda - a)$ and $\delta_r(\lambda - a)$ are all finite, we then have $\alpha_l(\lambda - a) = \delta_l(\lambda - a)$ and $\alpha_r(\lambda - a) = \delta_r(\lambda - a)$. The result follows from these equations and the above two inequalities. □

Lemma 2.1.4. *([Ric60], p. 32) Let $a \in \mathcal{A}$. Then L_a and R_a are invertible in $B(\mathcal{A})$ if and only if a is invertible in $\mathcal{A}$.*

Proof. We only give a proof for L_a as the other case for R_a follows analogously. If a is invertible in $\mathcal{A}$,then

$$L_{a^{-1}} L_a x = L_a L_{a^{-1}} x = x$$

for all x in $\mathcal{A}$ which implies that L_a is invertible in $B(\mathcal{A})$ with the inverse given by $L_{a^{-1}}$

Conversely, if L_a is invertible in $B(\mathcal{A})$, then

$$TL_a = L_aT = I$$

for some T in $B(\mathcal{A})$ where I is the identity in $B(\mathcal{A})$. Let $x = T1$. Then $ax = L_aT1 = 1$ which implies that $L_aL_x = L_{ax} = L_1 = I$. Therefore $xa = L_xL_a = 1$ and thus a is invertible in $\mathcal{A}$ and its inverse is x. □

Lemma 2.1.5. *Let $a \in \mathcal{R}(\mathcal{A})$ and $k \in \mathbb{N}$. Then for any nonzero complex number λ, $(\lambda - a)^k\mathcal{A} = r_k\mathcal{A}$.*

Proof. Let c_k be the Moore-Penrose inverse of $(\lambda - a)^k$. Then

$$r_k(\lambda - a)^k = (\lambda - a)^k \tag{2.2}$$

and

$$r_k = (\lambda - a)^k c_k. \tag{2.3}$$

Let $y \in (\lambda - a)^k\mathcal{A}$, that is

$$\begin{aligned} y &= (\lambda - a)^k d \text{ for some } d \in \mathcal{A} \\ &= r_k(\lambda - a)^k d \text{ by equation 2.2} \\ &\in r_k\mathcal{A}. \end{aligned}$$

That is $(\lambda - a)^k\mathcal{A} \subseteq r_k\mathcal{A}$.

For the reverse inclusion, let $z \in r_k\mathcal{A}$, that is

$$\begin{aligned} z &= r_k b \text{ for some } b \in \mathcal{A} \\ &= (\lambda - a)^k c_k b \text{ by equation 2.3} \\ &\in (\lambda - a)^k\mathcal{A}. \end{aligned}$$

□

We can now state and prove the main result of this section.

Theorem 2.1.3. *Let $a \in \mathcal{R}(\mathcal{A})$, $\lambda \neq 0$ a complex number and $k = \alpha_l(\lambda - a) = \alpha_r(\lambda - a) = \delta_l(\lambda - a) = \delta_r(\lambda - a)$. Then n_k and $1 - r_k$ are of finite rank and moreover $r_k a r_k = a r_k$, $n_k a n_k = a n_k$ and we can decompose the algebra $\mathcal{A} = n_k\mathcal{A} \oplus r_k\mathcal{A}$ such that $(\lambda - a)n_k$ is nilpotent and $(\lambda - a)r_k$ is invertible in $r_k\mathcal{A}$.*

Proof. The direct sum decomposition follows from Proposition 2.1.3 and Remark 2.1.1.

For any $k \in \mathbb{N}$,

$$\begin{aligned} r_k a r_k - a r_k &= (r_k - 1) a r_k \\ &= (r_k - 1) a (\lambda - a)^k c_k \\ &= (r_k - 1)(\lambda - a)^k a c_k \\ &= r_k (\lambda - a)^k a c_k - (\lambda - a)^k a c_k \\ &= (\lambda - a)^k a c_k - (\lambda - a)^k a c_k \\ &= 0 \end{aligned}$$

Therefore $r_k a r_k = a r_k$. Similarly $n_k a n_k = a n_k$.

To see that $(\lambda - a) n_k$ is nilpotent, let c_k denote the Moore-Penrose inverse of $(\lambda - a)^k$. Then

$$\begin{aligned} (\lambda - a) n_k &= (\lambda - a)[1 - c_k (\lambda - a)^k] \\ &= (\lambda - a) - (\lambda - a) c_k (\lambda - a)^k \end{aligned}$$

and

$$[(\lambda - a) n_k]^2 = (\lambda - a)^2 - (\lambda - a)^2 c_k (\lambda - a)^k.$$

Inductively, we obtain

$$\begin{aligned} [(\lambda - a) n_k]^k &= (\lambda - a)^k - (\lambda - a)^k c_k (\lambda - a)^k \\ &= 0. \end{aligned}$$

Since the ascent and descent of $L_{(\lambda-a)}$ are finite and equal to k, by [CPY74], Lemma 3.4.2 $L_{(\lambda-a)}$ restricted to $r_k\mathcal{A}$ is invertible. That is $L_{(\lambda-a)r_k}$ is invertible in $B(r_k\mathcal{A})$. Since by Lemma 2.1.4 the left regular representations preserves the spectrum, it follows that $(\lambda - a)r_k$ is invertible in $r_k\mathcal{A}$. □

Remark 2.1.3. The containment of the ideals $n_k\mathcal{A}$ for all k and $r_k\mathcal{A}$ for all k can be expressed in terms of the idempotents n_k and r_k. For idempotents e and f in $\mathcal{A}$ we say $f \leq e$ if $ef = f$ which is equivalent to $f\mathcal{A} \subseteq e\mathcal{A}$. In fact $ef = f$ and $fe = e$ if and only if $e\mathcal{A} = f\mathcal{A}$, see [Pal94], Proposition 8.2.4. Moreover if e and f are hermitian idempotents such that $e\mathcal{A} = f\mathcal{A}$ it follows that $e = f$. To see this we observe that from $e = fe$ and $f = ef$ we have $(e-f)^2 = 0$. That is $e - f$ is nilpotent, hence $r(e-f) = 0$. Since $e - f$ is hermitian, we have

$$\begin{aligned} \|e-f\| &= r(e-f) \\ &= 0. \end{aligned}$$

Therefore $e = f$.

2.2 On the multiplicity of isolated spectral points

In the first chapter we defined the multiplicity m_λ of an isolated spectral point of an element u of $\mathcal{A}$ to be the rank of the corresponding Riesz idempotent e_λ corresponding to λ. If $\mathcal{A} = B(X)$ for some Banach space X and $u \in \mathcal{A}$, m_λ is just the algebraic multiplicity of λ that is m_λ is the dimension of $Ker((\lambda - u)^\alpha)$ where α is the ascent of $(\lambda - u)$.

In Propostion 2.1.3 we showed that for a Riesz element u of a semiprime Banach algebra $\mathcal{A}$ the ascent and descent of $L_{(\lambda-u)}$, with $\lambda \neq 0$ coincide and they are finite. We also know from Lemma 2.1.4 that $\sigma_{\mathcal{A}}(u) = \sigma_{B(\mathcal{A})}(L_u) = \sigma_{B(\mathcal{A})}(R_u)$, hence if λ is an isolated spectral point of u then it is also an isolated spectral point of L_u. In [Puh78] Puhl defined the multiplicity of an isolated spectral point λ of a nuclear element u to be the order of the ideal $Ker(L_{(\lambda-u)^\alpha})$ which he denotes by $n(\lambda, u)$ where α is the ascent of $L_{(\lambda-u)}$. We will show that, in fact, $m_\lambda = n(\lambda, u)$.

Proposition 2.2.1. *Let λ be an isolated spectral point of u. Then $L_{e_{\lambda(u)}} = e_{\lambda(L_u)}$ and $R_{e_{\lambda(u)}} = e_{\lambda(R_u)}$.*

Proof. We only give a proof for the left multiplaction operator as the other case follows analogously. For all $x \in \mathcal{A}$ we have

$$\begin{aligned} e_{\lambda(L_u)}x &= \left(\frac{1}{2\pi i}\int_{|\mu-\lambda|=\epsilon}(\mu - L_u)^{-1}d\mu\right)x \\ &= \frac{1}{2\pi i}\int_{|\mu-\lambda|=\epsilon} L^{-1}_{(\mu-u)}xd\mu \\ &= \frac{1}{2\pi i}\int_{|\mu-\lambda|=\epsilon} L_{(\mu-u)^{-1}}xd\mu \\ &= \frac{1}{2\pi i}\int_{|\mu-\lambda|=\epsilon} (\mu-u)^{-1}xd\mu \\ &= L_{e_{\lambda(u)}}x. \end{aligned}$$

□

Remark 2.2.1. It is a well-known fact from spectral theory of Riesz operators on a Banach space X that if T is a Riesz operator and λ is a nonzero eigenvalue of T then $Ker((\lambda - T)^{\alpha}) = P_{\lambda}X$ where P_{λ} is the Riesz projection corresponding to λ and α is the ascent of $\lambda - T$ (See [CPY74], p. 49). From Proposition 2.2.1 $Ker(L^{\alpha}_{(\lambda-u)}) = L_{e_{\lambda}}\mathcal{A} = e_{\lambda}\mathcal{A}$. So the order of $Ker(L^{\alpha}_{(\lambda-u)})$ is the order of $e_{\lambda}\mathcal{A}$ which equals the rank of e_{λ}. Therefore $m_{\lambda} = n(\lambda, u)$.

In fact from Proposition 2.2.1 again we have $Ker(R^{\alpha}_{(\lambda-u)}) = R_{e_{\lambda}}\mathcal{A} = \mathcal{A}e_{\lambda}$. From this it follows that if $u \in \mathcal{R}(\mathcal{A})$ then the algebraic multiplicity of $\lambda \in \sigma_{\mathcal{A}}(u)$ is $rank(n_k) = rank(1 - r_k) = n(\lambda, u) = m_{\lambda}$.

2.3 Barnes idempotents in C^*-algebras

In this section $\mathcal{A}$ will denote a C^*-algebra, a^* the adjoint of a and $|a| := (a^*a)^{\frac{1}{2}}$. We will establish a spectral characterization of the hermitian Barnes idempotents for regular elements of the algebra corresponding to their respective Moore-Penrose inverses.

Lemma 2.3.1. *(cf. [HM93], Theorem 3 and [Str94], Corollary 4) An element $a \in \mathcal{A}$ is regular if and only if* $\inf(\sigma(|a|)\backslash\{0\}) > 0$.

Clearly for a regular element a, if $0 \in \sigma(|a|)$ then 0 is an isolated spectral point of $|a|$.

For a regular element a of $\mathcal{A}$ we let n_a be the Riesz idempotent corresponding to $\{0\}$ and we define n_{a^*} analogously. If $0 \notin \sigma(|a|)$ then $n_a = 0$. We let $r_a = 1-n_{a^*}$ and $r_{a^*} = 1-n_a$. We will later see that n_a and $1-r_a$ are exactly the hermitian Barnes idempotents for a corresponding to the Moore-Penrose inverse of a, hence the notation does not violate that of Remark 2.1.2.

Lemma 2.3.2. *Let a, $x \in \mathcal{A}$. Then* $\||a|x\|^2 = \|ax\|^2$.

Proof.

$$\begin{aligned} \||a|x\|^2 &= \|(|a|x)^*(|a|x)\| \\ &= \|x^*|a||a|x\| \\ &= \|x^*a^*ax\| \\ &= \|(ax)^*(ax)\| \\ &= \|ax\|^2. \end{aligned}$$

□

Remark 2.3.1. Let a be a regular element of $\mathcal{A}$. Since by holomorphic functional calculus $|a|n_a = 0$ it follows from Lemma 2.3.2 that $an_a = 0$.

Proposition 2.3.1. *Let a be a regular element of $\mathcal{A}$. If $ax = 0$ for some $x \in \mathcal{A}$ then $(1 - n_a)x = 0$.*

Proof. If $ax = 0$ then by Lemma 2.3.2 $|a|x = 0$. So for all $\lambda \in \rho(|a|)$

$$\begin{aligned} x &= (\lambda - |a|)^{-1}(\lambda - |a|)x \\ &= \lambda(\lambda - |a|)^{-1}x. \end{aligned}$$

That is

$$\frac{x}{\lambda} = (\lambda - |a|)^{-1}x.$$

Therefore,

$$\begin{aligned} n_a x &= \frac{1}{2\pi i}\int_\Gamma (\lambda - |a|)^{-1} d\lambda x \\ &= \frac{1}{2\pi i}\int_\Gamma (\lambda - |a|)^{-1} x d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{x}{\lambda} d\lambda \\ &= \frac{1}{2\pi i} x \int_\Gamma \frac{1}{\lambda} d\lambda \\ &= x. \end{aligned}$$

That is $n_a x = x$ which implies that $(1 - n_a)x = 0$. □

As a corollary of Proposition 2.3.1 we deduce that for a regular element a of $\mathcal{A}$ the projection n_a is the hermitian right Barnes idempotent for a corresponding to the Moore-Penrose inverse of a.

Corollary 2.3.1. *Let b be the Moore-Penrose inverse of a. Then $ba = (1-n_a)$.*

Proof. Since $aba = a$, it follows that $a(1 - ba) = 0$. Thus from Proposition 2.3.1 we have $n_a(1 - ba) = 1 - ba$. From Remark 2.3.1 $an_a = 0$, it follows that $(1 - ba)n_a = n_a$. Since $1 - ba$ and n_a are self adjoint idempotents, by Remark 2.1.3 we have $1 - ba = n_a$. □

From the above results we see that in a C*-algebra, if a is regular and b is its Moore-Penrose inverse in $\mathcal{A}$, then the hermitian projections $p_1 = ab$ and

$p_2 = ba$ are exactly the projections r_a and r_{a^*}. We call n_a the null projection of a and r_a the range projection of a. See, for instance, [MS97], Section 1 for a comprehensive study of kernel and range projections in C^*-algebras.

An analogue of the following result was proven in Section 2.1 using different techniques, see Proposition 2.1.1. We use the techniques of this section to give a different reformulation of the result using the spectral characterization of Barnes idempotents.

Theorem 2.3.1. *([Str94], Theorem 10) An element $a \in \mathcal{A}$ is invertible modulo $\mathcal{F}$ if and only if 1) a is regular and 2) n_a, $n_{a^*} \in \mathcal{F}$.*

Proof. If a is invertible modulo $\mathcal{F}$, then clearly it is invertible modulo $\mathcal{K}$, hence $0 \notin \sigma_K(|a|)$. But since $\sigma(|a|)\setminus\sigma_K(|a|)$ consists only of isolated eigenvalues which cluster on $\sigma(|a|)$, 0 must be (if it exists as an element of $\sigma(|a|)$) an isolated point of $\sigma(|a|)$. Hence a is regular. Suppose $b \in \mathcal{A}$ such that $1 - ab$, $1 - ba \in \mathcal{F}$. Since $\mathcal{F}$ is a two-sided ideal, it follows that $n_a = (1 - ba)n_a \in \mathcal{F}$. Similarly $n_{a^*} \in \mathcal{F}$. Conversely, if 1) and 2) hold, let b be the Moore-Penrose inverse for a, then $ab = r_a = 1 - n_{a^*}$ and $ba = r_{a^*} = 1 - n_a$. Hence a is invertible modulo $\mathcal{F}$. □

We refer the reader to [BMSW82] for more details on Riesz elements of a Banach algebra. Since invertibility modulo $\mathcal{F}$ and invertibility modulo $\mathcal{K}$ are equivalent, a is Riesz if and only if $\lambda - a$ is Fredholm for every $\lambda \neq 0$. In fact $(\lambda - a)^k$ is regular for every $k \in \mathbb{N}$. Hence, if we let $n_k = n_{(\lambda-a)^k}$ and $r_k = r_{(\lambda-a)^k}$, then n_k and r_k are self-adjoint idempotents in $\mathcal{A}$. By Corollary 2.3.1 it follows that that n_k and $1 - r_k$ are exactly the hermitian right and left Barnes idempotents of $(\lambda - a)^k$ respectively corresponding to the Moore-Penrose inverse c_k of $(\lambda - a)^k$.

Chapter 3

Traces and Determinants

As mentioned in the introduction the theory of trace and determinant plays an integral role in the study of operators on a Banach space, especially with regard to its connections to Fredholm theory. In the case of Banach algebras it seems that no complete theory exists for ideals larger than the socle of a semisimple Banach algebra, see for instance [Puh78, AdTM96]. In [Puh78] Puhl did, however, extend the trace defined on the socle to the ideal $\mathcal{N}$ of nuclear elements for algebras possessing the quasi-approximation property. This property seems fairly reasonable for in the case where $\mathcal{A} = B(X)$ the property is equivalent to X or X^* having the approximation property. In this chapter we will show that the determinant on the socle of a simiprime Banach algebra

$\mathcal{A}$ can also be continuously extended to the ideal $\mathcal{N}$ if the Banach algebra $\mathcal{A}$ possesses the quasi-approximation property. It turns out that our determinant restricted to the ideal $\mathcal{F}$ of finite rank elements is spectral. In fact we show that this determinant satisfies all the desired properties which hold in the classical theory of bounded linear operators on a Banach space possessing the approximation property, see for instance [Pie87].

3.1 Traces

It seems reasonable to call a function tr defined on an ideal I of $\mathcal{A}$ which contains $\mathcal{F}$ a **trace** if it satisfies the following properties:

$\tau 1$ $\operatorname{tr}(u) = \lambda_1$ *if* u *is of rank one and* $u^2 = \lambda_1 u$

$\tau 2$ $\operatorname{tr}(uv) = \operatorname{tr}(vu)$ *for all* $u, v \in I$

$\tau 3$ $\operatorname{tr}(u+v) = \operatorname{tr}(u) + \operatorname{tr}(v)$ *for all* $u, v \in I$

$\tau 4$ $\operatorname{tr}(\alpha u) = \alpha \operatorname{tr}(u)$ *for all* $\alpha \in \mathbb{C}$ *and* $u \in I$.

The above properties correspond to the axioms suggested by Pietsch, see [Pie87] Definition 4.2.1. Our first step is to introduce the spectral trace on the ideal $\mathcal{F}$ itself.

Definition 3.1.1. *Let* $x \in \mathcal{F}$ *and* $\{\lambda_i(x)\}_{i=1}^N$ *be the spectral points of* x *repeat-*

ed according to their respective algebraic multiplicities. We let

$$\mathrm{Tr}(x) := \sum_{i=1}^{N} \lambda_i(x).$$

We note that B. Aupetit and du T. Mouton used the same definition for elements of the socle of a semisimple Banach algebra.

Theorem 3.1.1. *(cf. [AdTM96], Theorem 3.1) Let f be an analytic function from a domain D of $\mathbb{C}$ into $\mathcal{F}$. Then* $\mathrm{Tr}(f(\lambda))$ *is holomorphic on D.*

Lemma 3.1.1. *Let $x, y \in \mathcal{F}$. Then* $\mathrm{Tr}(x+y) = \mathrm{Tr}(x)+\mathrm{Tr}(y)$.

Proof. By Theorem 3.1.1, $f(\alpha) = \mathrm{Tr}(x + \alpha y)$ is entire. In fact,

$$\begin{aligned}
\lim_{|\alpha|\to\infty} \frac{f(\alpha)}{\alpha} &= \lim_{|\alpha|\to\infty} \mathrm{Tr}(\frac{x}{\alpha} + y) \\
&= \lim_{|\lambda|\to 0} \mathrm{Tr}(\lambda x + y) \\
&= \mathrm{Tr}(y).
\end{aligned}$$

From Liouville's theorem, $f(\alpha)$ has to be a polynomial of degree one. That is $f(\alpha) = \mathrm{Tr}(\alpha y + x) = \alpha\mathrm{Tr}(y) + k$ for some constant k. Letting $\alpha = 0$ we get $\mathrm{Tr}(x) = k$, hence $\mathrm{Tr}(x+y) = \mathrm{Tr}(y) + \mathrm{Tr}(x)$. □

Remark 3.1.1. It follows from the fact that $\sigma_{\mathcal{A}}(xy)\backslash\{0\} = \sigma_{\mathcal{A}}(yx)\backslash\{0\}$ that $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ for all $x, y \in \mathcal{F}$. If x is a rank one element, that is for every $y \in \mathcal{A}$ there is a scalar λ_y such that $xyx = \lambda_y x$, then $\sigma_{\mathcal{A}}(x) = \{0, \lambda_1\}$, see [Puh78], Lemma 2.7. These facts along with Lemma 3.1.1 imply that the spectral trace satisfies the properties $\tau_1, \dots, \tau_4$ of a trace.

In [Puh78] Puhl introduced a trace on $\mathcal{F}$ which is continuous in the nuclear norm and showed that it extends continuously to the ideal $\mathcal{N}$ of nuclear elements of $\mathcal{A}$ provided the semiprime Banach algebra $\mathcal{A}$ has some approximation property. That property is defined as follows:

Property 3.1.1. *A semiprime Banach algebra $\mathcal{A}$ has the quasi-approximation property (q.a.p) if for each minimal idempotent $e \in \mathcal{A}$ the Banach space $\mathcal{A}e$ has the approximation property (respectively $e\mathcal{A}$).*

Remark 3.1.2. If $\mathcal{A}$ is commutative, then it possesses Property 3.1.1. To see this, let $e \in \mathcal{A}$ be a minimal idempotent. Since $e\mathcal{A} = e\mathcal{A}e = \mathbb{C}e$, it follows that $\mathcal{A}$ is isomorphic to $\mathbb{C}$ which implies that $\mathcal{A}$ has Property 3.1.1.

We also remark that if $\mathcal{A} = B(X)$ for some Banach space X, then $\mathcal{A}$ possesses property 3.1.1 if and only if X or X^* has the approximation property. To see this we let $P \in B(X)$ be a minimal idempotent. We show that $\mathcal{A}P \cong X$ (respectively $P\mathcal{A} \cong X^*$). Recall that P is a rank one projection. Let $P(X) := X_1$. Choose $0 \neq x_0 \in X_1$ with $\|x_0\| = 1$ such that X_1 is the norm closure of the linear span of $\{x_0\}$. Let ψ be defined as follows $\psi : \mathcal{A}P \to X$ with $\psi(SP) := Sx_0, \quad S \in \mathcal{A}$. Since P is a projection, ψ is well defined and moreover it is both linear and one to one and clearly of norm less or equal to one. To show that it is surjective, fix $x \in X$ and define a map $S_x x_0 := x$. Extend S_x linearly to X_1 by $S_x(\lambda x_0) = \lambda x$. Clearly S_x is linear and it is bounded. By Hahn-Banach we extend S_x to all of X. As such $S_x \in B(X)$ and moreover $\psi(S_x P) = S_x x_0 = x$. The Closed Graph Theorem ensures that ψ is a Banach space isomorphism. □

In [Puh78] Puhl defined a trace on the socle of a semiprime Banach algebra as follows:

Definition 3.1.2. *For a rank one element u, let $\tau(u) := \lambda_1$ where $u^2 = \lambda_1 u$. If u is of rank n and $u = \sum_{i=1}^{n} u_i$ we let $\tau(u) := \sum_{i=1}^{n} \tau(u_i)$.*

Puhl proved that if $u_i \in \mathcal{F}_1$ satisfy $\sum_{i=1}^{n} u_i = 0$, then $\sum_{i=1}^{n} \tau(u_i) = 0$, see [Puh78] Lemma 4.3. Thus τ is well-defined on $\mathcal{F}$. That $\tau(u)$ is a linear functional on $\mathcal{F}$ follows directly from the above definition. We will show that

$\tau(uv) = \tau(vu)$ for all $u, v \in \mathcal{F}$.

Theorem 3.1.2. *([Puh78], Theorem 4.5.) Suppose $\mathcal{A}$ is a semiprime Banach algebra. Then $\tau(u)$ is a linear funtional on $\mathcal{F}$. Moreover $\tau(uv) = \tau(vu)$ for all $u, v \in \mathcal{F}$.*

Proof. We only prove the second statement of the theorem. Let $u \in \mathcal{F}$ and $y \in \mathcal{F}_1$. Then $\lambda_u uy = uyuy = \tau(uy)uy$ and $\lambda_u yu = yuyu = \tau(yu)yu$. So $\tau(yu) = \tau(uy)$. Therefore for $v \in \mathcal{F}$ we have $\tau(uv) = \sum_{i=1}^{n} \tau(uv_i) = \sum_{i=1}^{n} \tau(v_i u) = \tau(vu)$. □

Remark 3.1.3. In the light of Theorem 3.1.2, we deduce that τ satisfies the stated properties of a reasonable trace. In fact Puhl showed that the trace τ defined on $\mathcal{F}$ is spectral. That is if $u \in \mathcal{F}$ and $\{\lambda_i(u)\}_{i=1}^{N}$ is the sequence of the nonzero spectral points of u repeated according to their respective multiplicities, then $\tau(u) = \sum_{i=1}^{N} \lambda_i(u) = \mathrm{Tr}(u)$, see [Puh78], Theorem 6.6. In fact on $\mathcal{F}$ there is a unique trace since for any trace tr and $u \in \mathcal{F}_n$ it follows that

$$\begin{aligned} \mathrm{tr}(u) &= \mathrm{tr}(u_1) + \ldots + \mathrm{tr}(u_n) \\ &= \tau(u_1) + \ldots + \tau(u_n) \\ &= \tau(u) \\ &= \mathrm{Tr}(u). \end{aligned}$$

We state the following result of Puhl without proof as we will need it in the sequel:

Theorem 3.1.3. *([Puh78], Theorem 5.12) Let $\mathcal{A}$ be a semiprime Banach algebra that possesses the q.a.p. Then the trace τ defined on $\mathcal{F}$ extends continuously to $\mathcal{N}$ and moreover for $u \in \mathcal{N}$, $|\tau(u)| \leq \nu(u)$.*

3.2 Determinants

In this section we establish a continuous extension of a determinant from the ideal $\mathcal{F}$ of finite rank elements to the ideal $\mathcal{N}$ of nuclear elements. We follow a similar approach as in the classical theory of determinant on operator ideals in Banach spaces [Pie87], Definition 4.3.1.

Let I be an ideal containing $\mathcal{F}$. Similarly as for traces it seems reasonable to call a function δ assigning to every element of the form $1 + x$ with $x \in I$ a complex number $\delta(1 + x)$ a **determinant** if it satisfies the following properties:

$\delta 1$ $\delta(1 + x) = 1 + \lambda_1$ *if x is of rank one and* $x^2 = \lambda_1 x$;

$\delta 2$ $\delta(1 + xy) = \delta(1 + yx)$ *whenever* $x \in \mathcal{A}$ *and* $y \in I$;

$\delta 3$ $\delta((1 + x)(1 + y)) = \delta(1 + x)\delta(1 + y)$ *for all* $x, y \in I$;

$\delta 4$ *$\delta(1+\lambda x)$ is an entire function of λ for fixed $x \in I$.*

As it is the case with traces it is natural to introduce a spectral determinant on $\mathcal{F}$. The properties of this determinant were fully studied by Aupetit and Mouton [AdTM96].

Definition 3.2.1. *Let $x \in \mathcal{F}$ and let $\{\lambda_i(x)\}_{i=1}^N$ be the spectral points of x repeated according to their respective algebraic multiplicities. We define*

$$\mathrm{Det}(1+x) := \prod_{i=1}^{N} (1+\lambda_i(x)).$$

Since $\sum_{\lambda\in\sigma(x)\setminus\{0\}} m_\lambda = rank(x)$ for all $x \in \mathcal{F}$, it follows directly from the above definition that

$$|\mathrm{Det}(1+x)| \le r(1+x)^{rank(x)} \le (1+r(x))^{rank(x)}. \tag{3.1}$$

Lemma 3.2.1. *Let $x \in \mathcal{F}$. Then $1+x$ is invertible if and only if* $\mathrm{Det}(1+x) \neq 0$.

Proof. Suppose $\mathrm{Det}(1+x) \neq 0$. So $1+\lambda \neq 0$ for all $\lambda \in \sigma(x)$. That is $-1 \notin \sigma(x)$. This says that $1+x$ is invertible. For the reverse implication, suppose $1+x$ is invertible, so $-1 \notin \sigma(x)$. That is $1+\lambda \neq 0$ for all $\lambda \in \sigma(x)$. Thus $\mathrm{Det}(1+x) \neq 0$. □

Theorem 3.2.1. *(cf. [AdTM96], Theorem 3.1) Let f be an analytic function from a domain D of $\mathbb{C}$ into $\mathcal{F}$. Then* $\operatorname{Det}(1 + f(\lambda))$ *is holomorphic on D.*

Lemma 3.2.2. *(cf. [AdTM96], Lemma 3.2) Let $f(\lambda, \mu)$ be a complex-valued function of two variables which is separately entire in λ, μ and such that $f(\lambda, \mu) \neq 0$ for all λ, μ in $\mathbb{C}$. Suppose moreover that there exist two positive constants A and B such that*

$$|f(\lambda, \mu)| \leq \exp(A|\lambda| + B|\mu|).$$

Then there are three complex constants α, β, γ such that

$$f(\lambda, \mu) = \exp(\alpha\lambda + \beta\mu + \gamma).$$

Lemma 3.2.3. *If $x \in \mathcal{F}$ and $\|x\| < 1$, then there exists $u \in \mathcal{F}$ such that* $1 - x = \exp(u)$.

Proof. Since $\|x\| < 1$, $1 - x$ is invertible in $\mathcal{A}$, hence $0 \notin \sigma(1 - x)$. Let $\sigma(x) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then $\sigma(1 - x) = \{1 - \lambda \, : \, \lambda \in \sigma(x)\} = \{1 - \lambda_1, \dots, 1 - \lambda_n\}$ and since $\|x\| < 1$, $|\lambda_i| < 1$ for $i = 1, \dots, n$. Now if λ_j is real for some j we must have $1 - \lambda_j > 0$. Hence $\sigma(1 - x)$ is entirely contained in a domain on which the logarithmic function ln is holomorphic. Let $u = \ln(1 - x) = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} x^n$. By functional calculus we have that $u \in \mathcal{F}$ and $1 - x = \exp(u)$. $\square$

Theorem 3.2.2. *(cf. [AdTM96], Theorem 3.3) Let $x, y \in \mathcal{F}$.*

(a) $\mathrm{Det}(\exp(x+y)) = \mathrm{Det}(\exp(x)\exp(y)) = \mathrm{Det}(\exp(x))\mathrm{Det}(\exp(y))$,

(b) $\mathrm{Det}((1+x)(1+y)) = \mathrm{Det}(1+x)\mathrm{Det}(1+y)$.

Proof. (a) We consider the function $f(\lambda,\mu) = \mathrm{Det}(\exp(\lambda x + \mu y))$. This function is well-defined because $a \in \mathcal{F}$ implies that $\exp(a) - 1 \in \mathcal{F}$. By Theorem 3.2.1, this function is separately holomorphic in λ, μ and it does not vanish because $\exp(\lambda x + \mu y)$ is invertible. We let $C = rank(\exp(\lambda x + \mu y) - 1)$. By subaddativity of the rank function, see Proposition 1.1.1 (a), and the inequality (3.1),

$$\begin{aligned} |f(\lambda,\mu)| &\leq r(\exp(\lambda x + \mu y))^{rank(\exp(\lambda x + \mu y) - 1)} \\ &\leq \exp(C(|\lambda|\|x\| + |\mu|\|y\|)). \end{aligned}$$

By Lemma 3.2.2, there are complex constants α, β, γ such that $f(\lambda,\mu) = \exp(\alpha\lambda + \beta\mu + \gamma)$. Since $f(0,0) = 1$, we can suppose that $\gamma = 0$. Letting $\lambda = 1, \mu = 0$ yields $\exp(\alpha) = \mathrm{Det}(\exp(x))$ and analogously we get $\exp(\beta) = \mathrm{Det}(\exp(y))$. Combining these two observations we obtain $\mathrm{Det}(\exp(x+y)) = f(1,1) = \mathrm{Det}(\exp(x))\mathrm{Det}(\exp(y))$.

(b) We choose $\lambda \in \mathbb{C}$ such that $\|\lambda x\|, \|\lambda y\| < 1$. By Lemma 3.2.3 there are $u, v \in \mathcal{F}$ such that $1 - \lambda x = \exp(u)$ and $1 - \lambda y = \exp(v)$. Thus by

(a) we have

$$\begin{aligned}\mathrm{Det}\,((1-\lambda x)(1-\lambda y)) &= \mathrm{Det}(\exp(u)\exp(v)) \\ &= \mathrm{Det}(\exp(u))\mathrm{Det}(\exp(v)) \\ &= \mathrm{Det}(1-\lambda x)\mathrm{Det}(1-\lambda y).\end{aligned}$$

However, by Theorem 3.2.1 the functions $\mathrm{Det}(1-\lambda x)(1-\lambda y)$, $\mathrm{Det}(1-\lambda x)$ and $\mathrm{Det}(1-\lambda y)$ are entire, as such the property is true for all λ and the proof is concluded. □

Hence the spectral determinant satisfies the properties $\delta_1, \ldots, \delta_n$.

Now let τ be the trace on $\mathcal{F}$ define by Puhl, see Definition 3.1.2.

Definition 3.2.2. *For a rank n element u of $\mathcal{A}$ we let*

$$\alpha_i(u) = \begin{vmatrix} \tau(u) & 1 & 0 & 0 & \ldots & 0 \\ \tau(u^2) & \tau(u) & 2 & 0 & \ldots & 0 \\ \tau(u^3) & \tau(u^2) & \tau(u) & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau(u^{i-1}) & \tau(u^{i-2}) & \tau(u^{i-3}) & \tau(u^{i-4}) & \ldots & i-1 \\ \tau(u^i) & \tau(u^{i-1}) & \tau(u^{i-2}) & \tau(u^{i-3}) & \ldots & \tau(u) \end{vmatrix}$$

and let $det(1+u) := 1 + \sum_{i=1}^{n} \frac{1}{i!}\alpha_i(u)$.

From the expansion of the i by i determinant $\alpha_i(u)$ where $u \in \mathcal{F}$, we get

$$\alpha_n(u) = \sum_{i=1}^{n} (-1)^{i+1} \frac{(n-1)!}{(n-i)!} \alpha_{n-i}(u) \tau(u^i).$$

Lemma 3.2.4. *Let u be a rank n element of $\mathcal{A}$ and $\lambda \in \mathbb{C}$. Then $\alpha_i(\lambda u) = \lambda^i \alpha_i(u)$ for all $i \leq n$.*

Proof. For any $\lambda \in \mathbb{C}$, $\alpha_1(\lambda u) = \lambda \alpha_1(u)$ and also $\alpha_2(\lambda u) = \lambda^2 \alpha_2(u)$. Suppose the statement is true for all $i < n$. Then

$$\begin{aligned} \alpha_n(\lambda u) &= \sum_{i=1}^{n} (-1)^{i-1} \frac{(n-1)!}{(n-i)!} \alpha_{n-i}(\lambda u) \tau((\lambda u)^i) \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{(n-1)!}{(n-i)!} \lambda^{n-i} \alpha_{n-i}(u) \lambda^i \tau(u^i) \\ &= \lambda^n \sum_{i=1}^{n} (-1)^{i-1} \frac{(n-1)!}{(n-i)!} \alpha_{n-i}(u) \tau(u^i) \\ &= \lambda^n \alpha_n(u). \end{aligned}$$

□

As a consequence of the above lemma, we deduce the following proposition:

Proposition 3.2.1. *Let $u \in \mathcal{F}$. Then $det(1 + \lambda u)$ is a polynomial in λ, hence an entire function.*

The next proposition shows that our determinant restricted to $\mathcal{F}$ is spectral and hence it enjoys all the properties $\delta 1, \ldots, \delta 4$.

Proposition 3.2.2. *Let $u \in \mathcal{F}_N$ and $\{\lambda_i(u)\}_{i=1}^N$ be the spectral points of u repeated according to their respective algebraic multiplicities. Then*

$$det(1+u) = \prod_{i=1}^{N}(1+\lambda_i(u)).$$

Proof. Let $\{\lambda_i(u)\}_{i=1}^N$ be the sequence of the nonzero spectral points of u as stated in the hypothesis. Consider the diagonal matrix T with diagonal entries $\lambda_1(u), \ldots, \lambda_N(u)$. Since $rank(T) = N$, it follows from Fredholm determinant theory for bounded linear operators (cf. [Pie87], Proposition 4.4.11) that

$$det(1+u) = 1 + \sum_{i=1}^{N} \frac{1}{i!}\alpha_i(u) = 1 + \sum_{i=1}^{N} \frac{1}{i!}\alpha_i(T) = \prod_{i=1}^{N}(1+\lambda_i(T)) = \prod_{i=1}^{N}(1+\lambda_i(u)).$$

□

Remark 3.2.1. As a corollary of the above proposition, the function *det* satisfies the properties $\delta_1, \ldots, \delta_4$ of a determinant for all $u \in \mathcal{F}$.

For a positive real number r, we let $B_r = \{x \in \mathcal{F} : \nu(x) \leq r\}$.

Proposition 3.2.3. *Let $\mathcal{A}$ be a semiprime Banach algebra. Then the function det defined on $\mathcal{F}$ satisfies the Lipschitz condition on B_r for some r.*

Proof. We first show that $det(1+\lambda u) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\tau(u^n)\lambda^n\right)$ for any $u \in \mathcal{F}$ and $|\lambda|$ sufficiently small. Suppose $rank(u) = N$ and let $\{\lambda_i(u)\}_{i=1}^{N}$ be the sequence of the nonzero spectral points of u repeated according to their respective algebraic multiplicities. Then by Proposition 3.2.2 and Remark 3.1.3

$$\begin{aligned}
det(1+\lambda u) &= \sum_{i=1}^{N} \frac{1}{i!}\alpha_i(\lambda u) \\
&= \prod_{i=1}^{N} (1+\lambda\lambda_i(u)) \\
&= \exp\left(\sum_{i=1}^{N} \ln(1+\lambda\lambda_i(u))\right) \\
&= \exp\left(\sum_{i=1}^{N}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\lambda^n\lambda_i(u)^n\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\lambda^n\left(\sum_{i=1}^{N}\lambda_i(u^n)\right)\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\tau(u^n)\lambda^n\right). \qquad (3.2)
\end{aligned}$$

Let $0 < r < 1$ and $u, x \in B_r$ be arbitrary. Clearly $\sigma(x)$ and $\sigma(u)$ are contained in $\{\lambda \in \mathbb{C} \mid |\lambda| < r\}$. Hence from the above computation we have

$$|det(1+u) - det(1+x)| = \left|\exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\tau(u^n)\right) - \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\tau(x^n)\right)\right|.$$

Since the exponential function satisfies the Lipschitz condition on the unit disc there exists a $c > 0$ such that

$$\begin{aligned} |det(1+u) - det(1+x)| &\leq c\sum_{n=1}^{\infty}\frac{1}{n}|\tau(u^n) - \tau(x^n)| \\ &= c\sum_{n=1}^{\infty}\frac{1}{n}|\tau(u^n - x^n)| \\ &\leq c\sum_{n=1}^{\infty}\frac{1}{n}\nu(u^n - x^n). \end{aligned} \tag{3.3}$$

Since $\nu(u^n - x^n) \leq \nu[(u^{n-1} - x^{n-1})u] + \nu[x^{n-1}(u-x)]$, if we let $p = \max\{\nu(u), \nu(x)\}$, it follows by induction that

$$\nu(u^n - x^n) \leq np^{n-1}\nu(u-x). \tag{3.4}$$

The inequalities (3.3) and (3.4) yield

$$\begin{aligned} |det(1+u) - det(1+x)| &\leq c\sum_{n=1}^{\infty}p^{n-1}\nu(u-x) \\ &\leq \frac{c}{1-r}\nu(u-x). \end{aligned}$$

□

Theorem 3.2.3. *Let $\mathcal{A}$ be a semiprime Banach algebra possessing the q.a.p. Then the determinant det admits a continuous extension in the nuclear norm from $\mathcal{F}$ to $\mathcal{N}$.*

Proof. Let $u \in \mathcal{N}$ and $u_n \in \mathcal{F}$ such $\nu(u_n - u) \to 0$. It follows from the density of $\mathcal{F}$ in $\mathcal{N}$ in the nuclear norm that $u = x + y$ with $y \in \mathcal{F}$, $\nu(x) < r$ and $\|x\| < 1$.

We let $y_n = u_n - y$ which is of course an element of $\mathcal{F}$. So, $\nu(y_n - x) \to 0$. From Proposition 3.2.3 the function *det* is uniformly continuous on B_r for some r. By using the fact that $\mathcal{F}$ is ν dense in $\mathcal{N}$ again, we can write $u = x + y$ where $\nu(x) < r$, $y \in \mathcal{F}$ and $\|x\| < 1$. Since $\nu(y_n) = \nu(u_n - y) \leq \nu(u_n - u) + \nu(x)$, it follows that $y_n \in B_r$ eventually. Therefore the sequence $\{det(1 + y_n)\}_{n=1}^{\infty}$ is Cauchy and hence it converges. Since $\|x\| < 1$ and $\|y_n - x\| \leq \nu(y_n - x) \to 0$, it follows that $1 + x$ and $1 + y_n$ are invertible in $\mathcal{A}$ for n sufficiently large. For such an n it follows that

$$
\begin{aligned}
det(1 + u_n) &= det(1 + y_n + y) \\
&= det(1 + y_n)det(1 + (1 + y_n)^{-1}y).
\end{aligned}
$$

Since $det(1 + y_n)$ converges, we should only show that $det(1 + (1 + y_n)^{-1}y)$ also converges. Suppose $y = \sum_{m=1}^{N} f_m$, with f_m being of rank one. We then have $(1 + y_n)^{-1}y = \sum_{m=1}^{N} (1 + y_n)^{-1} f_m$ and $(1 + x)^{-1}y = \sum_{m=1}^{N} (1 + x)^{-1} f_m$. Since $(1 + y_n)^{-1}y \to (1 + x)^{-1}y$ in $\mathcal{A}$, we have that

$$
\begin{aligned}
det(1 + (1 + y_n)^{-1}y) &= 1 + \sum_{i=1}^{N} \frac{1}{i!}\alpha_i((1 + y_n)^{-1}y) \\
&\to 1 + \sum_{i=1}^{N} \frac{1}{i!}\alpha_i((1 + x)^{-1}y) \\
&= det(1 + (1 + x)^{-1}y).
\end{aligned}
$$

Therefore $det(1 + (1 + y_n)^{-1}y)$ also converges and the result follows. □

Definition 3.2.3. *If $\mathcal{A}$ has the q.a.p and $u \in \mathcal{N}$ with $\nu(u_n - u) \to 0$, $u_n \in \mathcal{F}$, we let*

$$det(1+u) := \lim_{n\to\infty} det(1+u_n).$$

Lemma 3.2.5. *([Aup91], Theorem 3.2.1) Suppose that $\mathcal{A}$ is a Banach algebra, $x \in \mathcal{A}$ and $\|x\| < 1$. Then $1-x$ is invertible and $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$.*

Proof. Let $\|x\| = r < 1$ and let $s_n = \sum_{k=0}^{n} x^k$. For $n < m$, it follows that

$$\begin{aligned} \|s_n - s_m\| &\leq \sum_{k=n+1}^{m} \|x^k\| \\ &\leq \frac{r^{n+1}}{1-r}, \end{aligned}$$

therefore $\{s_n\}$ is Cauchy and since $\mathcal{A}$ is complete, $\{s_n\}$ converges to some element $a = \sum_{k=0}^{\infty} x^k$. Since $xs_n = s_{n+1} - 1$, it follows that $a(1-x) = (1-x)a = 1$. □

Proposition 3.2.4. *If $\mathcal{A}$ has the q.a.p and $u \in \mathcal{N}$ is fixed, the function $det(1+\lambda u)$ is an entire function.*

Proof. By Proposition 3.2.3 the function $det : \mathcal{F} \to \mathbb{C}$ is uniformly continuous on B_r for some r. Let $R > 0$ and $u \in \mathcal{N}$. We write $u = x + y$ where $y \in \mathcal{F}$ and $\nu(x) < \frac{r}{1+R}$. Since $\nu(x) < \frac{r}{1+R}$, we in fact can choose r so small such that for $|\lambda| \leq R$, $\|\lambda x\| < 1$. So, $1 + \lambda x$ is invertible in $\mathcal{A}$ and $det(1+\lambda u) =$

$det(1+\lambda x)det(1+\lambda(1+\lambda x)^{-1}y)$. We will show that $det(1+\lambda x)$ and $det(1+\lambda(1+\lambda x)^{-1}y)$ are both analytic in $|\lambda|<R$. Since $x\in\mathcal{N}$, there is a sequence $\{y_n\}$ in $\mathcal{F}$ such that $\nu(x-y_n)\to 0$. We in fact may assume that $\nu(y_n)<\frac{r}{1+R}$ for all n. So for $|\lambda|\leq R$ we have $\nu(\lambda y_n)<r$ and $det(1+\lambda y_n)\to det(1+\lambda x)$. Since $det(1+\lambda y_n)$ is a polynomial in λ and the function det is uniformly continuous on B_r, it follows that the convergence $det(1+\lambda y_n)\to det(1+\lambda x)$ is uniform on B_r. So $det(1+\lambda x)$ is analytic in $|\lambda|<R$. We show that $det(1+\lambda(1+\lambda x)^{-1}y)$ is also analytic in $|\lambda|<R$. By Lemma 3.2.5 we get

$$\begin{aligned} det(1+\lambda(1+\lambda x)^{-1}y) &= 1+\sum_{i=1}^{N}\frac{1}{i!}\alpha_i(\lambda(1+\lambda x)^{-1}y) \\ &= 1+\sum_{i=1}^{N}\frac{1}{i!}\alpha_i\left(\sum_{m=0}^{\infty}(-1)^m\lambda^{m+1}x^m y\right). \end{aligned}$$

In fact $\tau\left[\left(\sum_{m=0}^{\infty}(-1)^m\lambda^{m+1}x^m y\right)^n\right]$ is a power series in λ for any n. We illustrate this fact for $n=1$. Suppose $rank(y)=m$. That is $y=y_1+\ldots+y_m$ where y_i are of rank one. So $\tau\left[\sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}x^m y)\right]=\tau\left[\sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}x^m y_1)\right]+\ldots+\tau\left[\sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}x^m y_m)\right]$. Since for each summand $\sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}x^m y_i)$ one has $\sum_{m=0}^{\infty}\|((-1)^m\lambda^{m+1}x^m y_i)\|<\infty$, it follows that $\sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}x^m y_i)$

is a nuclear representation of a nuclear element, hence

$$\begin{aligned}\tau\left[\left(\sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}x^m y_i\right)\right] &= \sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}\tau\left(x^m y_i\right)) \\ &= \sum_{m=0}^{\infty}((-1)^m\tau\left(x^m y_i\right)\lambda^{m+1})\end{aligned}$$

which is a power series in λ. Since $\tau\left[\sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}x^m y)\right]$ is a finite sum of the $\tau\left[(\sum_{m=0}^{\infty}((-1)^m\lambda^{m+1}x^m y_i)\right]$ it should also be a power series.

We denote $\tau\left[(\sum_{m=0}^{\infty}(-1)^m\lambda^{m+1}x^m y)^n\right]$ by $\sum_n$. Therefore,

$$\alpha_k\left(\sum_{m=0}^{\infty}(-1)^m\lambda^{m+1}x^m y\right) = \begin{vmatrix} \sum_1 & 1 & 0 & 0 & \dots & 0 \\ \sum_2 & \sum_1 & 2 & 0 & \dots & 0 \\ \sum_3 & \sum_2 & \sum_1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{k-1} & \sum_{k-2} & \sum_{k-3} & \sum_{k-4} & \dots & k-1 \\ \sum_k & \sum_{k-1} & \sum_{k-2} & \sum_{k-3} & \dots & \sum_1 \end{vmatrix}$$

which is also analytic in $|\lambda| < R$. Therefore $\sum_{k=1}^{N}\frac{1}{k!}\alpha_k(\sum_{m=0}^{\infty}(-1)^m\lambda^{m+1}x^m y)$ is a finite sum of analytic functions, hence $det(1+\lambda(1+\lambda x)^{-1})$ is analytic in $|\lambda| < R$. Since R was arbitrarily chosen, $det(1+\lambda x)$ and $det(1+\lambda(1+\lambda x)^{-1})$ are entire functions and thus $det(1+\lambda u)$ is an entire function. □

Theorem 3.2.4. *For any $u \in \mathcal{N}$, the function $det(1+\lambda u)$ has a Taylor series* *expansion* $1+\sum_{n=0}^{\infty}\frac{\alpha_n(u)}{n!}\lambda^n$ *where*

$$\alpha_n(u)=\begin{vmatrix} \tau(u) & 1 & 0 & 0 & \dots & 0 \\ \tau(u^2) & \tau(u) & 2 & 0 & \dots & 0 \\ \tau(u^3) & \tau(u^2) & \tau(u) & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau(u^{n-1}) & \tau(u^{n-2}) & \tau(u^{n-3}) & \tau(u^{n-4}) & \dots & n-1 \\ \tau(u^n) & \tau(u^{n-1}) & \tau(u^{n-2}) & \tau(u^{n-3}) & \dots & \tau(u) \end{vmatrix}.$$

Proof. If $u \in \mathcal{F}$, then the result follows from the definition of *det*. Let $u \in \mathcal{N}$, with $\nu(u_n - u) \to 0$, where u_n is of finite rank for all n. Let $c_m(u_n)$ and $c_m(u)$ be the coefficients of λ^m in the respective Taylor expansions. Since the convergence $det(1+\lambda u_n) \to det(1+\lambda u)$ is uniform on B_r for some r, it follows that

$$\lim_{n\to\infty} c_m(u_n) = c_m(u), \quad m = 1, 2, \dots$$

By the continuity of the trace τ on $\mathcal{N}$ the result follows. □

Proposition 3.2.5. *Let* $u, v \in \mathcal{N}$. *Then* $det((1+u)(1+v)) = det(1+u)det(1+v)$.

Proof. Since multiplication is continuous and by Lemma 3.2.3 (b) $det((1+u)(1+v)) = det(1+u)det(1+v)$ if $u, v \in \mathcal{F}$, the result follows. □

In the classical case we know that if T is an element of an operator ideal that admits a continuous determinant, then $I+T$ is invertible if and only if the determinant of $I+T$ is not zero.

Proposition 3.2.6. *Let $u \in \mathcal{N}$. Then $det(1+u) \neq 0$ if and only if $1+u$ is invertible in $\mathcal{A}$.*

Proof. Suppose $1+u$ is invertible. Then $(1+u)a = 1$ for some $a \in \mathcal{A}$. In fact there is b in $\mathcal{N}$ such that $a = 1+b$. Thus $(1+u)(1+b) = 1$. It then follows from Proposition 3.2.5 that $det(1+u) \neq 0$.

By the density of $\mathcal{F}$ in $\mathcal{N}$ in the nuclear norm, we write $u = x+y$ with $x \in \mathcal{F}$, $\|y\| < \nu(y) < 1$. Since $y \in \mathcal{N}$ and $1+y$ is invertible in $\mathcal{A}$, it follows from the first part of the proof that $det(1+y) \neq 0$. Since $1+u = (1+y)(1+(1+y)^{-1}x)$, it follows that $det(1+u) = det(1+y)det(1+(1+y)^{-1}x)$. Now if $det(1+u) \neq 0$, then $det(1+(1+y)^{-1}x) \neq 0$. Since $1+(1+y)^{-1}x \in \mathcal{F}$, it follows by Lemma 3.2.1 that $1+(1+y)^{-1}x$ is invertible in $\mathcal{A}$, which implies that $(1+y)(1+(1+y)^{-1}x)$ is invertible in $\mathcal{A}$ and hence $1+u$ is invertible in $\mathcal{A}$. □

Remark 3.2.2. In the case of an operator ideal $\mathcal{U}$ between Banach spaces there is a one to one correspondence between continuous traces and continuous determinants. The correspondence is established by the following formula: Given a continuous determinant δ on $\mathcal{U}$ and $T \in \mathcal{U}$ then

$$\delta^{\cdot}(T) := \lim_{\lambda \to 0} \frac{\delta(1 + \lambda T) - 1}{\lambda}$$

defines a continuous trace on $\mathcal{U}$ [Pie87], Theorem 4.3.15. In fact the following formula holds:

$$\delta(1 + T) = \exp(\delta^{\cdot}(\ln(1 + T))) \text{ for all } T \in \mathcal{U}.$$

See also [Pie87], Proposition 4.6.2.

Proposition 3.2.7. *For $u \in \mathcal{N}$ it follows that*

$$det'(u) := \lim_{\lambda \to 0} \frac{det(1 + \lambda u) - 1}{\lambda} = \tau(u).$$

Proof. By Theorem 3.2.4 $det(1+\lambda u)$ has a Taylor series expansion $1+\sum_{n=0}^{\infty} \frac{\alpha_n(u)}{n!}\lambda^n$. Hence $det'(u) = \alpha_1(u) = \tau(u)$. □

Lemma 3.2.6. *If $u \in \mathcal{N}$ then $\exp(u) - 1 \in \mathcal{N}$. Moreover*

$$\nu(\exp(u) - 1) \leq \nu(u) \exp(\|u\|).$$

Proof. Since $\exp(u) - 1 = \sum_{n=1}^{\infty} \frac{u^n}{n!}$, it follows that $\exp(u) - 1 \in \mathcal{N}$ and for any natural number k we have

$$\begin{aligned} \nu\left(\sum_{n=k}^{\infty} \frac{u^n}{n!}\right) &= \nu\left(u \sum_{n=k}^{\infty} \frac{u^{n-1}}{n!}\right) \\ &\leq \nu(u) \sum_{n=k}^{\infty} \frac{\|u\|^{n-1}}{n!}. \end{aligned}$$

Hence if we let $k = 1$ the inequality follows. □

Proposition 3.2.8. *Let* $u, v \in \mathcal{N}$. *Then* $det(\exp(u+v)) = det(\exp(u)) det(\exp(v))$.

Proof. Without any loss we may assume that $3\|u + v\| \leq n$ for some n. So, by Lemma 3.2.5 $1 + \frac{1}{n}(u + v)$ is invertible and $\|(1 + \frac{1}{n}(u + v))^{-1}\| \leq \frac{3}{2} < 3$. We introduce x_n and y_n as follows:

$$\begin{aligned} (1 + \frac{1}{n}(u + v))(1 + x_n) &= (1 + \frac{1}{n}u)(1 + \frac{1}{n}v) \\ &= 1 + \frac{1}{n}(u + v) + \frac{1}{n^2}uv \qquad (3.5) \end{aligned}$$

and

$$1 + y_n = (1 + x_n)^n. \qquad (3.6)$$

From equation (3.6) we have $(1 + \frac{1}{n}(u + v))x_n = \frac{1}{n^2}uv$. That is $n^2 x_n =$

$(1+\frac{1}{n}(u+v))^{-1}uv$. Since x_n and uv are nuclear elements of $\mathcal{A}$, we have

$$\begin{aligned} n^2\nu(x_n) &= \nu\left((1+\frac{1}{n}(u+v))^{-1}uv\right) \\ &\leq \|(1+\frac{1}{n}(u+v))^{-1}\|\nu(uv) \\ &\leq 3\nu(uv). \end{aligned}$$

From equation (3.7) $y_n = (1+x_n)^n - 1$, that is

$$\begin{aligned} y_n &= \sum_{k=1}^{n}\binom{n}{k}x_n^k \\ &= x_n\sum_{k=1}^{n}\binom{n}{k}x_n^{k-1}. \end{aligned}$$

It follows that $y_n \in \mathcal{N}$ as $x_n \in \mathcal{N}$. Moreover

$$\begin{aligned} \nu(y_n) &\leq \nu(x_n)\sum_{k=1}^{n}\binom{n}{k}\|x_n\|^{k-1} \\ &\leq \nu(x_n)\sum_{k=1}^{n}\frac{n^k}{k!}\|x_n\|^{k-1} \\ &\leq n\nu(x_n)\exp(n\|x_n\|) \\ &\leq \frac{3\nu(uv)\exp(n\|x_n\|)}{n}. \end{aligned}$$

That is $\nu(y_n) \leq \frac{3\nu(uv)}{n}\exp(\frac{3\nu(uv)}{n})$ which implies that $\lim_{n\to\infty}\nu(y_n) = 0$. However

$$\begin{aligned} det\left[\left(1+\frac{1}{n}u\right)^n\right] det\left[\left(1+\frac{1}{n}v\right)^n\right] &= \left[det(1+\frac{1}{n}u)det(1+\frac{1}{n}v)\right]^n \\ &= \left[det\left(1+\frac{1}{n}(u+v)\right) det\left(1+x_n\right)\right]^n \\ &= det\left(\left[1+\frac{1}{n}(u+v)\right]^n\right)[det(1+x_n)]^n \\ &= det\left(\left[1+\frac{1}{n}(u+v)\right]^n\right) det(1+y_n). \end{aligned}$$

Now $\lim_{n\to\infty} \nu(y_n) = 0$ implies that $\lim_{n\to\infty} det(1+y_n) = 1$. It follows from the proof of Lemma 3.2.6 that $(1+\frac{1}{n}u)^n - 1 = \sum_{k=1}^n \frac{u^k}{k!}$ converges in the nuclear norm to $\exp(u)-1$ and similar results for $(1+\frac{1}{n}u)^n - 1$ and $(1+\frac{1}{n}(u+v))^n - 1$ hold. Taking limits on both sides of the above equation as n approaches infinity the desired formula follows. □

Proposition 3.2.9. *Let $u \in \mathcal{N}$. Then $det(\exp(u)) = \exp(\tau(u))$.*

Proof. Recall that $det'(u) = \tau(u)$ for all $u \in \mathcal{N}$. So that

$$det(1+\frac{1}{n}u) = 1 + \frac{1}{n}\tau(u) + \psi$$

where $\lim_{n\to\infty} n\psi = 0$. We let

$$R_n = \left(1+\frac{1}{n}\tau(u)+\psi\right)^n - \left(1+\frac{1}{n}\tau(u)\right)^n.$$

We want to show that $\lim_{n\to\infty}|R_n| = 0$. To accomplish that we take n such that $n|\psi| \leq 1$. This implies that

$$\begin{aligned} |R_n| &\leq \sum_{k=1}^{n} \binom{n}{k} \left|1 + \frac{1}{n}\tau(u)\right|^{n-k} |\psi|^k \\ &\leq \sum_{k=1}^{n} \frac{n^k}{k!} \left|1 + \frac{1}{n}\tau(u)\right|^{n-k} |\psi|^k \\ &= \sum_{k=1}^{n} \frac{1}{k!} \left|1 + \frac{1}{n}\tau(u)\right|^{n-k} |n\psi|^k \\ &\leq \sum_{k=1}^{n} \frac{1}{k!} \left|1 + \frac{1}{n}\tau(u)\right|^{n} |n\psi|^k. \end{aligned}$$

Since $n|\psi| \leq 1$, we have

$$|R_n| \leq \left|1 + \frac{1}{n}\tau(u)\right|^n n|\psi| \sum_{k=1}^{n} \frac{1}{k!}. \tag{3.7}$$

The first factor on the right hand side of the above inequality (3.8) approaches $\exp(\tau(u))$, the second factor approaches 0 and the third factor approaches $\exp(1)$, which gives $\lim_{n\to\infty}|R_n| = 0$. Hence

$$\begin{aligned} det(\exp(u)) - \exp(\tau(u)) &= \lim_{n\to\infty} det[(1 + \frac{1}{n}u)^n] - \lim_{n\to\infty}(1 + \frac{1}{n}\tau(u))^n \\ &= \lim_{n\to\infty}[det(1 + \frac{1}{n}u)]^n - \lim_{n\to\infty}(1 + \frac{1}{n}\tau(u))^n \\ &= \lim_{n\to\infty}\left[\left(1 + \frac{1}{n}\tau(u) + \psi\right)^n - \left(1 + \frac{1}{n}\tau(u)\right)^n\right] \\ &= \lim_{n\to\infty} R_n = 0. \end{aligned}$$

□

It would be interesting to have a similar result as the one mentioned in Remark 3.2.2. We do however have the following special case:

Theorem 3.2.5. *If* $u \in \mathcal{N}$ *with* $\nu(u) < 1$ *then*

$$det(1+u) = \exp(\tau(\ln(1+u))).$$

Proof. Since $\nu(u) < 1$ it follows that $\|u\| < 1$, hence $\sigma(1+u)$ is entirely contained in a domain on which $\ln(1+\lambda)$ is holomorphic. Hence $\ln(1+u)$ is well-defined and from the functional calculus $1+u = \exp(\ln(1+u))$. It follows from Proposition 3.2.9 that

$$det(1+u) = det(\exp(\ln(1+u))) = \exp(\tau(\ln(1+u))).$$

□

3.3 Concluding remarks

1. An interesting problem will be to consider other ideals containing $\mathcal{F}$ and to see whether the theory of traces and determinants can be extended. From [Pie87] an axiomatic approach was followed and applied to a number of important operator ideals on Banach spaces.

We give a short motivation for our problem: Recall from Chapter one that for $a \in \mathcal{A}$ and $n \in \mathbb{N}$, the nth approximation number of a is given by

$$\delta_n(a) := \inf\{\|a - x\| \; : \; rank(x) < n\}.$$

If we let

$$\mathcal{C}_1 := \{a \in \mathcal{A} \; : \; \sum_{n=1}^{\infty} \delta_n(a) < \infty\} \text{ and } \|a\|_1 := \sum_{n=1}^{\infty} \delta_n(a),$$

then $\mathcal{C}_1$ is an ideal containing $\mathcal{F}$. It is not known whether $\mathcal{C}_1 \subseteq \mathcal{N}$ in general. Important examples for which the above inclusion hold include all commutative Banach algebras, the algebra of bounded linear operators on a Banach space and C^*-algebras. In fact in the case of C^*-algebras (as for operators on Hilbert spaces) one has equality, that is

$$\mathcal{N} = \mathcal{C}_1 \text{ and } \|.\|_1 = \nu(.).$$

However if $\mathcal{A} = B(X)$, X a Banach space, it was shown by H. König, see [K̈86], Theorem 4.a.6, that the trace and also the determinant can be extended continuously from $F(X)$ to $\mathcal{C}_1(X)$ and moreover they are spectral on $\mathcal{C}_1(X)$.

If we can show that $\mathcal{C}_1 \subseteq \mathcal{N}$ is general, then we will be able to obtain the same result. The best that we can do is to show that for the ideal $\mathcal{C}_1 \cap \mathcal{N}$

one has a continuous extension of the determinant from $\mathcal{F}$ to $\mathcal{C}_1 \cap \mathcal{N}$ which is spectral.

2. Another problem is to extend Theorem 3.2.5 to the general case where for $u \in \mathcal{N}$ with $1 + u$ invertible one has

$$det(1 + u) = \exp(\tau(\ln(1 + u))).$$

The main issue will be to get $\ln(1 + u)$ to be defined ([Pie87], 4.6.2).

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Summary

Riesz theory and Fredholm determinants in Banach algebras

by

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In the classical theory of operators on a Banach space a beautiful interplay exists between Riesz and Fredholm theory, and the theory of traces and determinants for operator ideals. In this thesis we obtain a complete Riesz decomposition theorem for Riesz elements in a semiprime Banach algebra and

on the other hand extend the existing theory of traces and determinants to a more general setting of Banach algebras.

In order to obtain some of these results we use the notion of finite multiplicity of spectral points to give a characterization of the essential spectrum for elements in a Banach algebra. As an immediate corollary we obtain the well-known characterization of Riesz elements namely that their non-zero spectral points are isolated and of finite multiplicities. In the final chapter of the thesis we use Plemelj's type formulas to define a determinant on the ideal of finite rank elements and show that it extends continuously to the ideal of nuclear elements.

Opsomming

Riesz theory and Fredholm determinants in Banach algebras

deur

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Daar is a pragtige wisselwerking tussen Riesz- en Fredholm-teorie op 'n Banach-ruimte en die teorie van spore en determinante gedefinieer op sekere operator-ideale. Die proefskrif bevat 'n volledige Riesz-ontbinding stelling vir Riesz-elemente in 'n Banach algebra en daar word aangetoon hoe die bestaande

teorie van spore en determinante uitgebrei word na 'n meer algemene raamwerk. Hierdie resultate maak gebruik van 'n karakterisering van die essensiële spektrum in terme van spektraalwaardes met eindige multiplisiteite. Dit lei dan direk tot 'n bekende karakterisering van Riesz-elemente, naamlik dat die nie-nul spektraalwaardes geïsoleerd is met eindige multiplisiteite.

In die tweede deel van die proefskrif word gebruik gemaak van Plemelj-tipe formules om 'n determinant op die ideaal van eindige-rang elemente te definieer. Verder word aangetoon dat hierdie determinant onder sekere voorwaardes kontinu uitgebrei kan word na die ideaal van nukleêre elemente met al die verwagte eienskappe van 'n determinant.