Chapter 2

Regression Models in Renewal Theory

1 Introduction

In Chapter 1, five regression models in renewal theory were identified that have the potential to lead to the dissertation objectives. In this chapter the results of a thorough literature study on these models are discussed.

It was discovered that all the models have the same broad structure. First, a baseline function that describes the component's reliability as a function of time (usually the survivor function or hazard rate function) is estimated with either parametric or non-parametric techniques. Secondly, a functional term dependent on time and covariates is allowed to influence the baseline function (usually by multiplication) to estimate the total reliability of the component.

Throughout this chapter, the functional term is referred to as λ, which is a function of time and covariates, i.e. $\lambda(\overline{z(t)})$. Let $\overline{z(t)}$ denote a column vector of m measured covariates or $\overline{z(t)} = [z_1(t)\, z_2(t) \ldots .\, z_m(t)]^T$. For the sake of generality, covariates are assumed to be functions of time, although covariates may be time-independent. Further, suppose that $\overline{\gamma}$ is a row vector of regression coefficients associated with a specific model's covariate vectors i.e., $\overline{\gamma} = [\gamma_1\, \gamma_2 \ldots ..\, \gamma_m]$, estimated during model fitting procedures. The terminology is followed closely, except for the PWP model where some additional variables are needed to describe the model.

Each model is first introduced in mathematical terms and then some comments are made about the model's abilities, deficiencies and applicability. After the discussion of the different models, the model most suitable for this research project is selected with a proper motivation for the selection.

2 Accelerated Failure Time Model

The accelerated failure time model was introduced by Pike[31] in 1966 and is considered as the second most popular regression model used in renewal theory today. It is a fully parametric type of model and strives to estimate the survivor function. The model allows covariates to influence expected life time of a component directly, in a multiplicative manner.

2.1 Mathematical Model

Pike[31] presents the model as follows:

$$R(t,\overline{z(t)}) = R_0(\lambda(\overline{z(t)})\cdot t) \tag{2.1.}$$

In probabilistic terms the model can be written as:

$$R(t,\overline{z(t)}) = P\{T \geq t \mid \overline{z(t)}\} \tag{2.2.}$$

$R_0(t)$ is a parametric baseline survivor function estimated without considering covariates. The AFTM is then constructed by allowing covariates to influence life time by the functional term, $\lambda(\overline{z(t)})$. In (2.1.), it is required that $\lambda(\overline{0}) \equiv 1$ for the case where covariates do not have an effect on life time and $\lambda(\overline{z(t)}) > 0$ where covariates do play a role.

A popular form of the functional term is the log-linear function, i.e. $\lambda(\overline{z(t)}) = \exp(\overline{\gamma}\cdot\overline{z(t)})$, where $\overline{\gamma}$ is a vector of regression coefficients. In this case, $\lambda(\overline{z(t)}) > 1$ accelerates and $\lambda(\overline{z(t)}) < 1$ decelerates the rate at which a component moves through time with respect to the baseline survivor function.

Leemis[32] derived the general hazard rate function for the AFTM as:

$$h(t,\overline{z(t)}) = \lambda(\overline{z(t)})\cdot h_0(\lambda(\overline{z(t)})\cdot t) \tag{2.3.}$$

Newby[33] suggests the maximum likelihood method to estimate the model parameters although the method of moments have also been used successfully.

2.2 Comments

The theory of AFTMs has been developed in detail over the 33 years of the model's existence. Numerous theoretical publications on model estimation techniques, goodness-of-fit tests and extensions for the model to suit repairable systems reliability are found in the literature. A very good example of such a publication is Lin et. al.[35].

Not only has the AFTM a sound theoretical base, but it has also been applied widely on failure time data, especially in biomedical applications and more recently in reliability situations. Four relevant publications proving the abilities of the AFTM are:

1. Martorell et. al.[36]. In this paper the AFTM is used successfully to estimate the useful remaining life of nuclear power plants. Results are compared to methods not incorporating covariates. The authors conclude that this model is a useful maintenance management support tool.
2. Addison et. al.[37] used the AFTM to model unemployment duration data with attributes like employee age and profession. The results are compared to Cox's proportional hazards model (considered later in this chapter).
3. Shyur and Luxhoj[38] use Cox's PHM, AFTM and neural networks to model data obtained from ageing aircraft with success.
4. Publications where fatigue crack growth is modeled by the AFTM are encountered frequently in the literature. Principles of fracture mechanics as applied in fatigue crack growth are very suitable for the parametric approach of AFTMs. See [27] and [33].

Solomon[34] identified several cases where the AFTM was specified inappropriately because of many illustrations where accelerated failure time models seemed to arise naturally in practice. Newby[33] reports some of these misspecifications as well. Crowder[27] gives guidelines as to when the AFTM is appropriate.

The AFTM is established in regression type failure analyses although it has certain limitations. Newby[33] describes this model to be "an effective alternative to the proportional hazards model in appropriate cases" after a thorough study on both PHMs and AFTMs in 1988, thereby suggesting that the PHM is superior to the AFTM.

3 Proportional Hazards Model

Failure time data analysis underwent a total revolution after Cox[2] proposed this model in 1972. It was first intended for use in biomedicine but was soon modified to be suitable for the field of reliability. PHMs model the hazard rate of a component as the product of a baseline hazard rate dependent of time only and a functional term dependent on time and covariates.

The PHM was originally proposed as a semi-parametric model and regression parameters can be determined independently of the estimation of the baseline hazard rate although this only yields relative risks. For an absolute hazard rate, the baseline hazard rate has to be estimated. In general the PHM is used in its parameterized form to overcome numerical difficulties.

Extensions made to the original PHM by Prentice, Williams and Peterson[11] lead to the famous PWP model and extensions made by Pijnenburg[12] resulted in the additive hazards model, both considered later in this chapter.

3.1 Mathematical Model

The model is proposed by Cox[2] as:

$$h(t,\overline{z(t)}) = h_0(t) \cdot \lambda(\overline{z(t)}) \qquad (3.1.)$$

Analogous to AFTMs, the PHM consists of a baseline hazard rate, $h_0(t)$, which is influenced multiplicatively by a functional term $\lambda(\overline{z(t)}) \geq 0$, thereby including the effects of covariates. Inspection shows that the total hazard rate is identical and equal to the baseline hazard rate when the covariates have no influence on the component's risk to fail.

The assumption of the multiplicative effect of the covariates on the baseline hazard rate implies that the ratio of the hazard rates of any two items observed at any time t associated with covariate sets $\overline{z_1}$ and $\overline{z_2}$, respectively, will be a constant with respect to time and proportional to each other, i.e. $h(\overline{\gamma},\overline{z_1})\ \alpha\ h(\overline{\gamma},\overline{z_2})$. This property is referred to as the proportional hazards property of the model.

There are several possible forms for the functional term $\lambda(\overline{z(t)})$. Some are: the exponential form, $\exp(\overline{\gamma} \cdot \overline{z(t)})$; the logarithmic form, $\log(1+\exp(\overline{\gamma} \cdot \overline{z(t)}))$; the inverse linear form, $1/(1+\overline{\gamma} \cdot \overline{z(t)})$; or the linear form, $1+\overline{\gamma} \cdot \overline{z(t)}$. The exponential form of the functional term is used most widely and then equation (3.1.) becomes:

$$h(t;\overline{z(t)}) = h_0(t) \cdot \exp(\overline{\gamma} \cdot \overline{z(t)}) \qquad (3.2.)$$

where the regression vector $\overline{\gamma}$ and the baseline hazard function $h_0(t)$ needs to be determined. Methods to estimate $h_0(t)$ for the semi-parametric model in (3.2.) involve maximum likelihood theory and can be found in [2],[3] and [6].

The PHM is often used in its fully parameterized form to increase numerical practicability with the aid of the Weibull distribution, which is very suitable to model failure time data. Parameterization is done by approximating $h_0(t)$ with the Weibull representation of the hazard rate, i.e.:

$$h(t;\overline{z(t)}) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \cdot \exp(\overline{\gamma} \cdot \overline{z(t)}) \qquad (3.3.)$$

where β and η are the shape and scale parameters of the Weibull distribution respectively. Parameters of (3.3.) are generally determined by maximum likelihood methods, see [5] for example.

3.2 Comments

Proportional hazards modelling was probably the greatest contribution to time failure data analysis up to date and it is still the most popular model of its kind. The model was used extensively in biomedicine after its introduction and since the mid 1980's the model has been accepted more and more in the reliability modelling world.

The theoretical foundation of proportional hazards modelling and its successes are well established in the literature. Since its introduction in 1972, countless papers have been published on more efficient estimation methods, goodness-of-fit tests and minor extensions to the model.

Application of the model to practical failure time data is predominantly found in the field of biomedicine until the mid 1980's, especially the semi-parametric PHM. The reason for the popularity of the semi-parametric PHM is because no assumption needs to be made about the baseline hazard rate. Younes and Lachin[39] sum this up by stating: "In biomedical applications, little is generally known *a priori* about the shape of baseline functions, and models that assume specific parametric shapes (such as the accelerated failure time model) can be difficult to justify."

Bendell[1] expressed his disappointment in failure data analysts in 1985 with: "Why is it, however, that recorded applications of the PHM to date have until recently been almost entirely associated with medical data?", thereby criticizing them for overlooking this very logical approach to life data analysis. During this time many more publications on the PHM in reliability applications were made with data obtained from, amongst others, majorettes[7], marine gas turbines and ships' sonar[8], valves in light water reactor nuclear generating plants[9] and aircraft engines[10]. More recently the model was used with great success to model the reliability of locomotive diesel engines in Canada by using types and quantities of foreign particles in the engine oil[5] as covariates with the failure time data. It was also used on aircraft and marine engine failure data[4].

The PHM has certainly made good ground in the field of reliability modelling and has proved itself to be an excellent support tool for maintenance renewal decisions.

4 Prentice Williams Peterson Model

Prentice, Williams and Peterson[11] made a major extension to the original Cox PHM in 1981, with a model which will be referred to as the PWP model. The PWP model extends the PHM to handle situations where a specific item (component or system) experiences multiple failures during its life time, by allowing for stratification of the data in the model. The model is defined in such a way that it is suitable to model data generated by repairable systems and data from renewal situations (situations where the hazard rate is of primary importance). This model has three dimensions (compared to the two of the PHM) namely age, covariates and stratum which makes the PWP model extremely powerful.

4.1 Mathematical Model

Because of its complexity, the PWP model will only be introduced in very general terms. Suppose a specific item can experience more than one failure during its life time. For the moment it is not important whether the item is repaired to the as-good-as-new or as-bad-as-old condition. Define u as the long term life variable, where $u=0$ at the item's initial startup and let $N(u)$ be the counting process of the multiple item failures. Every short term item life, i.e. operational period in-between failures, is referred to as a stratum, S, where $S=N(u)+1$ at any instant u. Thus, the item enters the i^{th} stratum at occurrence of the $(i-1)^{th}$ failure, where $i=2,3,...$ and enters stratum 1 at $u=0$ when $N(0)=0$. Also define t, the time from the most recent failure to the current time u. Let $Z(u)$ be the covariate process observed from $u=0$.

From the above it is possible to define a concept used in repairable systems theory, that of *rate of occurrence of failure* (ROCOF):

$$v(u)=\frac{\mathrm{d}}{\mathrm{d}u}N(u) \qquad (4.1.)$$

Prentice, Williams and Peterson tried to estimate the ROCOF of items experiencing multiple failures with their model. They suggest two possibilities:

PWP model 1: v_1

Let the baseline function be a combination of the item's ROCOF and hazard rate by defining the function to be stratum-specific but dependent on long term time u. This leads to:

$$v_1[u \mid \{N(u), u \geq 0\}, \{Z(u), u \geq 0\}] = v_{0_s}(u) \cdot \exp(\overline{\gamma_s} \cdot \overline{z(u)}) \tag{4.2.}$$

where $\overline{\gamma_s}$ is a vector of stratum-specific regression coefficients.

PWP model 2: v_2

For model 2 the baseline function is the stratum-specific hazard rate (as a function of t) of the S^{th} stratum:

$$v_2[u \mid \{N(u), u \geq 0\}, \{Z(u), u \geq 0\}] = h_{0_s}(t) \cdot \exp(\overline{\gamma_s} \cdot \overline{z(u)}) \tag{4.3.}$$

Cox's PHM is the general case of v_2, where $h_{0_s}(\cdot) = h_0(\cdot)$ for all strata.

Prentice, Williams and Peterson used partial likelihood concepts similar to those used by Cox[2] to estimate the regression coefficients.

4.2 Comments

From the brief description above it should be clear that this model is extremely powerful. Surprisingly enough very little research has been done on this model up to date. Except for the original proposal of the model and an unpublished Ph.D. dissertation of Williams[40], only a few attempts have been made to utilize the endless advantages of this model. Two examples are Ascher[41], who investigated gas turbine engine reliability in 1982 with the PWP model and Dale[42] who has illustrated in 1983 how the model can be used for repairable systems reliability.

Authors who support the statement about the PWP model's extreme potential are, amongst others, Pijnenburg[12] and Ascher and Feingold[15]. Ascher and Feingold are of opinion that "the importance of the PWP model can scarcely be overemphasized".

5 Proportional Odds Model

The proportional odds model originated from epidemiological studies and was introduced by Bennett[43] in 1983 for use in biomedicine. This model is structurally similar to the PHM, but not a direct extension. It models the odds of an event occurring and unlike the PHM, does the effect of covariates in the POM model diminish as time approaches infinity. This diminishing property of the covariates means that the model is suitable for situations where a component adjusts to factors imposed on it or the factors only operate in early stages.

5.1 Mathematical model

For this model the odds of a failure occurring is defined in terms of the survivor function as:

$$\frac{F(\cdot)}{R(\cdot)} = \frac{1 - R(\cdot)}{R(\cdot)} \tag{5.1.}$$

This definition of 'odds' is used to introduce the POM:

$$\frac{1 - R(t, \overline{z(t)})}{R(t, \overline{z(t)})} = \varphi \cdot \frac{1 - R_0(t)}{R_0(t)} \tag{5.2.}$$

Equation (5.2.) states that the odds for a failure to happen under the influence of covariates are φ times higher than the odds of a failure without the effects of covariates. If φ increases, so does the probability of a shorter life time. Differentiation of (5.2.) with respect to time leads to:

$$\frac{h(t, \overline{z(t)})}{R(t, \overline{z(t)})} = \varphi \cdot \frac{h_0(t)}{R_0(t)} \tag{5.3.}$$

after using the coefficient rule. By rearranging the terms in (5.3.) and re-using (5.2.), a hazard ratio can be obtained:

$$\frac{h(t, \overline{z(t)})}{h_0(t)} = \varphi \cdot \frac{R(t, \overline{z(t)})}{R_0(t)} = \frac{1 - R(t, \overline{z(t)})}{1 - R_0(t)} \tag{5.4.}$$

Inspection shows that $\varphi|_{t=0} = \varphi$ and $\varphi|_{t=\infty} = 1$, from there the diminishing effect of the covariates.

Bennett derives the full unconditional likelihood for the model in his original paper to estimate the model parameters. Research done by Shen[44] provides more efficient estimation methods and methods to enable the model to handle suspended observations.

5.2 Comments

The POM has not been used very often in reliability modelling up to date although it has been fairly popular in biomedical data analyses since first publication. Its diminishing covariate effect property is probably the primary reason for its unpopularity in the reliability modelling field. It is argued that the effect of covariates describing components' reliability will seldom taper down close to failure or suspension.

6 Additive Hazards Model

Pijnenburg[12] proposed the additive hazards model in 1991. For this model a time dependent hazard rate is used as a baseline function and a functional term is added to the baseline function. Because of the addition, the functional term need not be positive as in the case of the PHM which immediately gives the model more flexibility. David and Moeschberger[45], Aranda-Ordaz[46] and Elandt-Johnson[47] are all of the opinion that this is a very valuable advantage. The AHM is also suitable for repairable systems when the ROCOF is modeled in an additive manner.

6.1 Mathematical Model

As before, suppose the time dependent hazard rate is denoted by $h_0(t)$ and the functional term which incorporates covariates is represented by $\lambda(\overline{z(t)})$. The additive hazard model is then:

$$h(t,\overline{z(t)}) = h_0(t) + \lambda(\overline{z(t)}) \qquad (6.1.)$$

There are many possibilities for the functional term. The most attractive form of the functional term is a polynomial. In practice the 1st order polynomial or straight line is used most often because of a lack of data availability, i.e. $\lambda(\overline{z(t)}) = \overline{\gamma} \cdot \overline{z(t)}$.

The behavior of the functional form gives the model its flexibility. If the measured covariates cause $\lambda(\overline{z(t)}) \geq 0$, it implies that the covariates have an accelerating effect on the wear out process of the component. A negative $\lambda(\overline{z(t)})$ would mean that the covariates are such that the expected wear out process is decelerated. If $\lambda(\overline{z(t)}) = 0$, the covariates have no additional effect on the wear out process.

Maximum likelihood can be used to estimate model parameters. Pijnenburg also provides a technique with which the additive assumption of the model can be evaluated.

6.2 Comments

Although Pijnenburg has shown the potential of the AHM with Davis'[47] bus engine data and Proschan's[48] aircraft air-conditioning system data, the model is not found very often in the literature. Pijnenburg is of opinion that "the AHM seems to be preferable to the PHM" for the mentioned data sets.

The model has not really been evaluated by the reliability modelling world and it is difficult to judge the model's potential based on publications.

7 Selection of Most Suitable Model

To be able to make an educated decision regarding the most suitable model for this research project, the following evaluation criteria were identified with which the different models could be compared (in order of importance):

(i) Theoretical foundation
(ii) Previous practical successes in reliability modelling
(iii) Potential to lead to the dissertation objectives
(iv) Achievability of numerical implementation
(v) Future potential in reliability modelling

With these criteria, a decision matrix can be constructed where different weights are allocated to the criteria and each model is evaluated with a mark out of 5 according to each criterion. The decision matrix is given on the next page in Table 7.1.

		Regression model				
Criterion	**Weight**	**AFTM**	**PHM**	**PWP**	**POM**	**AHM**
(i) Theoretical foundation	2^5	4	5	3	3	3
(ii) Previous practical successes in reliability modelling	2^4	5	5	1	1	3
(iii) Potential to lead to the dissertation objectives	2^3	4	5	4	1	4
(iv) Achievability of numerical implementation	2^2	4	4	2	4	4
(v) Future potential in reliability modelling	2^1	3	3	5	1	3
	Weighted total:	262	302	162	138	198

Table 7.1.: *Decision matrix*

The decision matrix shows that the proportional hazards model is the most suitable for this project and all research efforts will be focused on this model for the remainder of this dissertation.