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Geometric algebra via sheaf theory: A view towards symplectic geometry

by

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Declaration

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not previously been submitted by me for a degree at this or any other University.

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Date: May 2010

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Thinking of *Chibike*, my beloved son.

Title Geometric algebra via sheaf theory:
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Summary

This book provides detailed insight to the theory of abstract geometric algebra on sheaves in the framework of Abstract Differential Geometry (ADG). This is a new approach to geometric algebra based on sheaf theoretic methods to describe physical theories of geometric character without the use of ordinary algebra. Instead, an axiomatic treatment of geometric algebra is presented via sheaf theory (geometry).

Starting with a brief survey of the required sheaf theory, the exposition then moves on to pairings of sheaves of $\mathcal{A}$-modules (the abstraction of pairings of vector spaces): orthogonally convenient $\mathcal{A}$-pairings and theorems on ranks of free $\mathcal{A}$-modules, biorthogonality in pairings of $\mathcal{A}$-modules (the abstraction of biorthogonality in pairings of vector spaces), universal property of quotient $\mathcal{A}$-modules; and Witt's hyperbolic decomposition theorem for $\mathcal{A}$-modules (the abstraction of Witt's hyperbolic decomposition theorem for vector spaces). Having laid the basic groundwork, the main part of this work is devoted to the theory of symplectic and orthogonal geometry with their structures. A characterization of $\mathcal{A}$-transvections, in terms of $\mathcal{A}$-hyperplanes is given together with the associated matrix definition by taking the domain of the coefficients $\mathcal{A}$ to be a PID-algebra sheaf. Important topics such as affine Darboux theorem, orthosymmetric $\mathcal{A}$-bilinear forms, special features of orthogonal geometry, Witt's extension theorem, symplectic orthogonally convenient $\mathcal{A}$-modules, Lagrangian sub-$\mathcal{A}$-modules,

symplectic $\mathcal{A}$-transvections, and $\mathcal{A}$-symplectomorphisms as products of symplectic $\mathcal{A}$-transvections for vector spaces are also treated within the context of ADG.

In the course of our investigation, this study formulates more interesting results about the PID sheaf $\mathcal{A} \equiv \mathcal{C}^{\infty}_{X}$ which is not the requirement for manifolds in Classical Differential Geometry (CDG).

The book contains a wealth of detailed and interesting but rigorous computations which will appeal to researchers in mathematics, physicists, advanced undergraduate students and graduate students interested in applications of differential geometry to physical theories.

Preface

Several aspects of differential geometry with exposition to the treatment of classical geometry of $\mathcal{C}^{\infty}$-manifolds range from the standard theory of infinite-dimensional $\mathcal{C}^{\infty}$-manifolds to the more sophisticated aspects of "differential geometry" on topological spaces have developed into great independent theories. Linear algebra, topology, differential, algebraic geometry and algebraic topology are indispensable tools of the mathematician of our time.

The notion and theory of $\mathcal{A}$-module on an arbitrary topological space X are obtained essentially by a generalization of *sheaves of* $\mathbb{C}$-*vector spaces* (or else a $\mathbb{C}$-*vector space sheaves*) on X. The space X plays the secondary rôle as the carrier of the generalized "smooth" functions, which can be thought of as the sections of the structure sheaf $\mathcal{A}$, whereas the entire differential-geometric apparatus lives in $\mathcal{A}$. We refer to A. Mallios – E. E. Rosinger [58] for an application of these ideas to the multi-foam algebra, in problems of non-linear PDE's. Now, concerning the relevance of the foregoing to the classical theory of differential geometry, referred to as smooth viz., $\mathcal{C}^{\infty}$-manifolds (CDG), $\mathcal{A} \hookleftarrow_{\varepsilon} \mathbb{C}$ is expressed in terms of smooth differentiable $\mathcal{C}^{\infty}$-functions, sections of the respective classical "structure sheaf", (see, [57]), of the theory:

$$\mathcal{A} \equiv \mathcal{C}^{\infty}_{X}.$$

Hence, our axiomatic approach uses the basis tools of *sheaves of mod-*

ules on X over an appropriate *sheaf of* $\mathbb{C}$-*algebras* (alias, $\mathbb{C}$-*algebra sheaf*) by analogy with the classical case. Our study is quite general and essentially of an *algebraic topological* nature based on *sheaf theory.* Thus, it contains very special results from the classical aspect of finite and, partly, of infinite dimensional differential geometry of $\mathcal{C}^{\infty}$-manifolds with several other generalizations. More beautifully, each chapter of this thesis ushers in the reader with a very good brief introduction of what follows.

In order to make the whole treatment of "*Geometric Algebra via Sheaf Theory: A View Towards Symplectic Geometry*", as self contained as possible in Chapter I, we cover nearly all elements from sheaf and module theory for embarkation on advanced courses, specializing to the pertinent notions of *Sheaves and Presheaves* with *Algebraic Structures.* This is the type of sheaves we exclusively consider throughout the sequel. Concerning the prerequisite algebraic background for this, we mention that any standard course on algebra, groups, rings, fields, module and linear algebra will suffice. Most importantly, a good background on topological spaces at under graduate level is needed. Although we have kept the discussion as self-contained as possible, there are places where references to standard results are unavoidable; readers who are unfamiliar with such results should consult a standard text on abstract algebra and sheaf theory.

In Chapter II, we examine the theorems on ranks of free $\mathcal{A}$-modules including that for it's quotient. Next, we develop the foundations of pairings of sheaves of $\mathcal{A}$-modules. This situation serves as another example of the motto that says *"any good class of functions can be represented as continuous cross sections of an appropriate sheaf".* The $\mathcal{A}$-pairings are eventually *orthogonally convenient,* so that orthogonal reflexivity for free $\mathcal{A}$-modules can be applied; we further present this important case, to the extent that is necessary for the ensuing discussion. Furthermore, having in mind an application given in Chapter III, we also examine in the same the necessary material for *Biothorgonality.*

We apply the fundamental structure theorems to obtain important decomposition theorems for $\mathcal{A}$-modules in Chapter III.

In Chapter IV, we discuss in detail the structure of orthosymmetric pairings (alias, bilinear morphisms), ϕ, on modules over a *unital commutative torsion-free* $\mathbb{C}$*-algebra sheaf* $\mathcal{A}$ to obtain the only two types of geometry on ϕ; viz, *orthogonal and symplectic geometry.* Further detailed observation and application leads to special features of orthogonal geometry with further assumption that $\mathcal{A}$ be a PID $\mathbb{C}$-algebra sheaf.

Finally, in Chapter V, transvections in terms of $\mathcal{A}$-hyperplanes, and special features of symplectic $\mathcal{A}$-modules in the setting of *Abstract Geometric Algebra* is also given over an ordered $\mathbb{C}$-algebraized space $(X, \mathcal{A}, \mathcal{P})$. Also, analogue of Witt's extension theorem concerning $\mathcal{A}$-symplectomorphisms defined of appropriate Lagrangian sub-$\mathcal{A}$-modules is given, extensively.

This serves as a further application of the respective classical notion in finite-dimensional geometry *linear* and *multilinear* within the present abstract geometric framework.

Pretoria A.C. Anyaegbunam
May 2010

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To God be the glory

Contents

Chapter 1

Introduction

Sheaves were introduced more than 60 years ago by Jean Leray and their major useful virtue is that they unify and give a mechanism for dealing with many problems concerned with passage from local information to global information. This is very useful when dealing with, say, differential manifolds, since locally these look like Euclidean spaces, and hence localized problems can be dealt with by means of all tools of classical analysis. "Piecing together "solutions" of such local problems in a coherent manner to describe, e.g. global invariants, is most easily accomplished via sheaf theory and its associated cohomology theory". Most problems could be phrased and perhaps solved without sheaf theory, but notation would be enormously more complicated and difficult to comprehend, (see Wells [75]).

In this chapter we shall introduce the basic concepts of sheaves and presheaves to beginners, and shall give many of the fundamentals to be used throughout the thesis. Our major point of reference to the definitions is mostly from [50]. To write a few lines of introduction on a real corner stone of mathematics like sheaf theory is not an easy task. So, let us try with ... this introduction [67].

1.1 First Introduction: for students(and everybody else).

In the study of ordinary differential equations, when you face a Cauchy problem of the form

$$\{y^{(n)} = f(x, y, y', ..., y^{(n-1)}), y^{(i)}(x_0) = y_0^{(i)}\},$$

you know that the continuity of f is enough to get a local solution, i.e., a solution defined on an open neighbourhoood U_{x_0} of x_0. But, to guarantee the existence of a global solution, the stronger Lipschitz condition on f is required.

In complex analysis, we know that a power series $\sum a_n(z - z_0)^n$ uniformly converges on any compact space strictly contained in the interior of the convergence disc. This is equivalent to local uniform convergence: for any z in the open disc, there is an open neighbourhood U_z of z on which the series converges uniformly. But local uniform convergence does not imply uniform convergence on the whole disc. This gap between local uniform convergence and global uniform convergence is the reason why the theory of Weierstrass analytical functions exists.

These are only two simple examples, which are part of everybody's basic knowledge in mathematics, of the passage from local to global. *Sheaf theory is precisely meant to encode and study such a passage.*

Sheaf theory has its origin in complex analysis (see, for example, [26]) and in the study of cohomology of spaces [7] (see also, [30] for a historical survey of sheaf theory). Since local-to-global situations are pervasive in Mathematics, nowadays, sheaf theory deeply interacts also with mathematical logic [4, 29, 42, 48], algebraic geometry

[31, 32, 35, 36, 37], algebraic group theory [21], ring theory [28, 71], homological algebra [19, 27, 74] and, of course, category theory [44].

The references mentioned above are not at all exhaustive. Each item is a standard textbook in the corresponding area and the reader probably has already been in touch with some of them. We have listed them here because, just by having a quick glance at them, one can realize that sheaves play a relevant (sometimes crucial) role. In this way, we have no doubt that the reader will find motivations to attack sheaf theory directly from his favorite mathematical point of view.

1.2 Sheaves. Basic Notions

We begin this section with the fundamentals of sheaf theory that we need in the sequel. We also consider sheaves with algebraic structure in their fibres, as we will be working with these algebraic structures throughout this discussion, as well as, the basic geometries. Apart from the standard sources where this material can be found, we also refer to [50, Chapters I-II] for a full, self-contained and easy reading thereof, to the extent that this is needed in the present treatise.

We say that a map

$$\pi : \mathcal{E} \longrightarrow X \tag{1.1}$$

between topological spaces $\mathcal{E}$ and X is a sheaf, whenever π is a *local homeomorphism*. Explicitly, we mean that, for every $z \in \mathcal{E}$, there exists an (open) neighborhood V of z in $\mathcal{E}$, with $\pi(V)$ an (open) neighborhood of $\pi(z) \equiv x$ in X and such that the restriction of π to V

$$\pi|_V : V \longrightarrow \pi(V) \tag{1.2}$$

is a homeomorphism. It is easy to see that any local homeomorphism is a *continuous* and *open map*.

Example 1: Let $\mathcal{E} = (0, 1)$ in $\mathbf{R}$ equipped with the usual topology and $\mathcal{S}' = X$ be the unit circle in $\mathbf{R}^2$ with the induced topology. Define $\pi : \mathbf{R} \longrightarrow \mathcal{S}'$ by $\pi(z) = (cosz, sinz) = x \in \mathcal{S}'$. Then for every $z \in (0, 1)$, there exists an open set U containing z such that $\pi(U)$ is open in $\mathcal{S}'$ and $\pi|_U : U \longrightarrow \pi(U)$ is a homeomorphism.

Example 2: Let $\mathcal{E} = (-1, 0) \cup (0, 1)$ and $X = (0, 1)$ all with the usual topology on $\mathbf{R}$. Define $\pi : \mathcal{E} \longrightarrow X$ by $\pi(z) = |z| = x \in X$. Clearly, π is a local homeomorphism: let $z \in \mathcal{E}$ then we have two cases. Case 1: If $z \in (-1, 0)$, then $\pi|_{(-1,0)}$ is a homeomorphism. Case 2: If $z \in (1, 0)$, then $\pi|_{(1,0)}$ is a homeomorphism.

In this context, instead of π, we refer just to $\mathcal{E}$, by simply saying that $\mathcal{E}$ is a *sheaf over* X, the latter being now the *base space* and $\mathcal{E}$ the *sheaf space*, while π is still called the *projection map*.

On the other hand, given an element $x \in X$ (we assume, of course, that $x \in \text{im } \pi$), the set

$$\mathcal{E}_x := \pi^{-1}(\{x\}) \equiv \pi^{-1}(x) \tag{1.3}$$

is called the *fiber(or even stalk)* of $\mathcal{E}$ at $x \in X$. Hence, $\mathcal{E}$ is displayed as a disjoint union of its stalks, the relative topology from $\mathcal{E}$ on each fiber will then be the discrete one, by the hypothesis for π.

From Example 2 above, the $\mathcal{E}_x = \{-x, x\}$.

By the foregoing we have singled out a certain category, which is thus associated with a given topological space X, i.e., the *category of sheaves over* X, denoted in the following by

$$\mathcal{S}h_X.$$

The *objects* of $\mathcal{S}h_X$ are triples

$$(\mathcal{E}, \pi, X),$$

with π as in (1.1), while the *morphisms of* $\mathcal{S}h_X$ are the continuous maps between any two given objects, say, $(\mathcal{E}, \pi, X)$ and $(\mathcal{F}, \rho, X)$ in $\mathcal{S}h_X$ such that the maps are "fiber preserving"' in the sense that

$$\rho \circ \phi = \pi$$

or equivalently,

$$\phi(\mathcal{E}_x) \subseteq \mathcal{F}_x, \; x \in X \; .$$

Thus, the last relation defines, for every $x \in X$, a corresponding *restriction* map on each fiber of $\mathcal{E}$

$$\phi_x := \phi|_{\mathcal{E}_x} : \mathcal{E}_x \longrightarrow \mathcal{F}_x, \; x \in X, \tag{1.4}$$

so that the given continuous map ϕ, as in (1.4), may also be construed as a family of maps $(\phi_x)_{x \in X}$, with $\phi_x : \mathcal{E}_x \longrightarrow \mathcal{F}_x, x \in X$, in such a way that the resulting map $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ defined by the relation

$$\phi(z) := \phi_x(z), \text{ with } z \in \mathcal{E}_x \subseteq \mathcal{E} \text{ and } \pi(z) = x$$

be continuous. It is proved that a *sheaf morphism* $\phi : \mathcal{E} \longrightarrow \mathcal{F}$, displays $\mathcal{E}$ again as a sheaf over $\mathcal{F}$, that is, ϕ is still a *local homeomorphism*. On the other hand, ϕ is *one-to-one* and/or *onto* if, and only if, this holds true *fiber-wise* (viz. for each one of the maps (1.4)).

Now, a continuous map $s : U \longrightarrow \mathcal{E}$ over an open $U \subseteq X$ is called a *continuous local section* of a given sheaf $\mathcal{E}$ over X, whenever, we have

$$\phi \circ s = id_U = id_X|_U. \tag{1.5}$$

When $U = X$, one then speaks of a *continuous global section of* $\mathcal{E}$. We denote the corresponding sets of sections of $\mathcal{E}$, as above, by $\Gamma(U, \mathcal{E}) \equiv \mathcal{E}(U)$ and $\Gamma(X, \mathcal{E}) \equiv \mathcal{E}(X)$, respectively.

A *continuous section* s, as before, is always an (one-to-one, any section, whatsoever, here of π, according to (1.5), is one-to-one) open map, defining thus a *(local) homeomorphism* of the open set $U \subseteq X$ onto the open set $s(U) \subseteq \mathcal{E}$. Furthermore, sets of the form $s(U)$, where U runs over the open subsets of X and $s \in \Gamma(U, \mathcal{E})$, provide a basis of the topology of $\mathcal{E}$. On the other hand, for every $z \in \mathcal{E}$, there exists an open $U \subseteq X$, with $\pi(z) = x \in U$, and a section $s \in \Gamma(U, \mathcal{E})$, such that

$$z = s(x) \ (\equiv s_x) \ . \tag{1.6}$$

In this regard, we usually call $s(x) \equiv s_x$ *the germ of s at x* (terminology that will be justified later on), so that, in view of (1.6), one can also speak of $\mathcal{E}$, as the *sheaf of sections of germs of its sections*; here it is important to notice that if two local sections of $\mathcal{E}$ agree (viz. take the same value) at a point, then *they also do so on a whole (open) neighborhood of that point.*

1.3 Presheaves. Sheafification

A presheaf on a topological space X is a variable set indexed by the open subsets of X. More precisely, it is a contravariant functor

$$S : \mathrm{O}pen(X)^{op} \longrightarrow \mathrm{Sets},$$

where $\mathrm{O}pen(X)^{op}$ is the ordered set of open subsets of X considered as a category and the Sets is the category of sets.
The intuition comes from the case where $S(U)$ is the set of smooth (in some sense) functions defined on U an open subset of the topological space X, while an inclusion $V \subseteq U$ gives map $S(U) \longrightarrow S(V)$ which restricts a function on U to one on V.
Think, also, of the presheaf of continuous functions $\mathcal{C} : \mathrm{O}pen(X)^{op} \longrightarrow \mathrm{Sets}$; and let $\mathcal{C}(U)$ denote the set of all continuous real-values functions $h : U \longrightarrow \mathbf{R}$; the assignment $h \mapsto h|_V$ restricting each h to the

subset V is a function $\mathcal{C}(U) \longrightarrow \mathcal{C}(V)$ for each $V \subseteq U$. This makes $\mathcal{C}$ a contravariant functor on $Open(X)^{op}$ to Sets. This presheaf of $\mathbf{R}$-valued continuous functions on X is called the *sheaf of germs* $\mathcal{C}_X$ of continuous $\mathbf{R}$-valued functions on X.
Roughly speaking, a presheaf S is a sheaf when we can move from local elements to global elements, i.e., when we can paste together (compatible) elements $\{f_i \in S(U_i)\}_I$ to get a unique element $f \in S(\bigcup_I U_i)$. The above-mentioned presheaf $\mathcal{C}$ is a sheaf, whereas the presheaf $\mathcal{K}$ is not a sheaf. The first important result we want to discuss is the fact that the abstract notion of sheaf can be concretely represented by variable sets of the form "continuous functions“. More precisely, any sheaf is isomorphic to the sheaf of continuous sections of a suitable *é*tale map (= a local homoemorphism).

There is another equivalent way of giving the notion of a sheaf that also proves, several times more tractable, while the first definition given above is from a historical point of view, (in this respect, see the so-called *"sheaf of coefficients"*, Leray, Cartan, Oka, or even Weiestrass). Thus, one defines, firstly, a *presheaf* of sets on a topological space X, as contravariant functor from the category of open subsets of X to that one of the sets; so to every open $U \subseteq X$, one associates a set $E(U)$ and to any pair $U \supseteq V$ of open sets in X a ("restriction") map $\rho_V^U : E(U) \longrightarrow E(V)$, such that $\rho_U^U = id_U$ and $\rho_W^U = \rho_W^V \circ \rho_V^U$, for any open sets $U \supseteq V \supseteq W$ in X. Such a presheaf is said to be, a *complete presheaf* of sets, whenever (i) for any open $U \subseteq X$ and open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of U, two elements (*"sections"*) in $E(U)$, with $s_\alpha := \rho_{U_\alpha}^U(s) \equiv s|_{U_\alpha} = t|_{U_\alpha} \equiv t_\alpha := \rho_{U_\alpha}^U(t), \alpha \in I$, are in fact equal, viz. we then have that $s = t$, and (ii) with U and U_α, as before, if $(s_\alpha) \in \prod_\alpha E(U_\alpha)$, with $s_\alpha = s_\beta$, for any $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ (viz. $\rho_{U_{\alpha\beta}}^{U_\beta}(s_\alpha) = \rho_{U_{\alpha\beta}}^{U_\beta}(s_\beta)$), then $s_\alpha = \rho_{U_\alpha}^U(s)$, for some uniquely (due to (i)) defined $s \in E(U)$.

Thus, the *sections of a sheaf* (local homeomorphism), as defined above, over the open subsets of X *constitute a complete presheaf*, while this is, in effect a characteristic property, namely, the only way (up to an isomorphism) that complete presheaves arise; that is, one proves that any given *complete presheaf* of sets on a topological space X is isomorphic (we explain right below the definition of the later concept) to that *one defined by the sections of a sheaf* on X (cf., for instance [50, Chapt. I, Theorem 11.2]). Now, the latter sheaf, called also the *sheafification* of the given presheaf (completeness is not necessary to define the sheaf, we are seeking) is provided by the *fibers of the given presheaf*; thus, for every $x \in X$, we define

$$E_x := \varinjlim_{U \in \mathcal{V}(x)} E(U) \ . \tag{1.7}$$

(Here, when U varies over a basis $\mathcal{V}(x)$ of open neighborhoods of x, the sets $E(U)$ of the given presheaf, due to its very definition, provide an *inductive system of sets*, cf. Bourbaki ([8, p. 88]), hence, (1.7). Therefore, we further define the sheaf

$$\mathcal{E} := \sum_{x \in X} E_x, \tag{1.8}$$

with the obvious projection map onto X, as $\pi(\mathcal{E}_x) := x$, $x \in X$, while a *basis of the topology of* $\mathcal{E}$ is given by the family

$$\mathcal{B} := \{\tilde{s}(V) : s \in \mathcal{E}(U), \text{ for any open } V \subseteq U\} \tag{1.9}$$

such that

$$\tilde{s}(x) := \rho_U^x(s), \text{ with } s \in E(U) \text{ and } x \in U, \tag{1.10}$$

where $\rho_U^x : E(U) \longrightarrow E_x \subseteq \mathcal{E}$ is the corresponding canonical map provided by (1.7). Indeed, the *topology of* $\mathcal{E}$ is the *strongest* one (*"final topology"), making the maps* $\tilde{s} : U \longrightarrow \mathcal{E}$, for the various open $U \subseteq X$ and $s \in \mathcal{E}(U)$, *continuous*, in this regard, cf. [50, Chapt. I; (7.12) and Theorem 3.1].

Thus, we have an *identification* of the notions of *sheaves* and *complete presheaves*, this correspondence is supplied by the *section functor* Γ and the *sheafification functor* $\mathbf{S}$, respectively; yet, each one of these two functors is adjoint to the other, so that we have the following *isomorphism of sets* (cf. also (1.15) below for the notation applied)

$$\mathrm{Hom}_{\mathcal{S}h_X}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{C}o\mathcal{P}\mathcal{S}h_X}(\Gamma(\mathcal{E}), \Gamma(\mathcal{F})), \tag{1.11}$$

for any two given sheaves $\mathcal{E}$, $\mathcal{F}$ on X. In this context, we define a *morphism* between two given *presheaves* of sets on X, say,

$$E \equiv (E(U), \rho_V^U) \text{ and } F \equiv (F(U), \lambda_V^U), \tag{1.12}$$

as a family of maps

$$\phi \equiv (\phi_U) \in \prod_U \mathrm{Hom}(E(U), F(U)), \tag{1.13}$$

with U ranging over the open sets in X, such that we have

$$\lambda_V^U \circ \phi_U = \phi_V \circ \rho_V^U, \tag{1.14}$$

for any open sets $U \supseteq V$ in X. Thus, a morphism of presheaves is an *injection, or surjection, or a bijection* (viz. *isomorphism*), respectively, if this is the case for each one of the individual maps ϕ_U, as above. The notation "Hom" in (1.13) stands for *sets of maps* between the pertinent sets, while that one in (1.11) stands for *sets of morphisms* of the respective objects of the categories involved, namely, categories of sheaves, $\mathcal{S}h_X$, and categories of complete presheaves, $\mathcal{C}o\mathcal{P}\mathcal{S}h_X$, both on X. In this concern, it is very important to note that in the case of a given *complete presheaf* E on X (cf. (1.12)) and the respective *complete presheaf of sections of its sheafification* $\mathbf{S}(E) \equiv \mathcal{E}$, denoted henceforth by

$$\Gamma(\mathcal{E}) \equiv (\mathcal{E}(U), \tau_V^U), \tag{1.15}$$

one has the following canonical *isomorphisms* (set-theoretic bijections)

$$\rho_U : E(U) \longrightarrow \mathcal{E}(U) \equiv \Gamma(U, \mathcal{E}), \tag{1.16}$$

for any open $U \subseteq X$ (by setting $\rho_U(s) := \tilde{s}$, $s \in E(U)$; cf. (1.10) above). Truly, the previous isomorphisms constitute, another *characterization of making* the given presheaf E *complete* (see e.g., [50, Chapt. I; p. 51, Proposition 11.1]). (We refer also to [72], [73], [69], [70], [11], [46], [30], [23], [22], [13], and [67] e.t.c.)

The category $\mathcal{C}o\mathcal{PS}h_X$ of complete presheaves on X is a *full subcategory* of the category $\mathcal{PS}h_X$ of presheaves on X. The $\mathcal{C}o\mathcal{PS}h_X$ is *reflective* in $\mathcal{S}h_X$ since the section (inclusion) functor $\Gamma : \mathcal{S}h_X \longrightarrow \mathcal{C}o\mathcal{PS}h_X$ which is *fully faithful and essentially surjective* has a left adjoint $\mathbf{S} : \mathcal{C}o\mathcal{PS}h_X \longrightarrow \mathcal{S}h_X$. The functor $\mathbf{S}$ may be called a *reflector*, and it preserves finite limits.

This makes sheaves "behave like sets", allows considering $\mathcal{C}o\mathcal{PS}h_X$ as a model of set theory without the axiom of choice in an appropriate non-classical logic, and therefore suggests to develop "sheaf-based mathematics", where all sets are replaced with sheaves over a fixed space – good comment by G. Janelidze, external examiner.

1.4 Sheaves and Presheaves with Algebraic Structures

The sheaves and presheaves that will be encountered in the sequel will have algebraic structure in their fibers, usually the one of a $\mathbb{C}$-*algebra*. Thus, as a generic example of such an object, we consider first that one of *sheaf of groups* (or else *group sheaf*) $\mathcal{G}$ on a topological space X. This means that $\mathcal{G} \equiv (\mathcal{G}, \pi, X)$ is a sheaf on X whose fibers are groups, in such a manner that *"the group operation is continuous"* or by taking the *"fiber product"* $\mathcal{G} \times_X \mathcal{G} := \{(z, z') \in \mathcal{G} \times \mathcal{G} : \pi(z) = \pi(z')\} \equiv \mathcal{G} \circ \mathcal{G}$, the map $\mathcal{G} \circ \mathcal{G} \ni (z, z') \longmapsto z + z' \in \mathcal{G}_x \subseteq \mathcal{G}$, with $\pi(z) = \pi(z') = x \in X$, is continuous. Yet, ones proves that the operations of *"taking opposites"* and that of *subtraction* are also *continuous maps*, *"fiber-wise"*, as above (for more details see for instance, [50, Chapt. II; Lemma 1.2]). Furthermore, one defines the *zero-section* of $\mathcal{G}$, $x \longmapsto 0_x \in \mathcal{G}_x$ (viz. the neutral element of $\mathcal{G}_x, x \in X$), which is, in fact, *continuous*. Hence, its image, denoted just by 0 is a *subsheaf of* $\mathcal{G}$, that is, an *open subset of* $\mathcal{G}$, (also see e.g., [50, Chapt. II; Lemma 1.1]).

Analogously, one defines a *sheaf of rings or even ring sheaf (with identity)* on X, as being a sheaf of (abelian) groups on X, whose fibers are (unital) rings, such that the ring *multiplication* is *"continuous"* in the above sense, while the *identity section*, $x \longmapsto 1_x, x \in X$, is continuous, defining thus a *global continuous section*. In the same manner one defines a *sheaf of* $\mathbb{C}$*-algebras*, or even $\mathbb{C}$*-algebra sheaf* $\mathcal{A} \equiv (\mathcal{A}, \tau, X)$ on X (with an *identity* or not), as being a sheaf of rings on X, such that the *"scalar multiplication"* $\mathbb{C} \times \mathcal{A} \ni (\lambda, z) \longmapsto \lambda \cdot z \in \mathcal{A}_x \subseteq \mathcal{A}$ be *continuous*, where $x = \tau(z) \in X$, while $\mathbb{C}$ carries the *discrete topology*.

A basic notion for the sequel that emerges here is thus that one of a pair

$$(X, \mathcal{A}), \tag{1.17}$$

consisting of a topological space X and a $\mathbb{C}$-*algebra* sheaf on X, such that the stalks $\mathcal{A}_x, x \in X$ are, particularly, associative commutative and unital $\mathbb{C}$-algebras; we shall call hence forth such a pair $\mathbb{C}$-*algebraized space*.

On the other hand, given a $\mathbb{C}$-algebra sheaf $\mathcal{A} \equiv (\mathcal{A}, \tau, X)$ as before, an $\mathcal{A}$-*module* $\mathcal{E}$ on X (called also *sheaf of* $\mathcal{A}$-*modules*) is a sheaf $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ whose individual stalks $\mathcal{E}_x, x \in X$, are, particularly, $\mathcal{A}_x$-modules of abelian groups on X, such that the *"exterior module multiplication"* $\mathcal{A} \circ \mathcal{E} \ni (\alpha, z) \longmapsto \alpha \cdot z \in \mathcal{E}_x \subset \mathcal{E}$ is also continuous, where $\tau(\alpha) = \pi(z) = x \in X$.

Now, if we have a sheaf, as above, equipped with a particular algebraic structure, this is actually inherited (point-wise) on any individual set of local sections of the sheaf over an open set of the base space. Thus, for any $\mathbb{C}$-algebra sheaf $\mathcal{A}$, as before, and every open $U \subseteq X$, the set $\mathcal{A}(U) \equiv \Gamma(U, \mathcal{A})$ is, in fact, a (unital commutative associative) $\mathbb{C}$-*algebra*. Indeed, a given sheaf of sets $\mathcal{E}$ on a *topological space* X is endowed with a *particular algebraic structure* if, and only if, this is the case for its *complete presheaf of sections* $\Gamma(\mathcal{E})$ on X (see [50, Chapt.II; (1.67) p. 104]).

Hence, one actually proves, in general, that if a *given presheaf carries some* particular *algebraic structure* in the sense that individual sets carry it, then this passes over to the corresponding stalks and also to its *sheafification*. In particular, if

$$A \equiv (A(U), \tau_V^U) \textit{ and } E \equiv (E(U), \rho_V^U) \tag{1.18}$$

is a *presheaf of (unital)* $\mathbb{C}$-*algebras and of* $A(U)$-*modules on* X, respectively, then by considering the corresponding *sheafifications* $\mathcal{A} \equiv \mathbf{S}(A)$ and $\mathcal{E} \equiv \mathbf{S}(E)$, we conclude that $\mathcal{E}$ is an $\mathcal{A} -$ *module* on X (loc. cit. [50, Chapt. II; Proposition 1.1, p. 104]).

Now, by considering *morphisms of sheaves with algebraic structures*, we require, in addition to the usual definition of such maps, as above, that they also *preserve the algebraic structures* concerned *"fiber-wise"*. Thus, given a morphism $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ between $\mathcal{A}$-modules, we can consider the $\mathcal{A}$-*modules* im $\phi \subseteq \mathcal{E}$ and ker $\phi \subseteq \mathcal{F}$.

We also speak of such a morphism ϕ as an $\mathcal{A}$-*morphism*.

In the case of *presheaves with algebraic structures*, the corresponding *morphisms* (of presheaves, cf. (1.13)) are decreed to *preserve the algebraic structure* concerned *via* each one of their *constituent maps*.

1.5 Vector Sheaves. $\mathcal{H}om_{\mathcal{A}}$

Given the $\mathcal{A}$-*modules* $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{F} \equiv (\mathcal{F}, \rho, X)$, we define their *Whitney* or *direct sum* by the relation

$$\mathcal{E} \oplus \mathcal{F} := \{(z, z') \in \mathcal{E} \times \mathcal{F} : \pi(z) = \rho(z')\}.$$

Therefore, the map $\sigma : \mathcal{E} \oplus \mathcal{F} \ni (z, z') \longmapsto \sigma(z, z') := \pi(z) = \rho(z') \in X$ is a *local homeomorphism*, such that we have, fiber-wise, $(\mathcal{E} \oplus \mathcal{F})_x = \mathcal{E}_x \oplus \mathcal{F}_x$, $x \in X$, hence, $\mathcal{E} \oplus \mathcal{F}$ is an $\mathcal{A}$-*module* on X too. Thus, a given $\mathcal{A}$-module $\mathcal{E}$, as before, is said to be a *free* $\mathcal{A}$-*module of finite rank* $n \in \mathbf{N}$, whenever we have

$$\mathcal{E} = \mathcal{A}^n,$$

within an $\mathcal{A} -$ *isomorphism*, where $\mathcal{A}^n$ stands for the *n-fold Whitney sum* of $\mathcal{A}$ with itself. Hence, we set the following basic

Definition 1.1 *A given $\mathcal{A}$-module $\mathcal{E}$ on X is said to be locally free of finite rank $n \in \mathbf{N}$, whenever for any point $x \in X$, there exists an (open) neighborhood U of x, such that we have the following $\mathcal{A}|_U$-isomorphisms;*

$$\mathcal{E}|_U = \mathcal{A}^n|_U = (\mathcal{A}|_U)^n. \tag{1.19}$$

We shall also write for the *rank of* $\mathcal{E}$, $n = \operatorname{rank}_{\mathcal{A}} (\mathcal{E}) \equiv \operatorname{rank} \mathcal{E}$. Henceforth, a locally free sheaf of finite rank will be called a *vector* *sheaf* on X. In particular, if the rank is 1, then our vector sheaf is named a *line sheaf* on X.

Thus, given a vector sheaf $\mathcal{E}$ on X, with rank $\mathcal{E} = n$, we actually obtain an open covering of X, on the individual members of which (1.19) holds true. We call such an open covering of X a *local frame* of $\mathcal{E}$. Furthermore, every open set $U \subseteq X$ of the covering in question, for which (1.19) is valid, is called a *local gauge of* $\mathcal{E}$.

On the other hand, for any two $\mathcal{A}$-*modules* $\mathcal{E}$ and $\mathcal{F}$ on X, we define the *sheaf of germs* of $\mathcal{A}$-*morphisms* of $\mathcal{E}$ in $\mathcal{F}$, denoted by $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$, as the sheaf on X, generated by the complete presheaf

$$U \longmapsto Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{F}|_U), \ \textit{for any open } U \subseteq X.$$

In this context, one obtains the following $\mathcal{A}|_U$-*isomorphism*

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})|_U = \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{F}|_U), \ \textit{for any open } U \subseteq X.$$

Thus, for two *vector sheaves* $\mathcal{E}, \mathcal{F}$ on X, $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is a *vector sheaf* on X too, with *rank the product of those of the factors.*

Of particular importance for the sequel, is the sheaf on X, that is provided when, for any given $\mathcal{A}$-*module* $\mathcal{E}$ on X, we consider its corresponding *endomorphism* $\mathcal{A}$-*algebra sheaf* on X, defined by

$$\mathcal{E}nd_{\mathcal{A}}\, \mathcal{E} \equiv \mathcal{E}nd\, \mathcal{E} := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}).$$

In particular, if $\mathcal{E}$ is a *vector sheaf* on X, with $\mathrm{rank}_{\mathcal{A}}\mathcal{E} = n \in \mathbf{N}$, then $\mathcal{E}nd\, \mathcal{E}$ is a vector sheaf too, with *rank* n^2. Moreover, *the same* $\mathcal{A}$-*algebra sheaf* $\mathcal{E}nd\, \mathcal{E}$ is *self dual*, a property that exhibits, otherwise, a *free* $\mathcal{A}$-*module* on X. On the other hand, in the case of a *line sheaf* $\mathcal{L}$ on X, the above $\mathcal{A}$-algebra sheaf is ($\mathcal{A}$-isomorphic to) $\mathcal{A}$ itself, viz. we have

$$\mathcal{E}nd\, \mathcal{L} = \mathcal{A}. \tag{1.20}$$

When considering (continuous) sections of the above, one defines

$$Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(X).$$

In particular, we thus set;

$$End\ \mathcal{E} := (\mathcal{E}nd\ \mathcal{E})(X) \text{ and } (\mathcal{E}nd\ \mathcal{L})(X) \equiv End\ \mathcal{L} = \mathcal{A}(X),$$

the last equality which is an $\mathcal{A}(X)$-isomorphism is valid for $\mathcal{L}$, a given *line sheaf* on X (cf. (1.20)).

1.6 Quotients of $\mathcal{A}$-modules

Given two $\mathcal{A}$-modules $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{F} \equiv (\mathcal{F}, \rho, X)$ on a topological space X such that $\mathcal{E}$ is a *sub-$\mathcal{A}$-module* of $\mathcal{F}$, let $\mathcal{S}$ be the sheaf generated by the presheaf $U \to \mathcal{F}(U)/\mathcal{E}(U)$, for each U open in X . Then $\mathcal{S}$ is called the *quotient sheaf of* $\mathcal{F}$ by $\mathcal{E}$ and is denoted by $\mathcal{F}/\mathcal{E}$.

1.7 Riemannian $\mathcal{A}$-metrics

Now, according to our general pattern, every classical function that is of importance to us should be replaced by a section of an appropriate sheaf; "*any good class of functions can be represented simply as continuous cross sections*" (S. Mac Lane [47, p. 357]). "In particular, if the classical function involved is "numerical", in our case the corresponding section should take values from our "*arithmetic*", that is, it has to be a section of our *structure sheaf* $\mathcal{A}$" (Mallios [49]).

We assume from here onwards, even if it is not explicitly stated, that we are given a fixed ordered $\mathbb{R}$-algebraized space

$$(X, \mathcal{A} := \mathcal{P} \cup -\mathcal{P})$$

with $\mathcal{A}$ a sheaf of $\mathbb{R}$-algebras, commutative and unital on a given topological space X, and $\mathcal{P} \cup -\mathcal{P} := \mathcal{A} \cup \mathcal{A}^- \subseteq \mathcal{A}$, a subsheaf of $\mathcal{A}$ on X, see ([50], p. 316).

Thus, to provide substance to the latter expression, we introduce the notion of an *ordered algebraized space*, which, by definition, is a $\mathbb{C}$-algebraized space $(X, \mathcal{A})$ for which there also exists a *subsheaf* (viz. an open subset) $\mathcal{P}$ of $\mathcal{A}$, defining a *preorder in* $\mathcal{A}$; that is, one has i) $\lambda \cdot \mathcal{P} \subseteq \mathcal{P}, \lambda \in \mathbf{R}_+ \hookrightarrow \mathcal{A}$, ii) $\mathcal{P} + \mathcal{P} \subseteq \mathcal{P}$ and iii) $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$. The same $\mathcal{P}$ yields a *(partial) order* in $\mathcal{A}$, if, moreover, one has iv) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. (The previous relations are meant *fiber(or section-)wise*). Hence, the elements of $\mathcal{P} \subseteq \mathcal{A}$ are, by definition, considered as the *positive elements of* $\mathcal{A}$, denoting their set by $\mathcal{A}^+ := \mathcal{P}$, and analogously for the *negative elements of* $\mathcal{A}$, by setting $\mathcal{A}^- := -\mathcal{P}$.

Having introduced the latter notion, we come now to the very important case of symmetric $\mathcal{A}$-bilinear morphisms over an ordered $\mathbb{C}$-algebraized space $(X, \mathcal{A} := \mathcal{P} \cup -\mathcal{P})$ which follows from the corresponding formulae of the classical theory of modules (cf. e.g. [1], p. 371; [3], p. 148; [33], p. 97; [9], p. 22; [16], p.17; [43], p. 577; as well as, [6], p. 353.)

Definition 1.2 *Suppose we are given an ordered $\mathbb{C}$-algebraized space $(X, \mathcal{A})$; and let $\mathcal{E}$ be an $\mathcal{A}$-module on X, and suppose ϕ is a symmetric $\mathcal{A}$-bilinear morphism over $(X, \mathcal{A})$ on $\mathcal{E}$, then ϕ is called*

1. ***positive semi-definite;*** *if for any local section $s \in \mathcal{E}(U)$ over an open $U \subseteq X$, we have $\phi(s, s) \in \mathcal{P}(U) \equiv \mathcal{A}^+(U) \subseteq \mathcal{A}(U)$ such that $\phi(s, s) = 0$, if $s = 0$.*

2. ***negative semi-definite;*** *if for any local section $s \in \mathcal{E}(U)$ over an open $U \subseteq X$, we have $\phi(s, s) \in -\mathcal{P}(U) \equiv \mathcal{A}^-(U) \subseteq \mathcal{A}(U)$*

such that $\phi(s,s) = 0$, *if* $s = 0$.

3. ***positive definite***; *if for any local section* $s \in \mathcal{E}(U)$ *over an open* $U \subseteq X$, *we have* $\phi(s,s) \in \mathcal{P}(U) \equiv \mathcal{A}^+(U) \subseteq \mathcal{A}(U)$ *such that* $\phi(s,s) = 0$, *if, and only if* $s = 0$.

4. ***negative definite***; *if for any local section* $s \in \mathcal{E}(U)$ *over an open* $U \subseteq X$, *we have* $\phi(s,s) \in -\mathcal{P}(U) \equiv \mathcal{A}^-(U) \subseteq \mathcal{A}(U)$ *such that* $\phi(s,s) = 0$, *if, and only if* $s = 0$.

A given $\mathcal{A}$-module $\mathcal{E}$ on X equipped with an $\mathcal{A}$-metric ϕ, as above, is said to be a *Riemannian* $\mathcal{A}$*-module* whenever ϕ is *strongly non-degenerate*; this means that

$$\tilde{\phi} : \mathcal{E} \cong \mathcal{E}^* \equiv \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}),$$

up to an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules concerned, the map $\tilde{\phi}$ is defined by ϕ according to the relation;

$$\tilde{\phi}(s)(t) \equiv \phi_s(t) := \phi(s,t),$$

for any $s, t \in \mathcal{E}(U)$ and open $U \subseteq X$.

Now, we are ready for what follows in the following chapters.

Chapter 2

Pairings of sheaves of modules

In this chapter, we generalize the notion of *pairings of vector spaces* to the category $\mathcal{A}$-$\mathcal{M}od_X$ of sheaves of modules over a topological space X. Pairings of vector spaces are studied extensively in [3]; most of our results are modelled on corresponding results in [3]. Particular care is taken when defining *left* or *right kernels* and *orthogonality* of sub-$\mathcal{A}$-modules in pairings of sheaves of modules. Kernels and orthogonal to given sub-$\mathcal{A}$-modules turn out to be sub-$\mathcal{A}$-modules.

In the context of pairings of vector spaces, under certain conditions, orthogonality is *reflexive,* i.e., given a bilinear map $\phi : V \oplus W \longrightarrow \mathbb{K}$, where V and W are $\mathbb{K}$-vector spaces, then if ϕ is *non-degenerate* and G and H subspaces of V and W, respectively, one has: $G^{\perp\top} = G$ and $H^{\top\perp} = H$. It is a standard result, see for instance [15, p. 365, Corollaire 2], [16, p. 6, Proposition 1.2.1], or [33, p. 95, Theorem 1]. For convenience, we recall that $G^{\perp}$ is called the orthogonal of G (with respect to ϕ) and is the subspace of W consisting of all vectors $w \in W$ such that $\phi(G, w) = 0$, and, similarly, $H^{\top}$ is called the orthogonal of H and is the subspace of V made up of all vectors $v \in V$ such that $\phi(v, H) = 0$. It is an open question to see whether this very important classical property of orthogonality is preserved for any non-degenerate bilinear $\mathcal{A}$-morphism; we shall however show that in the subclass of *orthogonally convenient pairings of $\mathcal{A}$-modules,* orthogonality is reflexive for non-degenerate bilinear $\mathcal{A}$-morphisms.

2.1 Theorems on ranks of free $\mathcal{A}$-modules

We shall assume that the reader is familiar with basic properties of module theory as the latter is part of the required background for our dissertation. Our main references for module theory include [1] and [6].

Definition 2.1 Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules on a topological space X and $F : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow$ **Set** the forgetful functor. An $\mathcal{A}$-bilinear form on $\mathcal{E} \oplus \mathcal{F}$ is a sheaf morphism

$$\phi : F(\mathcal{E}) \oplus F(\mathcal{F}) \longrightarrow F(\mathcal{A})$$

satisfying

(1) $\phi_U(\alpha_1 s_1 + \alpha_2 s_2, t) = \alpha_1 \phi_U(s_1, t) + \alpha_2 \phi_U(s_2, t)$, and

(2) $\phi_U(s, \alpha_1 t_1 + \alpha_2 t_2) = \alpha_1 \phi_U(s, t_1) + \alpha_2 \phi_U(s, t_2)$

for any open $U \subseteq X$ and sections $\alpha_1, \alpha_2 \in \mathcal{A}(U)$, $s, s_1, s_2 \in \mathcal{E}(U)$, and $t, t_1, t_2 \in \mathcal{F}(U)$.

Lemma 2.2 *Let $\phi : \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ be an $\mathcal{A}$-bilinear form on $\mathcal{E}$ and $\mathcal{F}$, then ϕ induces an $\mathcal{A}$-valued sheaf morphism (or $\mathcal{A}$-morphism), viz.*

$$\phi^{\mathcal{E}} : \mathcal{F} \longrightarrow \mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}), \tag{2.1}$$

given by

$$\phi^{\mathcal{E}}_U(t)(s) := \phi_V(s, t|_V), \tag{2.2}$$

where U runs over the open subsets of X, $t \in \mathcal{F}(U)$ and $s \in \mathcal{E}(V)$, with V an open subset of U. Likewise, ϕ gives rise to a similar $\mathcal{A}$-morphism

$$\phi^{\mathcal{F}} : \mathcal{E} \longrightarrow \mathcal{F}^*. \tag{2.3}$$

One obtains that

$$\phi^{\mathcal{E}} \in Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{E}^*) \quad and \quad \phi^{\mathcal{F}} \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{F}^*).$$

Proof. Assume that $(\mathcal{F}(U), \sigma_V^U)$ and $(\mathcal{E}^*(U), \kappa_V^U)$ are the (complete) presheaves of sections of $\mathcal{F}$ and $\mathcal{E}^*$, respectively. It is easy to see that

$$\kappa_V^U \circ \phi_U^{\mathcal{E}} = \phi_V^{\mathcal{E}} \circ \sigma_V^U,$$

where U, V are open in X and such that $V \subseteq U$. In fact, for $t \in \mathcal{F}(U)$ and $s \in \mathcal{E}(W)$, where $W \subseteq V$ is an open subset of X, $\kappa_V^U(\phi_U^{\mathcal{E}}(t))(s) = \phi_W(s, t|_W)$. On the other hand, $\phi_V^{\mathcal{E}}(t|_V)(s) = \phi_W(s, t|_W)$. The preceding shows the correctness of our assertion for (2.1). In a similar way, one shows that $\phi^{\mathcal{F}}$ is an $\mathcal{A}$-morphism. ∎

The pair $((\mathcal{E}, \mathcal{F}; \phi); \mathcal{A}) \equiv (\mathcal{E}, \mathcal{F}; \mathcal{A})$ is called a **pairing** of **$\mathcal{A}$-modules** $\mathcal{E}$ and $\mathcal{F}$ or just an **$\mathcal{A}$-pairing** of $\mathcal{E}$ and $\mathcal{F}$. The $\mathcal{A}$-morphisms $\phi^{\mathcal{E}} \equiv \phi^L$ and $\phi^{\mathcal{F}} \equiv \phi^R$ are sometimes called **left** and **right insertion $\mathcal{A}$-morphisms,** respectively.

Theorem 2.3 *If $\mathcal{F}$ and $\mathcal{G}$ are sub-$\mathcal{A}$-modules of an $\mathcal{A}$-module $\mathcal{E}$, then*

$$\mathbf{S}(\Gamma(\mathcal{G})/\Gamma(\mathcal{F} \cap \mathcal{G})) \simeq \mathbf{S}((\Gamma\mathcal{F} + \Gamma\mathcal{G})/\Gamma\mathcal{F}). \tag{2.4}$$

Proof. Let $\Gamma\mathcal{E} \equiv (\mathcal{E}(U), \rho_V^U)$ be the complete presheaf of sections of $\mathcal{E}$,

$$\Gamma(\mathcal{E})/\Gamma(\mathcal{F}) \equiv (\mathcal{E}(U)/\mathcal{F}(U), \sigma_V^U)$$

a generating presheaf of the quotient $\mathcal{A}$-module $\mathcal{E}/\mathcal{F}$, and

$$\phi : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E})/\Gamma(\mathcal{F})$$

the canonical $\Gamma(\mathcal{A})$-morphism. For every open $U \subseteq X$, the restriction ψ_U of ϕ_U to the sub-$\mathcal{A}(U)$-module $\mathcal{G}(U)$ of $\mathcal{E}(U)$ is the $\mathcal{A}(U)$-morphism $\psi_U : \mathcal{G}(U) \longrightarrow \mathcal{E}(U)/\mathcal{F}(U)$ given by $\psi_U(r) := r + \mathcal{F}(U)$, $r \in \mathcal{G}(U)$. Clearly, $\bigcup\{r + \mathcal{F}(U) : \ r \in \mathcal{G}(U)\} = \mathcal{G}(U) + \mathcal{F}(U)$, where $\mathcal{G}(U) + \mathcal{F}(U)$ means the sub-$\mathcal{A}(U)$-module of $\mathcal{E}(U)$, generated by $\mathcal{G}(U) \cup \mathcal{F}(U)$. (In this regard, cf. Adkins-Weintraub [1, p.116, Definition 2.13] or Blyth [6, p.11, Theorem 2.3].) If $r, s \in \mathcal{G}(U)$ such that $r - s \in \mathcal{F}(U)$, we obviously have $r + \mathcal{F}(U) = s + \mathcal{F}(U)$. Therefore the cosets $r + \mathcal{F}(U)$ are the stratification of $\mathcal{G}(U) + \mathcal{F}(U)$ by cosets of the sub-$\mathcal{A}(U)$-module $\mathcal{F}(U)$. This shows that $\psi_U(\mathcal{G}(U)) = (\mathcal{G}(U) + \mathcal{F}(U))/\mathcal{F}(U)$. Now, let us find the kernel, $\ker \psi_U$, of ψ_U. For all elements $r \in \mathcal{G}(U)$, we have $\psi_U(r) = \phi_U(r)$. But $\ker \phi_U = \mathcal{F}(U)$, so that $\ker \psi_U = \mathcal{F}(U) \cap \mathcal{G}(U) \equiv (\mathcal{F} \cap \mathcal{G})(U)$. Applying the First Isomorphism Theorem (cf. Adkins-Weintraub [1, p.113, Theorem 2.4]), which says that given two modules M and N over a ring R, and $f : M \longrightarrow N$ an R-module homomorphism, then $M/\ker f \cong Imf$, we have an $\mathcal{A}(U)$-isomorphism

$$\overline{\psi}_U : \mathcal{G}(U)/(\mathcal{F} \cap \mathcal{G})(U) \longrightarrow (\mathcal{F}(U) + \mathcal{G}(U))/\mathcal{F}(U). \tag{2.5}$$

On the other hand, correspondences

$$U \longmapsto \mathcal{G}(U)/(\mathcal{F} \cap \mathcal{G})(U), \ \ U \longmapsto (\mathcal{F}(U) + \mathcal{G}(U))/\mathcal{F}(U),$$

where U runs over the open subsets of X, along with obvious restriction maps, respectively, yield $\mathcal{A}$-presheaves, denoted $\Gamma(\mathcal{G})/\Gamma(\mathcal{F} \cap \mathcal{G})$ and $(\Gamma(\mathcal{F}) + \Gamma(\mathcal{G})/\Gamma\mathcal{F}$, respectively. Since (2.5) holds for any open $U \subseteq X$, $\Gamma(\mathcal{G})/\Gamma(\mathcal{F} \cap \mathcal{G})$ is $\mathcal{A}$-isomorphic to $(\Gamma(\mathcal{F}) + \Gamma(\mathcal{G}))/\Gamma(\mathcal{F})$. Finally, applying the sheafification functor $\mathbf{S}$ to the $\Gamma(\mathcal{A})$-isomorphism $\overline{\psi} : \Gamma(\mathcal{G})/\Gamma(\mathcal{F} \cap \mathcal{G}) \longrightarrow (\Gamma(\mathcal{F}) + \Gamma(\mathcal{G})/\Gamma(\mathcal{F})$, we get

$$\begin{aligned} \mathbf{S}(\Gamma(\mathcal{G})/\Gamma(\mathcal{F} \cap \mathcal{G})) \ &:= \ \mathcal{G}/(\mathcal{F} \cap \mathcal{G}) \\ &= \ \mathbf{S}(\Gamma(\mathcal{F}) + \Gamma(\mathcal{G}))/\mathcal{F} \\ &=: \ \mathbf{S}(\Gamma(\mathcal{F}) + \Gamma(\mathcal{G})/\Gamma(\mathcal{F})) \end{aligned}$$

within an $\mathcal{A}$-isomorphism. The proof is finished. ■

Corollary 2.4 *Let $\mathcal{E}$ be an $\mathcal{A}$-module, $\mathcal{F}$ and $\mathcal{G}$ sub-$\mathcal{A}$-modules of $\mathcal{E}$ such that $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$. Then*

$$\mathcal{E}/\mathcal{F} = \mathcal{G}$$

within an $\mathcal{A}$-isomorphism.

Proof. Given that $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$, Equation (2.4) becomes

$$\mathbf{S}(\Gamma(\mathcal{G}))/0 \cong \mathbf{S}(\Gamma(\mathcal{G})) \cong \mathcal{G} \cong \mathbf{S}(\Gamma(\mathcal{E}))/\mathcal{F} \cong \mathcal{E}/\mathcal{F},$$

i.e.

$$\mathcal{E}/\mathcal{F} \cong \mathcal{G}.$$

■

Before proceeding to some particular theorems regarding free $\mathcal{A}$-modules, we recall here from [51, p. 401, Definition 1.1] the notion of a *free sub-$\mathcal{A}$-module* $\mathcal{F}$ of a given free $\mathcal{A}$-module $\mathcal{E}$.

Definition 2.5 Let $\mathcal{E}$ be a free $\mathcal{A}$-module isomorphic to the standard free $\mathcal{A}$-module $\mathcal{A}^{(I)} := \oplus_I \mathcal{A}$, where I is an arbitrary indexing set. A sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is called **free** if it is isomorphic to a standard free $\mathcal{A}$-module $\mathcal{A}^{(J)}$, where $J \subseteq I$. A free sub-$\mathcal{A}$-module $\mathcal{G}$ of $\mathcal{E}$ such that $\mathcal{G} \cong \mathcal{A}^{(I \setminus J)}$ (so that $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$) is called **supplementary** to $\mathcal{F}$ in $\mathcal{E}$. We also say that $\mathcal{F}$ is **complemented** by $\mathcal{G}$ in $\mathcal{E}$.

Lemma 2.6 *If $\mathcal{F}$ is a free sub-$\mathcal{A}$-module of a free $\mathcal{A}$-module $\mathcal{E}$, then, for every open $U \subseteq X$,*

$$(\mathcal{E}/\mathcal{F})(U) = \mathcal{E}(U)/\mathcal{F}(U)$$

with an $\mathcal{A}(U)$-isomorphism.

Proof. Suppose that $\mathcal{E} \simeq \mathcal{A}^{(I)}$ and $\mathcal{F} \simeq \mathcal{A}^{(K)}$, with $K \subseteq I$. Then, by Corollary 2.4 and Definition 2.5, the quotient $\mathcal{A}$-module $\mathcal{E}/\mathcal{F}$ is free; whence $\mathcal{E}/\mathcal{F} \simeq \mathcal{A}^{(J)}$, where $I = J \oplus K$. On the other hand, since for any open $U \subseteq X$, $\mathcal{E}(U) \simeq \mathcal{A}^{(I)}(U) \simeq \mathcal{A}(U)^{(I)}$ and $\mathcal{F}(U) \simeq \mathcal{A}^{(K)}(U)$ with $\mathcal{F}(U) \subseteq \mathcal{E}(U)$, it follows that $\mathcal{E}(U)/\mathcal{F}(U) \simeq \mathcal{A}^{(J)}(U)$. Hence, the proof is finished. ■

Now, let us make the following definition, patterned after the standard notion of *corank* in the setting of vector spaces.

Definition 2.7 Let $\mathcal{E}$ be a free $\mathcal{A}$-module, and $\mathcal{F}$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$ complemented in $\mathcal{E}$ by a free sub-$\mathcal{A}$-module $\mathcal{G}$. The rank of $\mathcal{G} \simeq \mathcal{E}/\mathcal{F}$ is called the **corank** of $\mathcal{F}$, viz.

$$\operatorname{corank}_{\mathcal{E}} \mathcal{F} = \operatorname{rank} \mathcal{E}/\mathcal{F}.$$

Then, we have the following results, cf. [51].

Theorem 2.8 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, and $\mathcal{F}$ and $\mathcal{G}$ free sub-$\mathcal{A}$-modules of $\mathcal{E}$ such that $\mathcal{F} \cap \mathcal{G}$ and $\mathcal{F} + \mathcal{G}$ are free. Then,*

$$\begin{aligned} \operatorname{rank} \mathcal{F} + \operatorname{corank}_{\mathcal{E}} \mathcal{F} &= \operatorname{rank} \mathcal{E} \\ \operatorname{rank} (\mathcal{F} + \mathcal{G}) + \operatorname{rank} (\mathcal{F} \cap \mathcal{G}) &= \operatorname{rank} \mathcal{F} + \operatorname{rank} \mathcal{G} \\ \operatorname{corank}_{\mathcal{E}} (\mathcal{F} + \mathcal{G}) + \operatorname{corank}_{\mathcal{E}} (\mathcal{F} \cap \mathcal{G}) &= \operatorname{corank}_{\mathcal{E}} \mathcal{F} + \operatorname{corank}_{\mathcal{E}} \mathcal{G}. \end{aligned}$$

Theorem 2.9 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of arbitrary rank. Then, for any open subset $U \subseteq X$, rank $\mathcal{E}^*(U) =$ rank $\mathcal{E}(U)$. Fix an open set U in X. If $\psi \equiv (\psi_V)_{U \supseteq V,\ open} \in \mathcal{E}^*(U)$ and $\psi_U(s) = 0$ (which implies that $\psi_V(s|_V) = 0$ for any open $V \subseteq U$) for all $s \in \mathcal{E}(U)$, then $\psi = 0$; on the other hand if $\psi_U(s) = 0$ for all $\psi \in \mathcal{E}^*(U)$, then $s = 0$. Finally, let rank $\mathcal{E}(U) = n$. To a given basis $\{s_i\}$ of $\mathcal{E}(U)$, we can find a **dual basis** $\{\psi_i\}$ of $\mathcal{E}^*(U) \simeq \mathcal{E}(U)$, where*

$$\psi_{i,V}(s_j|_V) = \delta_{ij,V} \in \mathcal{A}(V)$$

for any open $V \subseteq U$.

Now, we introduce *kernels* and *orthogonal sub-$\mathcal{A}$-modules* in $\mathcal{A}$-pairings.

Definition 2.10 Let $((\mathcal{E}, \mathcal{F}; \phi); \mathcal{A})$ be a pairing of $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$, and $\mathcal{G}$ and $\mathcal{H}$ be sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively. The **ϕ-orthogonal** (or just the **orthogonal** if there is no confusion to fear) **of $\mathcal{G}$,** denoted $\mathcal{G}^{\perp_\phi} \equiv \mathcal{G}^{\perp}$, is the sub-$\mathcal{A}$-module of $\mathcal{F}$ such that, for every open $U \subseteq X$,

$$\mathcal{G}^{\perp}(U) := \{t \in \mathcal{F}(U) : \ \phi_V(\mathcal{G}(V), t|_V) = 0, \ \text{for any open} V \subseteq U\}.$$

Similarly, one defines the orthogonal of $\mathcal{H}$ to be the sub-$\mathcal{A}$-module, denoted $\mathcal{H}^{\top^\phi} \equiv \mathcal{H}^{\top}$, given by

$$\mathcal{H}^{\top}(U) := \{s \in \mathcal{E}(U) : \ \phi_V(s|_V, \mathcal{H}(V)) = 0, \ \text{for any open } V \subseteq U\},$$

where U is any open subset of X. The orthogonal of $\mathcal{E}$ (resp. $\mathcal{F}$) is also called the **kernel** of $\mathcal{E}$ (resp. $\mathcal{F}$). The $\mathcal{A}$-bilinear morphism ϕ is said to be **non-degenerate** if $\mathcal{E}^{\perp} = \mathcal{F}^{\top} = 0$, and **degenerate** otherwise.

It is immediate that if $\mathcal{G}$ and $\mathcal{H}$ are as in Definition 2.10, then $\mathcal{G}^{\perp}$ and $\mathcal{H}^{\top}$ are sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively.

Definition 2.11 Let $(\mathcal{E}, \phi)$ be a self $\mathcal{A}$-pairing such that $\mathcal{E}^{\perp_\phi} = \mathcal{E}^{\top^\phi}$. We call $\mathcal{E}^{\perp_\phi}$ the **radical sheaf** (or **sheaf of $\mathcal{A}$-radicals**, or simply **$\mathcal{A}$-radical**) of $\mathcal{E}$, and denote it by $\mathbf{rad}_{\mathcal{A}}\mathcal{E} \equiv \mathbf{rad}\ \mathcal{E}$. An $\mathcal{A}$-module $\mathcal{E}$ such that $\mathrm{rad}\ \mathcal{E} = 0$ is called **non-isotropic**; $\mathcal{E}$ is called totally isotropic *if ϕ is identically zero*, i.e., $\phi(\mathcal{E}, \mathcal{E}) = 0$. Analogously, a sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$ is called totally isotropic if $\phi = 0$ on $\mathcal{F}$. For any open $U \subseteq X$, a *non-zero section* $s \in \mathcal{E}(U)$ is called **isotropic** if $\phi_U(s,s) = 0$. Finally, the $\mathcal{A}$-radical of a sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is defined as

$$\mathrm{rad}\,\mathcal{F} := \mathcal{F} \cap \mathcal{F}^{\perp_\phi} = \mathcal{F} \cap \mathcal{F}^{\top^\phi},$$

under the assumption that $\mathcal{G}^{\perp_\phi} = \mathcal{G}^{\top^\phi}$ for every sub-$\mathcal{A}$-module $\mathcal{G}$ of $\mathcal{E}$. A sub-$\mathcal{A}$-module $\mathcal{F}$ is said to be **isotropic** if $\mathrm{rad}\,\mathcal{F} \neq 0$.

In the sequel, whenever there is no fear of confusion we will simply use $\perp$ (resp., $\top$) instead of the more accurate $\perp_\phi$ (resp. $\top^\phi$) to avoid useless cumbersomeness.

Lemma 2.12 *If $(\mathcal{E}, \mathcal{F}; \phi)$ is an $\mathcal{A}$-pairing of free $\mathcal{A}$-modules of finite rank, then for every open subset U of X,*

$$\mathcal{E}^{\perp}(U) = \mathcal{E}(U)^{\perp}, \;\; \mathcal{F}^{\perp}(U) = \mathcal{F}(U)^{\perp}, \tag{2.6}$$

where

$$\mathcal{E}(U)^{\perp} := \{t \in \mathcal{F}(U) : \phi_U(\mathcal{E}(U), t) = 0\} \tag{2.7}$$

and, similarly,

$$\mathcal{F}(U)^{\perp} := \{t \in \mathcal{E}(U) : \phi_U(t, \mathcal{F}(U)) = 0\}. \tag{2.8}$$

Proof. That $\mathcal{E}^{\perp}(U) \subseteq \mathcal{E}(U)^{\perp}$ is clear. Now, let $t \in \mathcal{E}(U)^{\perp}$ and $\{e_i^U\}_{i=1}^n$ be a basis of $\mathcal{E}(U)$. Since $\phi_U(e_i^U, t)|_V = \phi_V(e_i^U|_V, t|_V) = 0$ and $\{e_i^U|_V\}_{i=1}^n$ being a basis of $\mathcal{E}(V)$, we have $\phi_V(s, t|_V) = 0$, for any $s \in \mathcal{E}(V)$. Therefore, $\mathcal{E}(U)^{\perp} \subseteq \mathcal{E}^{\perp}(U)$, and hence the equality $\mathcal{E}^{\perp}(U) = \mathcal{E}(U)^{\perp}$.

The second equality in (2.6) is shown in a similar way. ∎

A particular case of $\mathcal{A}$-pairings is the following:

Definition 2.13 The $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{E}^*; \nu)$ such that, for every open $U \subseteq X$,

$$\nu_U(r, \psi) := \psi_U(r),$$

where $\psi \in \mathcal{E}^*(U) := \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$ and $r \in \mathcal{E}(U)$, is called the **canonical $\mathcal{A}$-pairing** of $\mathcal{E}$ and $\mathcal{E}^*$. $(\mathcal{E}, \mathcal{E}^*; \nu)$ is called the **canonical free $\mathcal{A}$-pairing** if $\mathcal{E}$ is a free $\mathcal{A}$-module.

For every open $U \subseteq X$, let (ε_i^U) and (ε_i^{*U}) be bases (of sections) of $\mathcal{E}(U)$ and $\mathcal{E}^*(U)$, respectively, where $\mathcal{E}$ and $\mathcal{E}^*$ are free $\mathcal{A}$-modules of finite rank *canonically paired into* $\mathcal{A}$. The family $\phi \equiv (\phi_U)_{X \supseteq U,\ open}$ such that

$$\phi_U(\varepsilon_i^U) := \varepsilon_i^{*U}$$

is an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{E}^*$. Furthermore, the kernel of ϕ is exactly the same as the left kernel of the canonical $\mathcal{A}$-pairing $(\mathcal{E}^*, \mathcal{E}; \mathcal{A})$. Indeed, $\ker \phi = 0 = \mathcal{E}^{*\top}$.

While the notion of orthogonality with respect to arbitrary $\mathcal{A}$-bilinear forms generalizes orthogonality in canonical $\mathcal{A}$-pairings, the former may be related with the latter through Lemma 2.14 below, in which we use $\top$ to indicate the left orthogonal of sub-$\mathcal{A}$-modules in the canonical free $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{E}^*; \nu_{\mathcal{E}} \equiv \nu)$ as well as in $(\mathcal{F}, \mathcal{F}^*; \nu_{\mathcal{F}} \equiv \nu)$, a practice which will be applied in subsequent chapters for right orthogonal in canonical free $\mathcal{A}$-pairings as well.

Lemma 2.14 *Let $(\mathcal{E}, \mathcal{F}; \phi)$ be a free $\mathcal{A}$-pairing, $\mathcal{G}$ and $\mathcal{H}$ free sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then,*

$$\mathcal{G}^{\perp_\phi} \simeq (\phi^{\mathcal{F}}(\mathcal{G}))^\top, \tag{2.9}$$

and

$$\mathcal{H}^{\top^\phi} \simeq (\phi^{\mathcal{E}}(\mathcal{H}))^\top. \tag{2.10}$$

Proof. Let U be an open subset of X. Since $\mathcal{G}$ is free, we have from (2.6) that $\mathcal{G}^{\perp\phi}(U) = \mathcal{G}(U)^{\perp\phi}$; so for a section $t \in \mathcal{F}(U)$ to be in $\mathcal{G}^{\perp\phi}(U)$ it is necessary and sufficient that

$$\phi_U(\mathcal{G}(U), t) = 0.$$

But

$$(\phi_U^R(\mathcal{G}(U)))^\top = \{t \in \mathcal{F}(U) : \ \phi_U^R(\mathcal{G}(U))(t) := \phi_U(\mathcal{G}(U), t) = 0\},$$

therefore (2.9) holds as required.

In a similar way, one shows (2.10). ■

We end this section with a theorem that relates orthogonality to duality. We refer the reader to [51] for the proof of the theorem.

Theorem 2.15 *Let $\mathcal{F}$ and $\mathcal{E}$ be $\mathcal{A}$-modules paired into a $\mathbb{C}$-algebra sheaf $\mathcal{A}$, and assume that $\mathcal{E}^\perp = 0$. Moreover, let $\mathcal{F}_0$ be a sub-$\mathcal{A}$-module of $\mathcal{F}$ and $\mathcal{E}_0$ a sub-$\mathcal{A}$-module of $\mathcal{E}$. There exist natural $\mathcal{A}$-isomorphisms into:*

$$\mathcal{E}/\mathcal{F}_0^\perp \longrightarrow \mathcal{F}_0^*, \tag{2.11}$$

and

$$\mathcal{E}_0^\perp \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*. \tag{2.12}$$

2.2 Orthogonally convenient $\mathcal{A}$-pairings

We shall now focus on a particular subfamily of the family of $\mathcal{A}$-pairings, which grants *orthogonal reflexivity* for free $\mathcal{A}$-modules. The members of the afore-mentioned subfamily are called *orthogonally convenient $\mathcal{A}$-pairings;* the latter are characterized among other things by a very important result that complements Theorem 2.15 of the previous section. This result consists, on one hand, of the $\mathcal{A}$-isomorphism

$\mathcal{E}/\mathcal{F}_0^{\top} \simeq \mathcal{F}_0^*$, where $(\mathcal{E}, \mathcal{F}; \phi)$ is orthogonally convenient and $\mathcal{F}_0$ a sub-$\mathcal{A}$-module of $\mathcal{F}$, and, on the other hand, of the $\mathcal{A}$-isomorphism $\mathcal{E}_0^{\perp} \simeq (\mathcal{E}/\mathcal{E}_0)^*$, where $(\mathcal{E}, \mathcal{E}^*; \nu)$ is the canonical free $\mathcal{A}$-pairing. There is an analog of this result in the setting of vector spaces; to this effect, see, for instance, [3, p. 21, Theorem 1.11, and p. 23, Theorem 1.12].

Definition 2.16 A pairing $(\mathcal{E}, \mathcal{F}; \phi)$ of free $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ into the $\mathbb{C}$-algebra sheaf $\mathcal{A}$ is called an **orthogonally convenient pairing** if given free sub-$\mathcal{A}$-modules $\mathcal{E}_0$ and $\mathcal{F}_0$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, their orthogonal $\mathcal{E}_0^{\perp}$ and $\mathcal{F}_0^{\top}$ are free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively.

Lemma 2.17 *Let $(\mathcal{E}, \mathcal{F}; \phi)$ be an orthogonally convenient $\mathcal{A}$-pairing, $\mathcal{E}_0$ and $\mathcal{F}_0$ free sub-$\mathcal{A}$-modules of the (free $\mathcal{A}$-modules) $\mathcal{E}$ and $\mathcal{F}$, respectively. Then $\mathcal{E}/\mathcal{F}_0^{\top}$ and $\mathcal{F}/\mathcal{E}_0^{\perp}$ are free sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$ respectively, and for every open $U \subseteq X$,*

$$(\mathcal{E}/\mathcal{F}_0^{\top})(U) = \mathcal{E}(U)/\mathcal{F}_0^{\top}(U) \tag{2.13}$$

and

$$(\mathcal{F}/\mathcal{E}_0^{\perp})(U) = \mathcal{F}(U)/\mathcal{E}_0^{\perp}(U). \tag{2.14}$$

Proof. Since $\mathcal{F}_0^{\top}$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, Lemma 2.6 applies and one obtains the $\mathcal{A}(U)$-isomorphism (2.13). In a similar way, one shows (2.14) . ■

Theorem 2.18 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of finite rank. The canonical pairing $(\mathcal{E}, \mathcal{E}^*; \phi)$ is orthogonally convenient.*

Proof. First, we notice by Theorem 2.8 that both kernels, i.e. $\mathcal{E}^{\perp}$ and $(\mathcal{E}^{\top})^{\perp}$, are 0. Let $\mathcal{E}_0$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and consider the map (2.12) of Theorem 2.15: $\mathcal{E}_0^{\perp} \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*$. It is an $\mathcal{A}$-isomorphism into, and we shall show that it is onto. Fix an open set U in X, and let $\psi \in (\mathcal{E}/\mathcal{E}_0)^*(U) := \mathcal{H}om_{\mathcal{A}|_U}((\mathcal{E}/\mathcal{E}_0)|_U, \mathcal{A}|_U)$. Let us consider the family $\overline{\psi} \equiv (\overline{\psi}_V)_{U \supseteq V,\ open}$ where if V, W are open in U with $W \subseteq V$, then

$$\tau_W^V \circ \overline{\psi}_V = \overline{\psi}_W \circ \rho_W^V$$

(the $\{\rho_V^U\}$ and $\{\tau_V^U\}$ are the restriction maps for the (complete) presheaf of sections of $\mathcal{E}$ and $\mathcal{A}$, respectively) and

$$\overline{\psi}_V(r) := \psi_V(r + \mathcal{E}_0(V)), \quad r \in \mathcal{E}(V). \tag{2.15}$$

It is easy to see that $\overline{\psi}_V$ is $\mathcal{A}(V)$-linear for any open $V \subseteq U$. Thus,

$$\overline{\psi} \in \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U) =: \mathcal{E}^*(U).$$

Suppose $r \in \mathcal{E}_0(V)$, where V is open in U. Then

$$\overline{\psi}_V(r) = \psi_V(r + \mathcal{E}_0(V)) = \psi_V(\mathcal{E}_0(V)) = 0,$$

therefore

$$\nu_V(\mathcal{E}_0(V), \overline{\psi}|_V) = \overline{\psi}_V(\mathcal{E}_0(V)) = 0,$$

i.e. $\overline{\psi} \in \mathcal{E}_0^{\perp}(U)$. We contend that $\overline{\psi}$ has the given ψ as image under the map (2.12), and this will show the ontoness of (2.12) and that $\mathcal{E}_0^{\perp}$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}^*$.

Let us find the image of $\overline{\psi}$. Consider the pairing $(\mathcal{E}/\mathcal{E}_0, \mathcal{E}_0^{\perp}; \Theta)$ such that for any open $V \subseteq X$, we have

$$\Theta_V(r + \mathcal{E}_0(V), \alpha) := \nu_V(r, \alpha) = \alpha_V(r),$$

where $\alpha \in \mathcal{E}_0^{\perp}(V) \subseteq \mathcal{E}^*(V)$, $r \in \mathcal{E}(V)$. Clearly, the right kernel of this new pairing is 0. For $\alpha = \overline{\psi} \in \mathcal{E}_0^{\perp}(U) \subseteq \mathcal{E}^*(U)$, we have

$$\Theta_U(r + \mathcal{E}_0(U), \overline{\psi}) = \overline{\psi}_U(r)$$

where $r \in \mathcal{E}(U)$, and the map

$$\overline{\Theta}_U : \mathcal{E}_0^{\perp}(U) \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*(U)$$

given by

$$\overline{\psi} \longmapsto \overline{\Theta}_{U,\overline{\psi}} \equiv \big((\overline{\Theta}_{U,\overline{\psi}})_V\big)_{U\supseteq V,\ open}$$

and such that for any $r \in \mathcal{E}(V)$

$$(\overline{\Theta}_{U,\overline{\psi}})_V(r+\mathcal{E}_0(V)) := \overline{\Theta}_V(r+\mathcal{E}_0(V), \overline{\psi}|_V) = \overline{\psi}_V(r) = \psi_V(r+\mathcal{E}_0(V))$$

is the image. Thus the image of $\overline{\psi}$ is ψ, hence the map $\mathcal{E}_0^\perp(U) \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*(U)$, derived from (2.12), is onto, and therefore an $\mathcal{A}(U)$-isomorphism. Since $\mathcal{E}/\mathcal{E}_0$ is free by Theorem 2.15, so are $(\mathcal{E}/\mathcal{E}_0)^*$ and $\mathcal{E}_0^\perp$ free.

Now, let $\mathcal{F}_0$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}^* \cong \mathcal{E}$ (cf. Mallios [50, p.298, (5.2)]); on considering $\mathcal{F}_0$ as a free sub-$\mathcal{A}$-module of $\mathcal{E}$, according to all that precedes above $\mathcal{F}_0^\top$ is free in $\mathcal{E}^* \cong \mathcal{E}$, and so the proof is finished. ■

Now, if $(\mathcal{E}, \mathcal{F}; \phi)$ is an orthogonally convenient pairing, $\mathcal{E}_0$ and $\mathcal{F}_0$ free sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively, by Theorem 2.15, $\mathcal{E}/\mathcal{F}_0^\top$ and $\mathcal{E}/\mathcal{E}_0$ are free $\mathcal{A}$-modules. Since the maps in Theorem 2.15 are $\mathcal{A}$-isomorphisms into,

$$\operatorname{rank}(\mathcal{E}/\mathcal{F}_0^\top) \leq \operatorname{rank}\ \mathcal{F}_0^* = \operatorname{rank}\ \mathcal{F}_0 \tag{2.16}$$

and

$$\operatorname{rank}\ \mathcal{E}_0^\perp \leq \operatorname{rank}(\mathcal{E}/\mathcal{E}_0)^* = \operatorname{rank}(\mathcal{E}/\mathcal{E}_0). \tag{2.17}$$

Inequalities in (2.16) and (2.17) can also be written in the form

$$\operatorname{corank}\ \mathcal{F}_0^\top \leq \operatorname{rank}\ \mathcal{F}_0$$

and

$$\operatorname{rank}\ \mathcal{E}_0^\perp \leq \operatorname{corank}\ \mathcal{E}_0.$$

If we put $\mathcal{E}_0 = \mathcal{F}_0^\top$ in the last inequality and combine it with the first one, we get

$$\operatorname{rank}\ \mathcal{F}_0^{\top\perp} \leq \operatorname{corank}\ \mathcal{F}_0^\top \leq \operatorname{rank}\ \mathcal{F}_0. \tag{2.18}$$

But $\mathcal{F}_0$ is a free sub-$\mathcal{A}$-module of $\mathcal{F}_0^{\top\perp}$, so that rank $\mathcal{F}_0 \leq$ rank $\mathcal{F}_0^{\top\perp}$, and (3.11) becomes

$$\text{rank } \mathcal{F}_0^{\top\perp} = \text{corank } \mathcal{F}_0^{\top} = \text{rank } \mathcal{F}_0. \tag{2.19}$$

Let us consider formula (2.19) in the case where rank $\mathcal{F}_0$ is finite. We clearly have $\mathcal{F}_0^{\top\perp} = \mathcal{F}_0$ within an $\mathcal{A}$-isomorphism. The $\mathcal{A}$-module $\mathcal{F}_0$ is said to be *orthogonally reflexive*. In $\mathcal{A}$-morphism (2.11), both free $\mathcal{A}$-modules have the same finite dimension, the $\mathcal{A}$-isomorphism into is, therefore, onto and thus

$$\mathcal{E}/\mathcal{F}_0^{\top} = \mathcal{F}_0^{*}$$

within an $\mathcal{A}$-isomorphism. Hence, $\mathcal{E}/\mathcal{F}_0^{\top}$ may be regarded naturally as the dual $\mathcal{A}$-module of $\mathcal{F}_0$. For $\mathcal{A}$-morphism (2.12), put $\mathcal{E}_0 = \mathcal{F}_0^{\top}$; thus (2.12) becomes an $\mathcal{A}$-isomorphism

$$\mathcal{F}_0^{\top\perp} \simeq (\mathcal{E}/\mathcal{F}_0^{\top})^{*}.$$

Putting $\mathcal{F}_0 = \mathcal{F}$ in (3.12), we obtain

$$\text{corank } \mathcal{F}^{\top} = \text{rank } \mathcal{F}. \tag{2.20}$$

Now, assume in our orthogonally convenient pairing $(\mathcal{E}, \mathcal{F}; \phi)$ that the right kernel $\mathcal{E}^{\perp}$ is not 0. Let $\Psi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E} \oplus (\mathcal{F}/\mathcal{E}^{\perp}), \mathcal{A})$ such that

$$\Psi_U(s, t + \mathcal{E}^{\perp}(U)) := \phi_U(s, t),$$

where U is an open subset of X, $s \in \mathcal{E}$ and $t + \mathcal{E}^{\perp}(U) \in (\mathcal{F}/\mathcal{E}^{\perp})(U) \simeq \mathcal{F}(U)/\mathcal{E}^{\perp}(U)$, (cf. Lemma 2.17).

The element $t + \mathcal{E}^{\perp}(U)$ lies in the right kernel $\mathcal{E}^{\perp}(U) \cong \mathcal{E}(U)^{\perp}$, if $\phi_U(s, t) = 0$, for all $s \in \mathcal{E}(U)$. But this means $t \in \mathcal{E}^{\perp}(U)$, so that $t + \mathcal{E}^{\perp}(U) = \mathcal{E}^{\perp}(U)$. It follows that the right kernel of the $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{F}/\mathcal{E}^{\perp}; \Psi)$ is 0. The left kernel is obviously the old $\mathcal{F}^{\top}$. Applying (2.20), we have

$$\text{rank}(\mathcal{E}/\mathcal{F}^{\top}) = \text{rank}(\mathcal{F}/\mathcal{E}^{\perp}). \tag{2.21}$$

Suppose now that both kernels $\mathcal{E}^{\perp}$ and $\mathcal{F}^{\top}$ are zero, and that rank $\mathcal{F}$ is finite. (2.21) shows that rank $\mathcal{E}$ is also finite and rank $\mathcal{E} =$ rank $\mathcal{F}$. So whenever $\mathcal{E}^{\perp} = 0 = \mathcal{F}^{\top}$, by Theorem 2.15, we see that each of the free $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{E}$ is naturally the *dual* of the other.

Now, still under the condition $\mathcal{E}^{\perp} = 0 = \mathcal{F}^{\top}$ for the orthogonally convenient $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{F}; \phi)$, let us look at the correspondence $\mathcal{F}_0 \longmapsto \mathcal{F}_0^{\top}$ of a free sub-$\mathcal{A}$-module $\mathcal{F}_0$ of $\mathcal{F}$ and the free sub-$\mathcal{A}$-module $\mathcal{F}_0^{\top}$ of $\mathcal{E}$. Any free sub-$\mathcal{A}$-module $\mathcal{E}_0$ of $\mathcal{E}$ is obtainable from an $\mathcal{F}_0$; indeed we merely have to put $\mathcal{F}_0 = \mathcal{E}_0^{\perp}$. And if $\mathcal{F}_0 \ncong \mathcal{F}_1$, then $\mathcal{F}_0^{\top} \ncong \mathcal{F}_1^{\top}$. The correspondence $\mathcal{F}_0 \longleftrightarrow \mathcal{F}_0^{\top}$, where $\mathcal{F}_0$ is any free sub-$\mathcal{A}$-module of $\mathcal{F}$, is one-to-one, and also if $\mathcal{F}_0 \subseteq \mathcal{F}_1$ then $\mathcal{F}_0^{\top} \supseteq \mathcal{F}_1^{\top}$.

Let us collect all our results.

Theorem 2.19 *Let $(\mathcal{E}, \mathcal{F}; \phi)$ be an orthogonally convenient $\mathcal{A}$-pairing. Then,*

(a) rank $(\mathcal{F}/\mathcal{E}^{\perp}) =$ rank $(\mathcal{E}/\mathcal{F}^{\top})$; *in particular if one of the free $\mathcal{A}$-modules $\mathcal{F}/\mathcal{E}^{\perp}$ and $\mathcal{E}/\mathcal{F}^{\top}$ has finite rank, so has the other one, and the ranks are equal.*

(b) *If the right kernel $\mathcal{E}^{\perp}$ is zero, and $\mathcal{F}_0 \subseteq \mathcal{F}$ is a free sub-$\mathcal{A}$-module, then*

$$\text{rank } \mathcal{F}_0 = \text{corank } \mathcal{F}_0^{\top} = \text{rank } \mathcal{F}_0^{\top\perp}. \tag{2.22}$$

If rank $\mathcal{F}_0$ is finite, then $\mathcal{F}_0^{\top\perp} = \mathcal{F}_0$ and $\mathcal{E}/\mathcal{F}_0^{\top} = \mathcal{F}_0^{*}$ within an $\mathcal{A}$-isomorphism, i.e. each of the free $\mathcal{A}$-modules $\mathcal{F}_0$ and $\mathcal{E}/\mathcal{F}_0^{\top}$ is naturally the dual of the other.

(c) *If both kernels are zero, and rank $\mathcal{F}$ is finite, then $\mathcal{F} = \mathcal{E}$ within an $\mathcal{A}$-isomorphism. The correspondence $\mathcal{F}_0 \longmapsto \mathcal{F}_0^{\top}$ is a bijection between the free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and the free sub-$\mathcal{A}$-modules of $\mathcal{E}$, and it reverses any inclusion relation.*

In case our orthogonally convenient $\mathcal{A}$-pairing is the canonical pairing $(\mathcal{E}, \mathcal{E}^*; \nu)$ we observe the following. First, as mentioned in the proof of Theorem 2.18, $\mathcal{E}^\perp$ and $(\mathcal{E}^*)^\top$ are 0. Then, let $\mathcal{E}_0$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ of arbitrary rank; as in the proof of Theorem 2.18, one shows that $\mathcal{E}_0^\perp = (\mathcal{E}/\mathcal{E}_0)^*$ with an $\mathcal{A}$-isomorphism.

Fix an open set U in X. Let $r \notin \mathcal{E}_0(U)$, so $r + \mathcal{E}_0(U)$ is not the zero section of $\mathcal{E}(U)/\mathcal{E}_0(U)$. By Theorem 2.18, there exists a $\psi \equiv (\psi_V)_{U \supseteq V, \text{ } open}$ in $(\mathcal{E}/\mathcal{E}_0)^*(U)$ such that

$$\phi_U(\psi, r + \mathcal{E}_0(U)) \equiv \psi_U(r + \mathcal{E}_0(U)) \neq 0.$$

Referring to (2.15), the corresponding $\overline{\psi} \equiv (\overline{\psi}_V)_{U \supseteq V, \text{ } open}$ in $\mathcal{E}_0^\perp(U)$ gives

$$\overline{\psi}_U(r) := \psi_U(r + \mathcal{E}_0(U)) \neq 0.$$

This section r is, therefore, not orthogonal to all of $\mathcal{E}_0^\perp(U)$, and we deduce that a section orthogonal to all of $\mathcal{E}_0^\perp(U)$ must necessarily lie in $\mathcal{E}_0(U)$. The latter implies that $\mathcal{E}_0^{\perp\top}(U) \subseteq \mathcal{E}_0(U)$. But, $\mathcal{E}_0(U)$ is obviously contained in $\mathcal{E}_0^{\perp\top}(U)$, therefore $\mathcal{E}_0^{\perp\top}(U) = \mathcal{E}_0(U)$. Since U is arbitrary, $\mathcal{E}_0^{\perp\top} = \mathcal{E}_0$. Finally, set $\mathcal{F}_0 := \mathcal{E}_0^\perp$ in (2.22); so

$$\text{rank } \mathcal{E}_0^\perp = \text{corank } \mathcal{E}_0.$$

Thus, we see in our special case that there is a bijection $\mathcal{F}_0 \longmapsto \mathcal{F}_0^\top$ between free sub-$\mathcal{A}$-modules $\mathcal{F}_0 \subseteq \mathcal{E}^*$ of finite rank and free sub-$\mathcal{A}$-modules of $\mathcal{E}$ with finite corank. Indeed, if $\mathcal{F}_0$ is given and of finite rank, then by (2.22) corank $\mathcal{F}_0^\top =$ rank $\mathcal{F}_0$ is finite. Besides, if corank $\mathcal{E}_0$ is finite, then rank $\mathcal{E}_0^\perp =$ corank $\mathcal{E}_0$ is finite, and $\mathcal{F}_0 = \mathcal{E}_0^\perp$ up to an $\mathcal{A}$-isomorphism. Again let us collect our results.

Theorem 2.20 *Let $(\mathcal{E}, \mathcal{E}^*; \nu)$ be the canonical free $\mathcal{A}$-pairing, and let $\mathcal{E}_0$ be a free $\mathcal{A}$-module of $\mathcal{E}$. Then $\mathcal{E}_0^{\perp\top} = \mathcal{E}_0$ and $\mathcal{E}_0^{\perp} = (\mathcal{E}/\mathcal{E}_0)^*$ within an $\mathcal{A}$-isomorphism, and* $\mathrm{rank}\mathcal{E}_0^{\perp} = \mathrm{corank}\ \mathcal{E}_0$. *The correspondence $\mathcal{F}_0 \longmapsto \mathcal{F}_0^{\top}$ is a bijection between free sub-$\mathcal{A}$-modules $\mathcal{F}_0 \subseteq \mathcal{E}^*$ of finite rank and all the free sub-$\mathcal{A}$-modules of $\mathcal{E}$ with finite corank.*

Chapter 3

Biorthogonality and hyperbolic decomposition theorem

In this chapter, we show that results pertaining to *biorthogonality* in pairings of vector spaces, see to this end, for instance [15], do hold for biorthogonality in pairings of $\mathcal{A}$-modules. However, for the *rank formula*, the algebra sheaf $\mathcal{A}$ is assumed to be a **PID sheaf**, that is, for every open $U \subseteq X$, the algebra $\mathcal{A}(U)$ is a PID-algebra; in other words, given a free $\mathcal{A}$-module $\mathcal{E}$ and a sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$, $\mathcal{F}$ is *section-wise free.* See [61]. The rank formula relates the rank of an $\mathcal{A}$-morphism and the rank of the kernel (sheaf) of the same $\mathcal{A}$-morphism with the rank of the source free $\mathcal{A}$-module of the $\mathcal{A}$-morphism concerned. Also, in order to obtain an analog of the Witt's hyperbolic decomposition theorem, we assume that $\mathcal{A}$ is a PID.

Notation. We assume throughout this chapter, unless otherwise mentioned, that the pair $(X, \mathcal{A})$ is an *algebraized space* ([50, p. 96]), where $\mathcal{A}$ is a unital $\mathbb{C}$-algebra sheaf such that *every nowhere-zero section of $\mathcal{A}$ is invertible.* Furthermore, all free $\mathcal{A}$-modules are considered to be *torsion-free,* that is, for any open subset $U \subseteq X$ and nowhere-zero section $s \in \mathcal{E}(U)$, if $as = 0$, where $a \in \mathcal{A}(U)$, then necessarily

$a = 0$. Finally, left and right kernels in a canonical $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{E}^*; \nu)$ are simply denoted using superscripts $\perp$ and $\top$ instead of the more conventional ones $\perp_\nu$ and $\top^\nu$.

3.1 Universal property of quotient $\mathcal{A}$-modules

The universal property of quotient vector spaces (cf. [15, p. 15, Théorème]) is generalized to the category of $\mathcal{A}$-modules. From this result, we show in Theorem 3.6 that *duality* and *orthogonality* in the category of $\mathcal{A}$-modules do relate. Finally, in Theorem 3.8, we obtain that in a canonical free $\mathcal{A}$-pairing *orthogonality is reflexive* for any free sub-$\mathcal{A}$-module of the defining free $\mathcal{A}$-module of the $\mathcal{A}$-pairing.

Theorem 3.1 *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{A}$-modules.*

1. *Let $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ be a surjective $\mathcal{A}$-morphism. Then, if $\psi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ such that $\ker \phi \subseteq \ker \psi$, there exists a unique $\theta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ such that the diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\phi"] \arrow[dr, "\psi"'] & \mathcal{F} \arrow[d, dotted, "\theta"] \\ & \mathcal{G} \end{tikzcd}$$

commutes. In other words, the mapping $\theta \mapsto \theta \circ \phi$ is an $\mathcal{A}$-isomorphism from $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ onto the sub-$\mathcal{A}(X)$-module of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ consisting of $\mathcal{A}$-morphisms whose kernel contains $\ker \phi$.

2. *Let $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ be an injective $\mathcal{A}$-morphism. Then, if $\psi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ such that $\mathrm{Im}\psi \subseteq \mathrm{Im}\phi$, there exists a unique $\theta \in$*

$\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ *making the diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[d, dotted, "\theta"'] \arrow[dr, "\psi"] & \\ \mathcal{F} \arrow[r, "\phi"'] & \mathcal{G} \end{tikzcd}$$

commute. More precisely, the mapping $\theta \mapsto \phi \circ \theta$ *is an* $\mathcal{A}$*-isomorphism from* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ *onto the sub-*$\mathcal{A}(X)$*-module of* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ *consisting of* $\mathcal{A}$*-morphisms whose image is contained in* $\mathrm{Im}\phi$*.*

Proof. *Assertion* 1. **Uniqueness.** Let θ_1, $\theta_2 \in \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ be such that $\psi = \theta_1 \circ \phi$ and $\psi = \theta_2 \circ \phi$. Fix an open subset U in X; since ϕ_U is surjective, the equation $\theta_{1,U} \circ \phi_U = \theta_{2,U} \circ \phi_U$ implies that $\theta_{1,U} = \theta_{2,U}$. Thus, $\theta_1 = \theta_2$.

Existence. Fix an open subset U in X and consider an element (section) $t \in \mathcal{F}(U)$. Since ϕ_U is surjective, there exists an element $s \in \mathcal{E}(U)$ such that $t = \phi_U(s)$. Now, suppose there exists a $r \in \mathcal{F}(U)$ with $u \in \ker \psi_U$ and $v \notin \ker \psi_U$ as its pre-images by ϕ_U, i.e.

$$\phi_U(v) = r = \phi_U(u)$$

with $u \in \ker \psi_U$ and $v \notin \ker \psi_U$. Since ϕ_U is linear, $\phi_U(v - u) = 0$; so $v - u \in \ker \phi_U \subseteq \ker \psi_U$. But $u \in \ker \psi_U$, so $v \in \ker \psi_U$, which yields a *contradiction*. We conclude that such a situation cannot occur. Furthermore, the element $\psi_U(s)$ does only depend on t. Let θ_U be the $\mathcal{A}(U)$-morphism sending $\mathcal{F}(U)$ into $\mathcal{G}(U)$ and such that

$$\theta_U(t) = \psi_U(s);$$

that

$$\psi_U = \theta_U \circ \phi_U$$

is clear.

Next, let us consider the *complete presheaves of sections* of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, respectively, viz.

$$\Gamma(\mathcal{E}) \equiv (\Gamma(U, \mathcal{E}), \alpha_V^U), \ \ \Gamma(\mathcal{F}) \equiv (\Gamma(U, \mathcal{F}), \beta_V^U), \ \ \Gamma(\mathcal{G}) \equiv (\Gamma(U, \mathcal{G}), \delta_V^U).$$

Given open subsets U and V of X such that $V \subseteq U$, since $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$, one has

$$\psi_V \circ \alpha_V^U = \delta_V^U \circ \psi_U. \tag{3.1}$$

But $\psi_U = \theta_U \circ \phi_U$ and $\psi_V = \theta_V \circ \phi_V$, therefore, (4.9) becomes

$$\theta_V \circ \phi_V \circ \alpha_V^U = \delta_V^U \circ \theta_U \circ \phi_U$$

or

$$\theta_V \circ \beta_V^U \circ \phi_U = \delta_V^U \circ \theta_U \circ \phi_U. \tag{3.2}$$

Since ϕ_U is surjective, it follows from (3.2) that

$$\theta_V \circ \beta_V^U = \delta_V^U \circ \theta_U,$$

which means that $\theta \equiv (\theta_U)_{X \supseteq U, \ open}$ is an $\mathcal{A}$-morphism of $\mathcal{F}$ into $\mathcal{G}$ such that

$$\psi = \theta \circ \phi,$$

as required.

Assertion 2. **Uniqueness.** Let $\theta_1, \ \theta_2 \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ be such that $\psi = \phi \circ \theta_1$ and $\psi = \phi \circ \theta_2$. As ϕ is injective, the relation

$$\phi \circ \theta_1 = \phi \circ \theta_2$$

implies that $\theta_1 = \theta_2$, so uniqueness is obtained.

Existence. Fix an open subset U in X and consider an element $s \in \mathcal{E}(U)$; since $\mathrm{Im}\psi \subseteq \mathrm{Im}\phi$, there exists a $t \in \mathcal{F}(U)$ such that

$$\phi_U(t) = \psi_U(s). \tag{3.3}$$

But ϕ_U is injective, therefore such an element t is unique. Now, let θ_U be the mapping of $\mathcal{E}(U)$ into $\mathcal{F}(U)$ sending an element $s \in \mathcal{E}(U)$ to an element $t \in \mathcal{F}(U)$ such that (3.3) is satisfied. It is immediate that θ_U is $\mathcal{A}(U)$-linear, and one has

$$\psi_U = \phi_U \circ \theta_U.$$

Finally, let $\Gamma(\mathcal{E}) \equiv (\Gamma(U,\mathcal{E}), \alpha_V^U)$, $\Gamma(\mathcal{F}) \equiv (\Gamma(U,\mathcal{F}), \beta_V^U)$, $\Gamma(\mathcal{G}) \equiv (\Gamma(U,\mathcal{G}), \delta_V^U)$ be as above the complete presheaves of sections of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, respectively. Given open subsets U and V of X such that $V \subseteq U$, since $\psi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$, one has

$$\psi_V \circ \alpha_V^U = \delta_V^U \circ \psi_U. \tag{3.4}$$

But $\psi_U = \phi_U \circ \theta_U$ and $\psi_V = \phi_V \circ \theta_V$, therefore, we deduce from (3.4) that

$$\phi_V \circ \theta_V \circ \alpha_V^U = \delta_V^U \circ \phi_U \circ \theta_U$$

or

$$\phi_V \circ \theta_V \circ \alpha_V^U = \phi_V \circ \beta_V^U \circ \theta_U. \tag{3.5}$$

Since ϕ_V is injective, it is clear from (3.5) that

$$\theta_V \circ \alpha_V^U = \beta_V^U \circ \theta_U,$$

which is to say that $\theta \equiv (\theta_U)_{X \supseteq U,\ open}$ is an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{F}$ such that

$$\psi = \phi \circ \theta,$$

and the proof is complete. ∎

The *universal property of quotient $\mathcal{A}$-modules* is then obtained as a corollary of Theorem 3.1. More precisely, one has

Corollary 3.2 (***Universal property of quotient $\mathcal{A}$-modules***) *Let $\mathcal{E}$ be an $\mathcal{A}$-module, $\mathcal{E}'$ a sub-$\mathcal{A}$-module of $\mathcal{E}$, and ϕ the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto $\mathcal{E}/\mathcal{E}'$. The pair $(\mathcal{E}/\mathcal{E}', \phi)$ satisfies the following universal property:*

Given any pair $(\mathcal{F}, \psi)$ consisting of an $\mathcal{A}$-module $\mathcal{F}$ and an $\mathcal{A}$-morphism $\psi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ such that $\mathcal{E}' \subseteq \ker \psi$, there exists a unique $\mathcal{A}$-morphism $\widetilde{\psi} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}/\mathcal{E}', \mathcal{F})$ such that the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\phi"] \arrow[dr, "\psi"'] & \mathcal{E}/\mathcal{E}' \arrow[d, dotted, "\widetilde{\psi}"] \\ & \mathcal{F} \end{tikzcd}$$

commutes, i.e.

$$\psi = \widetilde{\psi} \circ \phi.$$

The kernel of $\widetilde{\psi}$ equals the image by ϕ of the kernel of ψ, and the image of $\widetilde{\psi}$ equals the image of ψ.

The mapping

$$\theta \mapsto \theta \circ \phi$$

is an $\mathcal{A}$-isomorphism of the $\mathcal{A}(X)$-module $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}/\mathcal{E}', \mathcal{F})$ onto the sub-$\mathcal{A}(X)$-module of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ consisting of $\mathcal{A}$-morphisms of $\mathcal{E}$ into $\mathcal{F}$ whose kernel contains $\mathcal{E}'$.

Proof. Apply assertion 1 of Theorem 3.1. ■

Similarly to the classical case (cf. [15, p. 15, Corollary 1]), we also have the following corollary, the proof of which is an easy exercise and is, for that reason, omitted.

Corollary 3.3 *Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules and $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$. Then,*

(1) $\mathcal{E}/\ker\phi = \mathrm{Im}\ \phi$ *within an $\mathcal{A}$-isomorphism.*

(2) *Given a sub-$\mathcal{A}$-module $\mathcal{F}'$ of $\mathcal{F}$, $\mathcal{E}' \equiv \phi^{-1}(\mathcal{F}')$ is a sub-$\mathcal{A}$-module of $\mathcal{E}$ containing $\ker\phi$; moreover, $\mathcal{F}' = \phi(\mathcal{E}')$ if ϕ is surjective.*

(3) *Conversely, if* $\mathcal{E}'$ *is a sub-*$\mathcal{A}$*-module of* $\mathcal{E}$ *containing* $\ker \phi$*, then* $\mathcal{F}' \equiv \mathrm{Im}\ \mathcal{E}'$ *is a sub-*$\mathcal{A}$*-module of* $\mathcal{F}$ *such that* $\mathcal{E}' = \phi^{-1}(\mathcal{F}')$.

As a further application of the universal property of quotient $\mathcal{A}$-modules, we have

Corollary 3.4 *Let* $\mathcal{E}$ *be a free* $\mathcal{A}$*-module, and* $\mathcal{E}_1$ *a free sub-*$\mathcal{A}$*-module of* $\mathcal{E}$*. Then, the* $\mathcal{A}$*-morphism* $\phi \equiv (\phi_U)_{X\supseteq U,\ open} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}^*, \mathcal{E}_1^*)$ *such that every* ϕ_U *maps an element* $(\psi_V)_{U\supseteq V,\ open}$ *of* $\mathcal{E}^*(U)$ *onto its restriction*

$$(\psi_V|_{\mathcal{E}_1(V)})_{U\supseteq V,\ open} \in \mathcal{E}_1^*(U)$$

is surjective, and has $\mathcal{E}_1^\perp \subseteq \mathcal{E}^*$ *as its kernel. Moreover,*

$$\mathcal{E}^*/\mathcal{E}_1^\perp = \mathcal{E}_1^*$$

within an $\mathcal{A}$*-isomorphism.*

Proof. That $\ker \phi = \mathcal{E}_1^\perp$ is clear. Now, let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ complementing $\mathcal{E}_1$. It follows (cf. [50, p. 137, relations (6.21), (6.22)] that

$$\mathcal{E}^* = \mathcal{E}_1^* \oplus \mathcal{E}_2^*,$$

so that if U is open in X and

$$\psi \equiv (\psi_V)_{U\supseteq V,\ open} \in \mathcal{E}_1^*(U) \quad \text{and} \quad \theta \equiv (\theta_V)_{U\supseteq V,\ open} \in \mathcal{E}_2^*(U),$$

then

$$\Omega \equiv \psi + \theta \in \mathcal{E}^*(U).$$

If V is open in U and $s \in \mathcal{E}(V)$, so s is uniquely written as $s = r + t$ where $r \in \mathcal{E}_1(V)$ and $t \in \mathcal{E}_2(V)$, then

$$\Omega_V(s) = \psi_V(r) + \theta_V(t).$$

Consequently,

$$\phi_U(\Omega) = (\Omega_V|_{\mathcal{E}_1(V)})_{U\supseteq V,\ open} = \psi;$$

thus ϕ_U is surjective. Hence, applying Corollary 3.3 (1), we obtain an $\mathcal{A}$-isomorphism

$$\mathcal{E}^*/\mathcal{E}_1^{\perp} \simeq \mathcal{E}_1^*.$$

■

Now, let us introduce the notion of $\mathcal{A}$-*projection.*

Definition 3.5 Let $\mathcal{E}$ be an $\mathcal{A}$-module, $\mathcal{F}$ and $\mathcal{G}$ two *supplementary sub-$\mathcal{A}$-modules* of $\mathcal{E}$. The $\mathcal{A}$-*endomorphism*

$$\pi^{\mathcal{F}} \equiv (\pi^{\mathcal{F}}_U)_{X\supseteq U,\ open} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{E}) := \mathrm{End}_{\mathcal{A}}(\mathcal{E})$$

such that, for any section $s \in \mathcal{E}(U) \equiv \Gamma(\mathcal{E})(U) := \Gamma(U,\mathcal{E})$,

$$\pi^{\mathcal{F}}_U(s) \equiv \pi^{\mathcal{F}}_U(r+t) := r,$$

where $s = r + t$ with $r \in \mathcal{F}(U)$ and $t \in \mathcal{G}(U)$, is called the **$\mathcal{A}$-projection onto $\mathcal{F}$ (parallel to $\mathcal{G}$)**. In a similar way, one defines the **$\mathcal{A}$-projection onto $\mathcal{G}$** (*parallel to* $\mathcal{F}$).

Theorem 3.6 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, $\mathcal{E}_1$ and $\mathcal{E}_2$ two free sub-$\mathcal{A}$-modules of $\mathcal{E}$ the direct sum of which is $\mathcal{A}$-isomorphic to $\mathcal{E}$, $\pi_1 \equiv \pi^{\mathcal{E}_1}$, $\pi_2 \equiv \pi^{\mathcal{E}_2}$ the corresponding $\mathcal{A}$-projections. Then,*

$$\mathcal{E}^* = \mathcal{E}_1^{\perp} \oplus \mathcal{E}_2^{\perp},$$

and the $\mathcal{A}$-projections $\pi'_1 \equiv \pi^{\mathcal{E}_1^{\perp}}$, $\pi'_2 \equiv \pi^{\mathcal{E}_2^{\perp}}$ associated with this direct decomposition are given by setting

$$\pi'_{1,U}(\alpha) := (\alpha_V \circ \pi_{2,V})_{U\supseteq V,\ open} \quad and \quad \pi'_{2,U}(\alpha) := (\alpha_V \circ \pi_{1,V})_{U\supseteq V,\ open}$$

for any $\alpha \equiv (\alpha_V)_{U\supseteq V,\ open} \in \mathcal{E}^*(U)$.

As a precaution, we signal that the proof of this theorem requires results pertaining to Theorem 2.9.

Proof. Fix an open set U in X. That $(\alpha_V \circ \pi_{2,V})_{U \supseteq V,\ open}$ and $(\alpha_V \circ \pi_{1,V})_{U \supseteq V,\ open}$ belong to $\mathcal{E}_1^{\perp}(U)$ and $\mathcal{E}_2^{\perp}(U)$, respectively, is obvious. For any open $V \subseteq U$, the relation

$$\alpha_V = \alpha_V \circ \pi_{1,V} + \alpha_V \circ \pi_{2,V}$$

shows that

$$\mathcal{E}^*(U) = \mathcal{E}_1^{\perp}(U) + \mathcal{E}_2^{\perp}(U).$$

Finally, suppose that there exists $\beta \equiv (\beta_V)_{U \supseteq V,\ open}$ in $\mathcal{E}_1^{\perp}(U) \cap \mathcal{E}_2^{\perp}(U)$; since $\beta_V(s) = 0$ for any open $V \subseteq U$ and any $s \in \mathcal{E}(V) = \mathcal{E}_1(V) \oplus \mathcal{E}_2(V)$, it follows that $\beta = 0$ (cf. Theorem 2.9). Thus,

$$\mathcal{E}^*(U) = \mathcal{E}_1^{\perp}(U) \oplus \mathcal{E}_2^{\perp}(U)$$

and hence

$$\mathcal{E}^* = \mathcal{E}_1^{\perp} \oplus \mathcal{E}_2^{\perp}$$

as claimed. ■

An interesting result may be derived from Theorem 2.15, viz.:

Theorem 3.7 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, $\mathcal{E}_1$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and ϕ the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto (the free sub-$\mathcal{A}$-module) $\mathcal{E}/\mathcal{E}_1$. The $\mathcal{A}$-morphism*

$$\Lambda \equiv (\Lambda_U)_{X \supseteq U,\ open} : (\mathcal{E}/\mathcal{E}_1)^* \longrightarrow \mathcal{E}^*$$

such that, given any open subset $U \subseteq X$ and a section $\psi \equiv (\psi_V)_{U \supseteq V,\ open} \in (\mathcal{E}/\mathcal{E}_1)^*(U) := \mathrm{Hom}_{\mathcal{A}|_U}((\mathcal{E}/\mathcal{E}_1)|_U, \mathcal{A}|_U)$,

$$\Lambda_U(\psi) := (\psi_V \circ \phi_V)_{U \supseteq V,\ open}$$

is an $\mathcal{A}$-isomorphism of $(\mathcal{E}/\mathcal{E}_1)^*$ onto $\mathcal{E}_1^{\perp}$, where $\mathcal{E}_1^{\perp}$ is the $\mathcal{A}$-orthogonal of $\mathcal{E}_1$ in the canonical $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{E}^*; \mathcal{A})$.

Proof. It is clear that Λ is indeed an $\mathcal{A}$-morphism. Now, let us fix an open set U in X and let us consider a section $\psi \equiv (\psi_V)_{U \supseteq V,\ open} \in (\mathcal{E}/\mathcal{E}_1)^*(U)$. Then, $\Lambda_U(\psi) = 0$ if for any open V in U and $s \in \mathcal{E}(V)$,

$$\Lambda_U(\psi)(s) = 0.$$

But

$$\Lambda_U(\psi)(s) = (\psi_V \circ \phi_V)(s) = \psi_V(\phi_V(s)) = 0,$$

therefore, by Theorem 2.8 ,

$$\psi_V = 0.$$

It follows that

$$\ker \Lambda_U = 0,$$

and consequently

$$\ker \Lambda = 0;$$

in other words, Λ is injective.

Next, for every $\psi \equiv (\psi_V)_{U \supseteq V,\ open} \in (\mathcal{E}/\mathcal{E}_1)^*(U)$, where U is a fixed open set in X,

$$\Lambda_U(\psi)(s) = (\psi_V \circ \phi_V)(s) = 0,$$

where s is any element in $\mathcal{E}_1(V)$; that is

$$\Lambda_U(\psi) \in \mathcal{E}_1^{\perp}(U),$$

from which we deduce that

$$\operatorname{Im} \Lambda \subseteq \mathcal{E}_1^{\perp}.$$

Finally, let us consider, for every open $V \subseteq U$, the following commutative diagram

$$\begin{tikzcd} \mathcal{E}(V) \arrow[r, "\phi_V"] \arrow[dr, "\psi_V \circ \phi_V"'] & (\mathcal{E}/\mathcal{E}_1)(V) \arrow[d, dotted, "\psi_V"] \\ & \mathcal{A}(V) \end{tikzcd}.$$

The *universal property of quotient $\mathcal{A}$-modules* (cf. Corollary 3.2) shows that, given an element $\sigma_V \in \mathrm{H}om_{\mathcal{A}(V)}(\mathcal{E}(V), \mathcal{A}(V))$ such that $\ker \phi_V \subseteq \ker \sigma_V$, i.e., $\sigma_V(\mathcal{E}_1(V)) = 0$, there is a unique $\psi_V \in \mathrm{H}om_{\mathcal{A}(V)}((\mathcal{E}/\mathcal{E}_1)(V), \mathcal{A}(V))$ such that

$$\sigma_V = \psi_V \circ \phi_V.$$

It is clear that the family $\sigma \equiv (\sigma_V)_{U \supseteq V,\ open}$ is an $\mathcal{A}$-morphism $\mathcal{E}|_U \longrightarrow \mathcal{A}|_U$ satisfying the property:

$$\sigma = \psi \circ \phi.$$

Thus, Λ is surjective and the proof is finished. ■

As a result, based essentially on everything above, we have

Theorem 3.8 *Let $(\mathcal{E}, \mathcal{E}^*; \mathcal{A})$ be the canonical free $\mathcal{A}$-pairing and $\mathcal{E}_1$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$. Then,*

(1) $(\mathcal{E}_1^\perp)^\top = \mathcal{E}_1$ *within an $\mathcal{A}$-isomorphism.*

(2) $\mathcal{E}_1$ *has finite rank if and only if $\mathcal{E}_1^\perp$ has finite corank in $\mathcal{E}^*$, and then one has*

$$\mathit{rank}\ \mathcal{E}_1 = \mathrm{corank}_{\mathcal{E}^*}\ \mathcal{E}_1^\perp.$$

(3) $\mathcal{E}_1$ *has finite corank in $\mathcal{E}$ if and only if $\mathcal{E}_1^\perp$ has finite rank, and*

$$\mathrm{corank}_{\mathcal{E}}\ \mathcal{E}_1 = \mathit{rank}\ \mathcal{E}_1^\perp.$$

Proof. *Assertion* (1). Let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$, complementing $\mathcal{E}_1$. By Theorem 3.6,

$$\mathcal{E}^* \simeq \mathcal{E}_1^\perp \oplus \mathcal{E}_2^\perp.$$

We already know that $\mathcal{E}_1 \subseteq (\mathcal{E}_1^\perp)^\top$. Now, consider a section $s \in (\mathcal{E}_1^\perp)^\top(U)$; there exist $r \in \mathcal{E}_1(U)$ and $t \in \mathcal{E}_2(U)$ such that $s = r+t$. The section t is orthogonal to $\mathcal{E}_2^{\perp}(U)$, and since r and s are orthogonal to $\mathcal{E}_1^{\perp}(U)$, we then have that t is orthogonal to $\mathcal{E}_1^{\perp}(U) \oplus \mathcal{E}_2^{\perp}(U) \simeq \mathcal{E}^*(U)$. It follows from Theorem 2.8 that $t = 0$; thus $(\mathcal{E}_1^{\perp})^{\top}(U) \subseteq \mathcal{E}_1(U)$, and hence $(\mathcal{E}_1^{\perp})^{\top} \subseteq \mathcal{E}_1$.

Assertion (2). Since $\mathcal{E}_1$ is free, it follows that $\mathcal{E}_1^* \simeq \mathcal{E}_1$ (cf. [50, p. 298, (5.2)]). Thus, $\mathcal{E}_1$ has finite rank if and only if $\mathcal{E}_1^*$ has finite rank, and

$$\text{rank } \mathcal{E}_1^* = \text{rank } \mathcal{E}_1.$$

But, by Corollary 3.4, $\mathcal{E}^*/\mathcal{E}_1^{\perp}$ is $\mathcal{A}$-isomorphic to $\mathcal{E}_1^*$, therefore

$$\text{rank } \mathcal{E}_1 = \text{corank}_{\mathcal{E}^*}\mathcal{E}_1^{\perp}.$$

Assertion (3). Let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ complementing $\mathcal{E}_1$. But $\mathcal{E}/\mathcal{E}_1$ is $\mathcal{A}$-isomorphic to $\mathcal{E}_2$ (cf. Corollary 2.4), therefore $\mathcal{E}/\mathcal{E}_1$ is free; consequently $\mathcal{E}/\mathcal{E}_1$ has finite rank if and only if $(\mathcal{E}/\mathcal{E}_1)^*$ has finite rank, and one has

$$(\mathcal{E}/\mathcal{E}_1)^* \simeq \mathcal{E}/\mathcal{E}_1$$

so that

$$\text{corank}_{\mathcal{E}}\mathcal{E}_1 = \text{rank } \mathcal{E}/\mathcal{E}_1 = \text{rank } (\mathcal{E}/\mathcal{E}_1)^*.$$

But, by Theorem 3.7, $(\mathcal{E}/\mathcal{E}_1)^* \simeq \mathcal{E}_1^{\perp}$ within an $\mathcal{A}$-isomorphism, so the assertion is corroborated. ■

3.2 Biorthogonality with respect to arbitrary $\mathcal{A}$-bilinear forms

In this section, we investigate usual results pertaining to biorthogonality in pairings of vector space in the setting of $\mathcal{A}$-pairings defined by arbitrary $\mathcal{A}$-bilinear morphisms. The section ends with the Witt's hyperbolic decomposition theorem for $\mathcal{A}$-modules.

Let us introduce a set of notions we will be concerned with in the sequel.

Definition 3.9 *An $\mathcal{A}$-module $\mathcal{E}$ is called a **generalized locally free $\mathcal{A}$-module** if there exist an open covering $\mathcal{U} \equiv (U_\alpha)_{\alpha \in I}$ of X and a number $n(\alpha) \in \mathbb{N}$ for every open set U_α such that*

$$\mathcal{E}|_{U_\alpha} = \mathcal{A}^{n(\alpha)}|_{U_\alpha}.$$

*The open covering $\mathcal{U}$ is called a **local frame.***

Example 3.10 Consider a free $\mathcal{A}$-module $\mathcal{E}$, where $\mathcal{A}$ is a PID-algebra sheaf. Then, every sub-$\mathcal{A}$-module of $\mathcal{E}$ is a generalized locally free $\mathcal{A}$-module.

Definition 3.11 *Let* $(\mathcal{E}, \mathcal{F}; \phi)$ be a free $\mathcal{A}$-pairing. The triple $(\mathcal{E}, \mathcal{F}; \phi)$ is called a **generalized orthogonally convenient $\mathcal{A}$-pairing** if for all generalized locally free sub-$\mathcal{A}$-modules $\mathcal{E}_0$ and $\mathcal{F}_0$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, their orthogonal $\mathcal{E}_0^{\perp\phi}$ and $\mathcal{F}_0^{\top\phi}$ are generalized locally free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively. If for all free sub-$\mathcal{A}$-modules $\mathcal{E}_0 \subseteq \mathcal{E}$ and $\mathcal{F}_0 \subseteq \mathcal{F}$, $\mathcal{E}_0^{\perp\phi}$ and $\mathcal{F}_0^{\top\phi}$ are free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, $(\mathcal{E}, \mathcal{F}; \phi)$ is called an **orthogonally convenient $\mathcal{A}$-pairing.**

Definition 3.12 Let $\mathcal{E}$ and $\mathcal{F}$ be free $\mathcal{A}$-modules. An $\mathcal{A}$-morphism $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is called **free** if Im ϕ is a free sub-$\mathcal{A}$-module of $\mathcal{F}$. The rank of Im ϕ is called the **rank** of ϕ, and is denoted **rank** ϕ.

We may now state the counterpart of the *fundamental theorem* of the classical theory, see [15, p. 54, Théorème 6.4].

Theorem 3.13 *Let $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ be a free $\mathcal{A}$-morphism. Then, the rank of ϕ is finite if and only if the kernel of ϕ has finite corank in $\mathcal{E}$. Moreover, one has*

$$\mathrm{rank}\ \phi := \mathit{rank}\ \mathrm{Im}\ \phi = \mathrm{corank}_{\mathcal{E}} \ker \phi.$$

Proof. Corollary 3.3(1) shows that the quotient free $\mathcal{A}$-module $\mathcal{E}/\ker \phi$ is $\mathcal{A}$-isomorphic to Im ϕ. ■

Corollary 3.14 *Let $\mathcal{A}$ be a PID algebra sheaf and $\mathcal{E}$, $\mathcal{F}$ free $\mathcal{A}$-modules. Then, if rank $\mathcal{E}$ is finite, every free $\mathcal{A}$-morphism $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ has finite rank, and*

$$\mathrm{rank}(\phi) + \mathit{rank}\ \ker(\phi) = \mathit{rank}\ \mathcal{E}. \tag{3.6}$$

The formula above is called the ***rank formula.***

Proof. Indeed, given that every $\mathcal{A}(U)$, where U is open in X, is a PID algebra, it follows that $\ker(\phi_U)$ is a free sub-$\mathcal{A}(U)$- module of the free $\mathcal{A}(U)$-module $\mathcal{E}(U)$. By elementary module theory (see, for instance, [1, p. 173, Proposition 8.8] or [6, p. 105, Corollary 2]), we have

$$\mathrm{rank}\ \ker(\phi_U) + \mathrm{rank}\ \mathrm{Im}(\phi_U) = \mathrm{rank}\ \mathcal{E}(U).$$

Since for any subsets U and V of X, rank $\ker(\phi_U) =$ rank $\ker(\phi_V)$, it follows that $\ker(\phi)$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and therefore

$$\mathrm{rank}\ \ker(\phi) + \mathrm{rank}\ \mathrm{Im}(\phi) = \mathrm{rank}\ \mathcal{E},$$

or

$$\mathrm{rank}\ \ker(\phi) + \mathrm{rank}(\phi) = \mathrm{rank}\ \mathcal{E}.$$

■

Theorem 3.15 *Let $(\mathcal{E}, \mathcal{E}^*; \mathcal{A})$ be the canonical free $\mathcal{A}$-pairing, and $\mathcal{F}$ a free sub-$\mathcal{A}$-module of $\mathcal{E}^*$. $\mathcal{F}$ has finite rank if and only if $\mathcal{F}^\top$ has finite corank in $\mathcal{E}$; moreover, one has*

$$rank\ \mathcal{F} = \mathrm{corank}_{\mathcal{E}}\mathcal{F}^\top; \quad (\mathcal{F}^\top)^\perp = \mathcal{F}.$$

Proof. The case $\mathcal{F} = 0$ is trivial.

Suppose that $\mathcal{F}$ has finite rank; let U be an open subset of X, $(e_1^{U*}, \ldots, e_n^{U*})$ a canonical (local) gauge of $\mathcal{F}$ (cf. [50, p. 291, (3.11) along with p. 301, (5.17) and (5.18)]), and $\phi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}^n)$ be such that if $s \in \mathcal{E}(U)$,

$$\phi_U(s) := (e_1^{U*}(s), \ldots, e_n^{U*}(s)).$$

It is clear that ϕ is indeed an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{A}^n$ whose kernel is $\mathcal{F}^\top$, which is a free sub-$\mathcal{A}$-module of $\mathcal{E}$ for the simple reason that *canonical free $\mathcal{A}$-pairings are orthogonally convenient,* (see Chap. II, Theorem 2.1). It is also clear that Im ϕ is $\mathcal{A}$-isomorphic to the free $\mathcal{A}$-module $\mathcal{A}^n$; thus, by Theorem 3.13, one has

$$\mathrm{rank}(\phi) := \mathrm{corank}_{\mathcal{E}}\mathcal{F}^\top = \mathrm{rank}\ \mathcal{F}. \tag{3.7}$$

According to Theorem 3.8(3), $(\mathcal{F}^\top)^\perp$ has finite rank, and

$$\mathrm{rank}\ (\mathcal{F}^\top)^\perp = \mathrm{corank}_{\mathcal{E}}\mathcal{F}^\top. \tag{3.8}$$

Since $\mathcal{F}$ is contained in $(\mathcal{F}^\top)^\perp$, we deduce from (3.7) and (3.8) that

$$\mathcal{F} = (\mathcal{F}^\top)^\perp.$$

Conversely, suppose that $\mathcal{F}^\top$ has finite corank in $\mathcal{E}$; then $(\mathcal{F}^\top)^\perp$ has finite rank, and thus $\mathcal{F}$ as well, as $\mathcal{F}$ is contained in $(\mathcal{F}^\top)^\perp$. ■

Given an $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{F}; \phi)$, if ϕ is non-degenerate (cf. Definition 2.10), then both insertion $\mathcal{A}$-morphisms ϕ^R and ϕ^L are injective.

Moreover, if $\mathcal{E}$ and $\mathcal{F}$ are both free $\mathcal{A}$-modules of finite rank, then, $\mathcal{E} = \mathcal{F}$ within an $\mathcal{A}$-isomorphism.

The case where $((\mathcal{E}, \mathcal{F}; \phi); \mathcal{A})$ is an *orthogonally convenient pairing* and ϕ is *degenerate* is interesting, for it yields the following result.

Theorem 3.16 *Given an orthogonally convenient pairing $((\mathcal{E}, \mathcal{F}; \phi); \mathcal{A})$, where $\mathcal{E}$ and $\mathcal{F}$ are free $\mathcal{A}$-modules of finite rank, the free quotient $\mathcal{A}$-modules $\mathcal{E}/\mathcal{F}^{\top\phi}$ and $\mathcal{F}/\mathcal{E}^{\perp\phi}$ have the same rank, i.e.*

$$\mathcal{E}/\mathcal{F}^{\top\phi} = \mathcal{F}/\mathcal{E}^{\perp\phi}$$

within an $\mathcal{A}$-isomorphism.

Proof. Since $((\mathcal{E}, \mathcal{F}; \phi); \mathcal{A})$ is *orthogonally convenient,* kernels $\mathcal{E}^{\perp\phi}$ and $\mathcal{F}^{\top\phi}$ are free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively. By [52], it follows that the quotient $\mathcal{A}$-modules $\mathcal{E}/\mathcal{F}^{\top\phi}$ and $\mathcal{F}/\mathcal{E}^{\perp\phi}$ are free, and for any open subset U of X,

$$(\mathcal{E}/\mathcal{F}^{\top\phi})(U) = \mathcal{E}(U)/\mathcal{F}^{\top\phi}(U) = \mathcal{E}(U)/\mathcal{F}(U)^{\top\phi}$$

and

$$(\mathcal{F}/\mathcal{E}^{\perp\phi})(U) = \mathcal{F}(U)/\mathcal{E}^{\perp\phi}(U) = \mathcal{F}(U)/\mathcal{E}(U)^{\perp\phi}$$

within $\mathcal{A}(U)$-isomorphism. Clearly, for a fixed open $U \subseteq X$, if $s \in \mathcal{E}(U)$ and $t, t_1 \in \mathcal{F}(U)$ such that $t - t_1 \in \mathcal{E}^{\perp\phi}(U)$, then

$$\phi_U(s,t) = \phi_U(s,t_1).$$

In the same vein, if $s = s_1 \mod \mathcal{F}^{\top\phi}(U)$ and $t = t_1 \mod \mathcal{E}^{\perp\phi}(U)$, then

$$\phi_U(s,t) = \phi_U(s_1,t_1).$$

Now, let us consider the $\mathcal{A}$-bilinear morphism

$$\overline{\phi} \equiv (\overline{\phi}_U)_{X \supseteq U,\ open} \equiv ((\overline{\phi})_U)_{X \supseteq U,\ open} : \mathcal{E}/\mathcal{F}^{\top\phi} \oplus \mathcal{F}/\mathcal{E}^{\perp\phi} \longrightarrow \mathcal{A},$$

induced by the $\mathcal{A}$-bilinear morphism ϕ, which is such that, for any open $U \subseteq X$ and sections $\overline{s} := \mathrm{cl}(s) \mod \mathcal{F}^{\top^{\phi}}(U)$, $\overline{t} := \mathrm{cl}(t) \mod \mathcal{E}^{\perp_{\phi}}(U)$ ($\mathrm{cl}(s)$ stand for the *equivalence class containing* s), one has

$$\overline{\phi}_U(\overline{s}, \overline{t}) := \phi_U(s,t).$$

It is clear that $\overline{\phi}_U(\overline{s}, \overline{t}) = 0$ for any $\overline{s} \in (\mathcal{E}/\mathcal{F}^{\top^{\phi}})(U) = \mathcal{E}(U)/\mathcal{F}^{\top^{\phi}}(U)$ is equivalent to $\phi_U(s,t) = 0$ for any $s \in \mathcal{E}(U)$; therefore $t \in \mathcal{E}^{\perp_{\phi}}(U) = 0$ and hence $\overline{t} = 0$. This implies that $(\mathcal{E}/\mathcal{F}^{\top^{\phi}})^{\perp_{\phi}} = 0$. Similarly, that $\overline{\phi}_U(\overline{s}, \overline{t}) = 0$ for any $\overline{t} \in (\mathcal{F}/\mathcal{E}^{\perp_{\phi}})(U) = \mathcal{F}(U)/\mathcal{E}^{\perp_{\phi}}(U)$ is equivalent to $\overline{s} = 0$, from which we deduce that $(\mathcal{F}/\mathcal{E}^{\perp_{\phi}})^{\top^{\phi}} = 0$. Hence, $\overline{\phi}$ is non-degenerate; so

$$\mathcal{E}/\mathcal{F}^{\top^{\phi}} = \mathcal{F}/\mathcal{E}^{\perp_{\phi}}$$

within an $\mathcal{A}$-isomorphism. ■

Theorem 3.17 *Let $\mathcal{A}$ be a PID algebra sheaf, $(\mathcal{E}, \mathcal{F}; \phi)$ an orthogonally convenient $\mathcal{A}$-pairing with rank $\mathcal{E}$ and rank $\mathcal{F}$ finite. Moreover, let ϕ^L and ϕ^R be the left and right insertion $\mathcal{A}$-morphisms associated with ϕ. Then,*

(1) *For every free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, one has*

1.1) $\phi^R(\mathcal{G}) \simeq (\mathcal{G}^{\perp_{\phi}})^{\perp}$ *and* $\phi^L(\mathcal{H}) \simeq (\mathcal{H}^{\top^{\phi}})^{\perp}$.

1.2) *rank* $\phi^R(\mathcal{G}) = \mathrm{corank}_{\mathcal{F}}\mathcal{G}^{\perp_{\phi}}$ *and rank* $\phi^L(\mathcal{H}) = \mathrm{corank}_{\mathcal{E}}\mathcal{H}^{\top^{\phi}}$.

(2) *$\mathcal{A}$-morphisms ϕ^L and ϕ^R have the same rank:*

$$\mathrm{rank}(\phi^L) = \mathrm{rank}(\phi^R), \tag{3.9}$$

which is the rank of ϕ.

Proof. *Assertion* (1). Since $(\mathcal{E}, \mathcal{F}; \phi)$ is orthogonally convenient, the sub-$\mathcal{A}$-module $\mathcal{G}^{\perp_{\phi}}$ is free, and thus

$$\mathcal{G}^{\perp_{\phi}}(U) \simeq \mathcal{G}(U)^{\perp_{\phi}}$$

for every open $U \subseteq X$. By Lemma 2.14,

$$\mathcal{G}^{\perp_\phi} = (\phi^L(\mathcal{G}))^\top$$

within an $\mathcal{A}$-isomorphism. Applying Theorem 3.15, and since rank $\mathcal{F}$ is finite, we have

$$(\mathcal{G}^{\perp_\phi})^\perp = \phi^L \mathcal{G}$$

within an $\mathcal{A}$-isomorphism. By the same theorem along with Theorem 3.8, it follows that

$$\text{rank } \mathcal{G}^{\perp_\phi} + \text{rank } \phi^L \mathcal{G} = \text{rank } \mathcal{F},$$

from which we deduce that

$$\text{rank } \phi^L \mathcal{G} = \text{corank}_{\mathcal{F}} \mathcal{G}^{\perp_\phi}.$$

In particular,

$$\text{rank}(\phi^L) = \text{corank}_{\mathcal{F}} \mathcal{E}^{\perp_\phi}. \tag{3.10}$$

In a similar way, one shows the claims related to the induced $\mathcal{A}$-morphism ϕ^R by using the fact that rank $\mathcal{E}$ is finite. The analog of (3.10) is

$$\text{rank}(\phi^L) = \text{corank}_{\mathcal{E}} \mathcal{F}^{\top^\phi}. \tag{3.11}$$

Assertion (2). That

$$\ker(\phi^L) \simeq \mathcal{E}^{\perp_\phi} \quad \text{and} \quad \ker(\phi^R) \simeq \mathcal{F}^{\top^\phi}$$

is immediate. Applying the *rank formula* (Corollary 3.14), we obtain

$$\text{rank}(\phi^R) := \text{rank } \phi^R(\mathcal{F}) = \text{rank } \mathcal{F} - \text{rank } \mathcal{E}^{\perp_\phi} = \text{corank}_{\mathcal{F}} \mathcal{E}^{\perp_\phi}, \tag{3.12}$$

and

$$\text{rank}(\phi^L) := \text{rank } \phi^L(\mathcal{E}) = \text{rank } \mathcal{E} - \text{rank } \mathcal{F}^{\top^\phi} = \text{corank}_{\mathcal{E}} \mathcal{F}^{\top^\phi}. \tag{3.13}$$

From (3.10), (3.11), (3.12) and (3.13), one gets (3.9). ■

Corollary 3.18 *Let $\mathcal{A}$ be a PID algebra sheaf and $(\mathcal{E}, \mathcal{F}; \phi)$ an orthogonally convenient $\mathcal{A}$-pairing with free $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ both of finite rank.*

(1) *For every free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, one has*

1.1) *rank* $\mathcal{G}^{\perp_\phi} \geq$ *rank* $\mathcal{F} -$ *rank* $\mathcal{G}$ *and rank* $\mathcal{H}^{\top^\phi} \geq$ *rank* $\mathcal{E} -$ *rank* $\mathcal{H}$

1.2) $(\mathcal{G}^{\perp_\phi})^{\top^\phi} \supseteq \mathcal{G}$ *and* $(\mathcal{H}^{\top^\phi})^{\perp_\phi} \supseteq \mathcal{H}$.

(2) *If ϕ is nondegenerate, then*

2.1) *rank* $\mathcal{G}^{\perp_\phi} +$ *rank* $\mathcal{G} =$ *rank* $\mathcal{F} =$ *rank* $\mathcal{E} =$ *rank* $\mathcal{H}^{\top^\phi} +$ *rank* $\mathcal{H}$

2.2) $(\mathcal{G}^{\perp_\phi})^{\top^\phi} \simeq \mathcal{G}$ *and* $(\mathcal{H}^{\top^\phi})^{\perp_\phi} \simeq \mathcal{H}$.

Proof. *Assertion* (1). Theorem 3.20 shows that

$$\text{rank } \phi^L(\mathcal{G}) = \text{corank}_{\mathcal{F}}\mathcal{G}^{\perp_\phi} = \text{rank } \mathcal{F} - \text{rank } \mathcal{G}^{\perp_\phi}.$$

On the other hand, by virtue of Corollary 3.14, one has

$$\text{rank } \phi^L(\mathcal{G}) = \text{rank } \mathcal{G} - \text{rank } (\ker \phi^L \cap \mathcal{G}).$$

It follows, in particular, that

$$\text{rank } \mathcal{G} \geq \text{rank } \phi^L(\mathcal{G}),$$

from which we have

$$\text{rank } \mathcal{G}^{\perp_\phi} \geq \text{rank } \mathcal{F} - \text{rank } \mathcal{G}.$$

Likewise, one shows the second inequality of 1.1).

Assertion (2). If ϕ is nondegenerate, rank $\mathcal{E} =$ rank $\mathcal{F}$; therefore ϕ^L is an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{F}^*$. Thus, rank $\phi^L(\mathcal{G}) =$ rank $\mathcal{G}$, and

$$\text{rank } \mathcal{G}^{\perp_\phi} = \text{rank } \mathcal{F} - \text{rank } \mathcal{G}.$$

Likewise, one has

$$\text{rank } \mathcal{H}^{\top_\phi} = \text{rank } \mathcal{E} - \text{rank } \mathcal{H}.$$

Applying relation 2.1) of Corollary 3.18 to the free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{G}^{\perp_\phi}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, we see that

$$\text{rank } (\mathcal{G}^{\perp_\phi})^{\top_\phi} = \text{rank } \mathcal{G}.$$

Since $\mathcal{G}$ is contained in $(\mathcal{G}^{\perp_\phi})^{\top_\phi}$, it follows that

$$(\mathcal{G}^{\perp_\phi})^{\top_\phi} = \mathcal{G}$$

within an $\mathcal{A}$-isomorphism. In a similar way, we show that $(\mathcal{H}^{\top_\phi})^{\perp_\phi} = \mathcal{H}$ within an $\mathcal{A}$-isomorphism. ■

We will soon turn to the hyperbolic decomposition theorem for $\mathcal{A}$-modules. However, the so-called theorem requires some preparations.

Lemma 3.19 *Let $(\mathcal{E}, \phi)$ be a symplectic $\mathcal{A}$-module (An $\mathcal{A}$-module with a symplectic $\mathcal{A}$-morphism (cf. Definition4.1)), U an open subset of X and $(r_1, \ldots, r_n) \subseteq \mathcal{E}(U)$ an arbitrary (local) gauge of $\mathcal{E}$. For any $r \equiv r_i$, $1 \leq i \leq n$, there exists a nowhere-zero section $s \in \mathcal{E}(U)$ such that $\phi_U(r, s)$ is nowhere zero.*

Proof. Without loss of generality, assume that $r_1 = r$. On the other hand, since the induced $\mathcal{A}$-morphism $\widetilde{\phi} \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^*)$ is one-to-one and both $\mathcal{E}$ and $\mathcal{E}^*$ have the same finite rank, it follows that the matrix D representing ϕ_U (see also [1, p. 357, Theorem 2.21, along with p. 356, Definition 2.19] or [15, p. 343, Proposition 20.3]), with respect to the basis $(r_1, \ldots, r_n)$, has a *nowhere-zero determinant*; so since

$$\det D = \sum_{i=1}^{n} (-1)^{1+i} \phi(r_1, r_i) \det D_{1i} = \phi(r_1, \sum_{i=1}^{n} (-1)^{1+i} \det D_{1i} r_i),$$

where D_{1i} is the minor of the corresponding $\phi(r_1, r_i)$, and $\det D$ nowhere zero, we thus have a section $s := \sum_{i=1}^{n}(-1)^{1+i} \det D_{1i} r_i \in \mathcal{E}(U)$ such that $\phi(r, s)$ is nowhere zero. ■

For the purpose of Theorem 3.21 below, we require the following result, see [52].

Theorem 3.20 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of finite rank, equipped with an $\mathcal{A}$-bilinear morphism $\phi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$. Then, every non-isotropic free sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is a direct summand of $\mathcal{E}$; viz.*

$$\mathcal{E} = \mathcal{F} \perp \mathcal{F}^{\perp_\phi}.$$

So, we have

Theorem 3.21 (***Hyperbolic Decomposition Theorem***) *Let $\mathcal{A}$ a PID algebra sheaf on X and $(\mathcal{E}, \mathcal{E}; \phi)$ an orthogonally convenient self-pairing, where ϕ is non-degenerate and skew-symmetric. Then, if $\mathcal{F}$ is a totally isotropic (free) sub-$\mathcal{A}$-module of rank k, there is a non-isotropic sub-$\mathcal{A}$-module $\mathcal{H}$ of $\mathcal{E}$ of the form*

$$\mathcal{H} := \mathcal{H}_1 \perp \cdots \perp \mathcal{H}_k,$$

where if $(r_{1,U}, \ldots, r_{k,U})$ is a basis of $\mathcal{F}(U)$ (with U an open subset of X), then $r_{i,U} \in \mathcal{H}_i(U)$ for $1 \leq i \leq k$.

Proof. Suppose that $k = 1$, i.e. $\mathcal{F} = \mathcal{A}$, within an $\mathcal{A}$-isomorphism. If $\mathcal{F}(X) = [r_X]$ with $r_X \in \mathcal{E}(X)$ a nowhere-zero section, then for every open $U \subseteq X$, $r_U \equiv r_X|_U$ generates the $\mathcal{A}(U)$-module $\mathcal{F}(U)$. Since ϕ_X is non-degenerate, by Lemma 3.19, there exists a nowhere-zero section $s_X \in \mathcal{E}(X)$ such that $\phi_X(r_X, s_X)$ is nowhere zero. The correspondence

$$U \longmapsto \mathcal{H}(U) := [r_U, s_U] \equiv [r_X|_U, s_X|_U],$$

where U runs over the open sets in X, along with the obvious restriction maps yields a *complete presheaf of* $\mathcal{A}(U)$*-modules* on X. Clearly, the pair $(\mathcal{H}, \overline{\phi})$, where $\overline{\phi}$ is the $\mathcal{A}$-bilinear $\overline{\phi} : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{A}$ such that

$$(r, s) \longmapsto \overline{\phi}_U(r, s) := \phi_U(r, s),$$

where $r, s \in \mathcal{H}(U)$, is *non-isotropic.* Hence, the theorem holds for the case $k = 1$. Let us now proceed by induction to $k > 1$. To this end, put $\mathcal{F}_{k-1} \simeq \mathcal{A}^{k-1}$ and $\mathcal{F}_k := \mathcal{F} \simeq \mathcal{A}^k$. Then, $\mathcal{F}_{k-1} \subsetneqq \mathcal{F}_k$, so $\mathcal{F}_k^{\perp_\phi} \subsetneqq \mathcal{F}_{k-1}^{\perp_\phi}$. Since *orthogonals of free sub-*$\mathcal{A}$*-modules in an orthogonally convenient* $\mathcal{A}$*-module are free sub-*$\mathcal{A}$*-modules,* the inclusion $\mathcal{F}_k^{\perp_\phi} \subsetneqq \mathcal{F}_{k-1}^{\perp_\phi}$ implies that, if $\mathcal{F}_{k-1}^{\perp_\phi} \simeq \mathcal{A}^m$ and $\mathcal{F}_k^{\perp_\phi} \simeq \mathcal{A}^n$ with $n < m$, then there exists a sub-$\mathcal{A}$-module $\mathcal{G} \subseteq \mathcal{F}_{k-1}^{\perp_\phi}$ such that $\mathcal{G} \simeq \mathcal{A}^{m-n}$. For every open $U \subseteq X$, pick a nowhere-zero section $s_{k,U} \in \mathcal{G}(U)$, and put $\mathcal{H}_k(U) = [r_{k,U}, s_{k,U}]$. The correspondence

$$U \longmapsto \mathcal{H}_k(U),$$

where U is open in X, along with the obvious restriction maps, is a complete presheaf of $\mathcal{A}(U)$-modules. Since $\phi_U(r_{i,U}, s_{k,U}) = 0$ for $1 \leq i \leq k-1$, $\phi_U(r_{k,U}, s_{k,U})$ is nowhere zero. Hence, $\mathcal{H}_k(U)$ is a non-isotropic $\mathcal{A}(U)$-plane containing $r_{k,U}$. By Theorem 3.20 $\mathcal{E} = \mathcal{H}_k \perp \mathcal{H}_k^{\perp_\phi}$. Since $r_{k,U}, s_{k,U} \in \mathcal{F}_{k-1}^{\perp_\phi}(U)$, $\mathcal{H}_k(U) \subseteq \mathcal{F}_{k-1}^{\perp_\phi}(U)$ for every open $U \subseteq X$; so $\mathcal{H}_k \subseteq \mathcal{F}_{k-1}^{\perp_\phi}$, which in turn implies that $\mathcal{F}_{k-1} \subseteq \mathcal{H}_k^{\perp_\phi}$. Apply an inductive argument to $\mathcal{F}_{k-1}$ regarded as a sub-$\mathcal{A}$-module of the non-isotropic $\mathcal{A}$-module $\mathcal{H}_k^{\perp_\phi}$. ∎

Chapter 4

Symplectic and orthogonal $\mathcal{A}$-modules

The study of a finite dimensional vector space with respect to a non-degenerate skew-symmetric form is called *symplectic geometry.* See [39, pp 372-384]. In the same vein, we call the study of a free $\mathcal{A}$-module $\mathcal{E}$ of finite rank, endowed with a symplectic $\mathcal{A}$-bilinear form ϕ *abstract symplectic geometry*, i.e., *symplectic geometry within the setting of abstract differential geometry.* If the $\mathcal{A}$-bilinear form is orthosymmetric (the definition of an orthosymmetric $\mathcal{A}$-bilinear form is adapted from the classical one) on an $\mathcal{A}$-module $\mathcal{E}$ ($\mathcal{E}$ is not here necessarily free), then ϕ is section-wise either symmetric (the corresponding geometry is called orthogonal) or skew-symmetric (the corresponding geometry is called symplectic), cf Theorem 4.4. The chapter ends with the structure of orthogonal geometry, of which the Cartan-Dieudonné theorem is the main result. Briefly, Cartan-Dieudonné's theorem stipulates that $\mathcal{A}$-isometries are products of symmetries with respect to non-isotropic hyperplanes. The structure of symplectic geometry is carried out in Chapter 5.

We will assume in this chapter that the pair $(X, \mathcal{A})$ is an *ordered* *algebraized space* with $\mathcal{A}$ a *unital commutative torsion-free* $\mathbb{C}$-*algebra sheaf* such that *all its nowhere-zero sections are invertible,* viz. if $s \in \mathcal{A}(U)$, where U is open in X, is such that $s|_V \neq 0$ for every open $V \subseteq U$, then $s \in \mathcal{A}(U)^\bullet \simeq \mathcal{A}^\bullet(U)$ ($\mathcal{A}^\bullet$ denotes the sheaf generated by the complete presheaf $U \longmapsto \mathcal{A}(U)^\bullet$, where U runs over the open subsets of X, and $\mathcal{A}(U)^\bullet$ consists of the invertible elements of the unital $\mathbb{C}$-algebra $\mathcal{A}(U)$; (cf. [50]). By a *torsion-free* $\mathbb{C}$-*algebra sheaf* $\mathcal{A}$, we mean that *if* $s, t \in \mathcal{A}(U)$ *are nowhere zero and* U *arbitrary, then* $s \cdot t$ *is nowhere zero.*

4.1 Affine Darboux theorem

Definition 4.1 Let $\mathcal{E}$ be an $\mathcal{A}$-module. A **symplectic $\mathcal{A}$-morphism** (or **symplectic $\mathcal{A}$-form**) on $\mathcal{E}$ is a **skew-symmetric** and **non-degenerate** $\mathcal{A}$-bilinear form on $\mathcal{E}$.

Theorem 4.2 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of rank $2n$, ϕ a skew-symmetric non-degenerate $\mathcal{A}$-bilinear form on $\mathcal{E}$, and I and J two (possibly empty) subsets of $\{1, \ldots, n\}$. Moreover, let $A = \{r_i \in \mathcal{E}(U) : \ i \in I\}$ and $B = \{s_j \in \mathcal{E}(U) : \ j \in J\}$ be such that r_i, s_j $(i \in I, \ j \in J)$ are nowhere zero, and*

$$\phi_U(r_i, r_j) = \phi_U(s_i, s_j) = 0, \quad \phi_U(r_i, s_j) = \delta_{ij}, \quad (i, j) \in I \times J. \qquad (4.1)$$

Then, there exists a basis $\mathfrak{B}$ of $(\mathcal{E}(U), \phi_U)$ containing $A \cup B$.

Proof. As in [25, pp. 12, 13, Theorem 1.15], we have three cases. With no loss of generality, we assume that $U = X$.

(1) <u>*Case:* $I = J = \emptyset$</u>. Since $\mathcal{A}^{2n} \neq 0$ (we already assumed that $\mathbb{C} \equiv \mathbb{C}_X \subseteq \mathcal{A}$), there exists an element

$$0 \neq r_1 \in \mathcal{E}(X) \simeq \mathcal{A}^{2n}(X) \simeq \mathcal{A}(X)^{2n}$$

(take e.g. the image (by the isomorphism $\mathcal{E}(X) \simeq \mathcal{A}^{2n}(X)$) of an element in the canonical basis of (sections) of $\mathcal{A}^{2n}(X)$). By virtue of Lemma 3.19, there exists a section $\overline{s}_1 \in \mathcal{E}(X)$ such that $\phi_V(r_1|_V, \overline{s}_1|_V) \neq 0$ for any open subset V in X. Thus, based on the hypothesis on $\mathcal{A}$, $\phi_X(r_1, \overline{s}_1)$ is invertible in $\mathcal{A}(X)$. Putting $s_1 := u^{-1}\overline{s}_1$, where $u \equiv \phi_X(r_1, \overline{s}_1) \in \mathcal{A}(X)$, one gets

$$\phi_X(r_1, s_1) = 1.$$

Now, let us consider

$$S_1 := [r_1, s_1],$$

that is, the $\mathcal{A}(X)$*-plane,* spanned by r_1 and s_1 in $\mathcal{E}(X)$, along with *its orthogonal complement in* $\mathcal{E}(X)$, i.e.,

$$S_1^{\perp} \equiv T_1 := \{t \in \mathcal{E}(X) : \ \phi_X(t, z) = 0, \ \text{for all } z \in S_1\}.$$

The sections r_1 and s_1 are linearly independent, for if $s_1 = ar_1$, with $a \in \mathcal{A}(X)$, then

$$1 = \phi_X(r_1, s_1) = \phi_X(r_1, ar_1) = a\phi_X(r_1, r_1) = 0,$$

a *contradiction.* So, $\{r_1, s_1\}$ is a basis of S_1. Furthermore, we prove that

$$(i) \ S_1 \cap T_1 = 0, \ \ (ii) \ S_1 + T_1 = \mathcal{E}(X).$$

Indeed, (i) since $\phi_X(r_1, s_1) \neq 0$, we have $S_1 \cap T_1 = 0$. On the other hand, (ii) for every $z \in \mathcal{E}(X)$, one has

$$z = (-\phi_X(z, r_1)s_1 + \phi_X(z, s_1)r_1) + (z + \phi_X(z, r_1)s_1 - \phi_X(z, s_1)r_1),$$

with

$$-\phi_X(z, r_1)s_1 + \phi_X(z, s_1)r_1 \in S_1,$$

and

$$z + \phi_X(z, r_1)s_1 - \phi_X(z, s_1)r_1 \in T_1.$$

Thus,

$$\mathcal{E}(X) = S_1 \oplus T_1.$$

The restriction $\phi_1 \equiv \phi_{1,X}$ of ϕ_X to T_1 *is non-degenerate* as $\mathcal{E}$ (in particular, $\mathcal{E}(X)$) is non-isotropic. (T_1, ϕ_1) *is thus a symplectic free* $\mathcal{A}(X)$*-module of rank* $2(n-1)$. Repeating the construction above $n-1$ times, we obtain a strictly decreasing sequence

$$(\mathcal{E}(X), \phi_X) \supseteq (T_1, \phi_1) \supseteq \cdots \supseteq (T_{n-1}, \phi_{n-1})$$

of symplectic free $\mathcal{A}(X)$-modules with rank $T_k = 2(n-k)$, $k = 1, \ldots, n-1$, and also an increasing sequence

$$\{r_1, s_1\} \subseteq \{r_1, r_2; s_1, s_2\} \subseteq \cdots \subseteq \{r_1, \ldots, r_n; s_1, \ldots, s_n\}$$

of gauges; each satisfying relations (4.1).

(2) Case $I = J \neq \emptyset$. We may assume without loss of generality that $I = J = \{1, 2, \ldots, k\}$, and let S be the submodule spanned by $\{r_1, \ldots, r_k; s_1, \ldots, s_k\}$. Clearly, $\phi_X|_S$ is non-degenerate; by Adkins-Weintraub [1, Lemma (2.31), p.360], it follows that $S \cap S^\perp = 0$. On the other hand, let $z \in \mathcal{E}(X)$. One has

$$z = (-\sum_{i=1}^{k} \phi_X(z, r_i)s_i + \sum_{i=1}^{k} \phi_X(z, s_i)r_i)$$

$$+(z + \sum_{i=1}^{k} \phi_X(z, r_i)s_i - \sum_{i=1}^{k} \phi_X(z, s_i)r_i),$$

with

$$-\sum_{i=1}^{k} \phi_X(z, r_i)s_i + \sum_{i=1}^{k} \phi_X(z, s_i)r_i \in S,$$

and

$$z + \sum_{i=1}^{k} \phi_X(z, r_i)s_i - \sum_{i=1}^{k} \phi_X(z, s_i)r_i \in S^\perp.$$

Thus,

$$S \oplus S^\perp = \mathcal{E}(X).$$

Based on the hypothesis on S_1 the restriction $\phi_X|_S$ is a symplectic $\mathcal{A}$-bilinear form. (It is also easily seen that the restriction $\phi_X|_{S^\perp}$ is skew-symmetric.) Moreover, since $S \oplus S^\perp$ and $\mathcal{E}(X)^\perp = 0$, if there exist $z_1 \in S^\perp$ such that $\phi_X(z_1, z) = 0$ for all $z \in S^\perp$, then $z_1 \in \mathcal{E}(X)^\perp = 0$, i.e., $z_1 = 0$. Thus, $\phi_X|_{S^\perp}$ is non-degenerate and hence a symplectic $\mathcal{A}$-form. Applying Case (1) , we obtain a symplectic basis of $S^\perp$, which we denote as

$$\{r_{k+1}, \ldots, r_n; s_{k+1}, \ldots, s_n\}.$$

Then,

$$\mathfrak{B} = \{r_1, \ldots, r_n; s_1, \ldots, s_n\}$$

is a symplectic basis of $\mathcal{E}(X)$ with the required property.

(3) *Case $J \setminus I \neq \emptyset$ (or $I \setminus J \neq \emptyset$).* Suppose that $k \in J \setminus I$; since ϕ_X is non-degenerate there exists $\overline{r}_k \in \mathcal{E}(X)$ such that $\phi_X(\overline{r}_k, s_k) \neq 0$ in the sense that $\phi_V(\overline{r}_k|_V, s_k|_V) \neq 0$ for any open $V \subseteq X$. In other words, the section $v \equiv \phi_X(\overline{r}_k, s_k) \in \mathcal{A}(X)$ is nowhere zero, and is therefore *invertible*. So, if $r_k := v^{-1}\overline{r}_k$, we have $\phi_X(r_k, s_k) = 1$. Next, let us consider the sub-$\mathcal{A}(X)$-module R, spanned by r_k and s_k, viz. $R = [r_k, s_k]$. As in Case (1), we have

$$\mathcal{E}(X) = R \oplus R^\perp.$$

Clearly, for every $i \in I$, $r_i \in R^\perp$. To show this, fix i in I, and assume that $r_i = ar_k + bs_k + x$, where $a, b \in \mathcal{A}(X)$ and $x \in R^\perp$. So, one has

$$0 = \phi_X(r_i, s_k) = a, \ \ 0 = \phi_X(r_i, r_k) = b,$$

which corroborates the claim that $r_i \in R^\perp$ for all $i \in I$. Furthermore, we also clearly have that for every $j \neq k$ in J, $s_j \in R^\perp$. Then $A \cup B \cup \{r_k\}$ is a family of linearly independent sections: the equality

$$a_k r_k + \sum_{i \in I} a_i r_i + \sum_{j \in J} b_j s_j = 0$$

implies that $a_k = a_i = b_j = 0$. Repeating this process as many times as necessary, we are lead back to Case (2), and the proof is finished. ■

Referring to Theorem 4.2, the basis $\mathfrak{B}$ is called a **symplectic $\mathcal{A}(U)$-basis** of $(\mathcal{E}(U), \phi_U)$.

4.2 Orthosymmetric $\mathcal{A}$-bilinear forms

The purpose of this section is to characterize the kinds of *geometries* defined by *symmetric* and *skew-symmetric* $\mathcal{A}$-bilinear morphisms on $\mathcal{A}$-modules. In classical theory, *given a vector space E and a bilinear form $B : E \oplus E \longrightarrow \mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, if B is orthosymmetric (that is $B(x, y) = 0$ if and only if $B(y, x) = 0$, for all $x, y \in E$), then B must either be symmetric or skew-symmetric.* Cf. [16, pp. 4, 5] and [33, pp. 90, 91]. If B is symmetric, the geometry is called *orthogonal.* On the other hand, if B is skew-symmetric, the geometry is called *symplectic.* No other case can occur if B is to be orthosymmetric.

In the present setting, this classical result above is verifiable *section-wise.* We will elaborate on the afore-mentioned result, but first we need the following definition.

Definition 4.3 An $\mathcal{A}$-bilinear form $\phi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ on an $\mathcal{A}$-module $\mathcal{E}$ is called **orthosymmetric** if the following is true:

$$\mathcal{E}^{\perp} = \mathcal{E}^{\top}. \tag{4.2}$$

Equivalently, Equation (4.2) means that, for every open $U \subseteq X$ and (local) sections $t \in \mathcal{E}(U)$, $s \in \mathcal{E}(V)$, with V an open subset of U, we have

$$\phi_V(s, t|_V) = 0 \text{ if, and only if, } \phi_V(t|_V, s) = 0.$$

It is clear that if ϕ is symmetric or skew-symmetric, then ϕ is orthosymmetric. The following theorem shows that the converse of the preceding statement is true section-wise.

Theorem 4.4 *Let $\mathcal{E}$ be an $\mathcal{A}$-module and $\phi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ an orthosymmetric $\mathcal{A}$-bilinear form. Then, section-wise ϕ is either symmetric or skew-symmetric.*

Proof. Let U be an open subset of X, and $r, s, t \in \mathcal{E}(U)$. Clearly, we have

$$\phi_U(r, \phi_U(r,t)s) - \phi_U(r, \phi_U(r,s)t) =$$
$$\phi_U(r,t)\phi_U(r,s) - \phi_U(r,s)\phi_U(r,t) = 0,$$

but

$$\phi_U(r, \phi_U(r,t)s - \phi_U(r,s)t) = 0$$

is equivalent to

$$\phi_U(\phi_U(r,t)s - \phi_U(r,s)t, r) = 0;$$

thus we obtain

$$\phi_U(r,t)\phi_U(s,r) = \phi_U(r,s)\phi_U(t,r). \tag{4.3}$$

For $t = r$, $\phi_U(r,r)\phi_U(s,r) = \phi_U(r,s)\phi_U(r,r)$. If

$$\phi_V(r|_V, s|_V) \neq \phi_V(s|_V, r|_V), \text{ for any open } V \subseteq U, \tag{4.4}$$

then, since $\mathcal{A}$ is torsion-free,

$$\phi_U(r,r) = 0.$$

(We note in passing that (4.4) suggests that both $\phi_V(r|_V, s|_V)$ and $\phi_V(s|_V, r|_V)$ are nowhere zero on V, because if, for instance, $\phi_V(r|_V, s|_V)(x) = 0$ for some $x \in V$ then $\phi_V(r|_V, s|_V) = 0$ on some open neighborhood $R \subseteq V$ of x (cf. Mallios [50, (3.7), p.13]), i.e., assuming that (ρ^U_V) and (σ^U_V) are the restriction maps for the presheaves of sections of $\mathcal{E}$ and $\mathcal{A}$, respectively, we have

$$\sigma^U_R(\phi_U(s,r)) = \phi_R(\rho^U_R(s), \rho^U_R(r)) \equiv \phi_R(s|_R, r|_R) = 0,$$

which, by hypothesis, is equivalent to $\phi_R(r|_R, s|_R) = 0$. That is a contradiction to (4.4).)

Similarly, as

$$\phi_U(s, \phi_U(s,t)r) - \phi_U(s, \phi_U(s,r)t) = 0,$$

which, obviously, leads to

$$\phi_U(s,t)\phi_U(r,s) = \phi_U(s,r)\phi_U(t,s); \tag{4.5}$$

one has, for $t = s$,

$$\phi_U(s,s)\phi_U(r,s) = \phi_U(s,r)\phi_U(s,s).$$

Using (4.4) and since $\mathcal{A}$ is torsion-free, we have

$$\phi_U(s,s) = 0.$$

We actually have *more* than just what we have obtained so far. Indeed, if (4.4) holds, then $\phi_U(t,t) = 0$ for all $t \in \mathcal{E}(U)$. We prove this statement as follows.

(A) Let $\phi_V(r|_V, t|_V) \neq \phi_V(t|_V, r|_V)$ for any open $V \subseteq U$. Since

$$\phi_U(t,r)\phi_U(s,t) = \phi_U(t,s)\phi_U(r,t), \tag{4.6}$$

by putting $s = t$, we have $\phi_U(t,t) = 0$.

(B) Suppose that there exists an open $W \subseteq U$ such that $\phi_W(r|_W, t|_W) = \phi_W(t|_W, r|_W)$. Then, by virtue of (4.3) and since $\phi_W(r|_W, s|_W) \neq \phi_W(s|_W, r|_W)$ everywhere on W, it follows that

$$\phi_W(r|_W, t|_W) = 0.$$

On the other hand, suppose that $\phi_V(s|_V, t|_V) \neq \phi_V(t|_V, s|_V)$ for any open $V \subseteq U$. Putting $r = t$ in (4.6), one gets $\phi_U(t, t) = 0$. Now, assume that there exists an open $T \subseteq U$ such that $\phi_T(s|_T, t|_T) = \phi_U(t|_T, s|_T)$ and for any open subset $V \subseteq U \setminus \overline{T}$, where $\overline{T}$ is the closure of T in X, $\phi_V(s|_V, t|_V) \neq \phi_V(t|_V, s|_V)$. By virtue of (4.5) and of

$$\phi_T(s|_T, r|_T) \neq \phi_T(r|_T, s|_T),$$

it follows that

$$\phi_T(s|_T, t|_T) = \phi_T(t|_T, s|_T) = 0.$$

Hence,

$$\phi_T(r|_T + t|_T, s|_T) = \phi_T(r|_T, s|_T) \neq \phi_T(s|_T, r|_T) = \phi_T(s|_T, r|_T + t|_T),$$

and if we substitute $r|_T + t|_T$ and $s|_T$ for $t|_V$ and $r|_V$ respectively in (A), we get

$$\phi_T(r|_T + t|_T, r|_T + t|_T) = 0.$$

But $\phi_T(r|_T, r|_T) = 0$ (since $\phi_U(r, r) = 0$ and $T \subseteq U$ is open), then if $\phi_T(r|_T, t|_T) = \phi_T(t|_T, r|_T) = 0$, one has

$$\phi_T(t|_T, t|_T) = 0. \tag{4.7}$$

If $\phi_T(r|_T, t|_T) \neq 0 \neq \phi_T(t|_T, r|_T)$ everywhere on T, and $\phi_T(r|_T, t|_T) \neq \phi_T(t|_T, r|_T)$, we deduce from (4.6), by putting $s = t$, $\phi_T(t|_T, t|_T) = 0$. If instead we have $\phi_T(r|_T, t|_T) = \phi_T(t|_T, r|_T)$, we will end up with

$$\phi_T(r|_T, t|_T) = \phi_T(t|_T, r|_T) = 0,$$

which leads to (4.7) as previously shown.

Next, $\phi_V(s|_V, t|_V) \neq \phi_V(t|_V, s|_V)$ for every open $V \subseteq U \setminus \overline{T}$, so

$\phi_V(t|_V, t|_V) = 0$ for every such V; coupling the latter observation with (4.7) and the fact that sections are continuous, one gets in this case too that $\phi_U(t,t) = 0$.

We have shown that there are only two cases: either $\phi_U(r,r) = 0$ for all $r \in \mathcal{E}(U)$, or for some $r \in \mathcal{E}(U)$, $\phi_U(r,r) \neq 0$, from which we deduce that $\phi_U(r,s) = \phi_U(s,r)$ for all $r, s \in \mathcal{E}(U)$.

Finally, we notice in ending the proof that if $\phi_U(r,r) = 0$ for all $r \in \mathcal{E}(U)$, then

$$\phi_U(r,s) = -\phi_U(s,r)$$

for all $r, s \in \mathcal{E}(U)$. ■

Scholium 4.5 In connection with the proof of Theorem 4.4, if there exists an open subset $L \subseteq T$ such that $\phi_L(r|_L, t|_L) = \phi_L(t|_L, r|_L) = 0$ and $\phi_V(r|_V, t|_V) \neq \phi_V(t|_V, r|_V)$ for every $V \subseteq T \setminus \overline{L}$, where $\overline{L}$ is the closure of L in X, then $\phi_L(t|_L, t|_L) = 0$ and $\phi_V(t|_V, t|_V) = 0$ for every open $V \subseteq T \setminus \overline{L}$. Hence, $\phi_T(t|_T, t|_T) = 0$.

Referring still to Theorem 4.4, if ϕ_U is symmetric, the geometry is called **orthogonal.** If ϕ_U is skew-symmetric, the geometry is called **symplectic.** No other case can occur if ϕ must be orthosymmetric. A *pairing* $(\mathcal{E}, \phi)$ is called *symmetric* if every ϕ_U is symmetric, and *skew-symmetric* if every ϕ_U is skew-symmetric. Suppose ϕ is skew-symmetric and $r, s \in \mathcal{E}(U)$ with $\phi_U(r,s) = 1$. Then $\phi_U(s,r) = -1$. If we restrict ϕ_U to $H = \text{span}\{r, s\}$, its matrix has the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Such a pair of sections are called **hyperbolic pair** and H is called a **hyperbolic plane**.

4.3 Special features of orthogonal geometry

In this section, $\mathcal{A}$ will be a PID $\mathbb{C}$-algebra sheaf, whereas $\mathcal{E}$ will stand for a special kind of $\mathcal{A}$-modules. But first let us recall the following notion.

Definition 4.6 (Mallios). A subpresheaf F of a presheaf of modules (or more precisely, $A(U)$-modules) E (cf. [50, p. 99, Definition 1.6]) is called a **free subpresheaf** if for every open U in X, $F(U)$ is a free sub-$A(U)$-module of $E(U)$.

Definition 4.7 A **convenient $\mathcal{A}$-module** is a self-pairing $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a free $\mathcal{A}$-module of finite rank and ϕ an orthosymmetric $\mathcal{A}$-bilinear form, such that the following conditions are satisfied:

(i) If $\mathcal{F}$ is a free subpresheaf of $\mathcal{A}(U)$-modules of $\mathcal{E}$, then $\mathcal{F}^{\perp} \equiv \mathcal{F}^{\perp_\phi}$ is a free subpresheaf of $\mathcal{A}(U)$-modules of $\mathcal{E}$;

(ii) Every free subpresheaf $\mathcal{F}$ of $\mathcal{A}(U)$-modules of $\mathcal{E}$ is orthogonally reflexive, i.e., $\mathcal{F}^{\perp\top} = \mathcal{F}^{\top\perp} = \mathcal{F}$;

(iii) The intersection of any two free subpresheaves of $\mathcal{A}(U)$-modules of $\mathcal{E}$ is a free subpresheaf of $\mathcal{A}(U)$-modules.

Note. Concerning the above definition of *convenient $\mathcal{A}$-modules*, by supposing that the (*coefficient-*) algebra sheaf $\mathcal{A}$ is a PID-*algebra sheaf*, we obtain that *every subpresheaf of $\mathcal{A}(U)$-modules of a free $\mathcal{A}$-module is free.* So in that context, conditions (i) and (iii) in Definition 4.7 are satisfied. Now, concerning condition (ii) of the same definition, the *reflexivity* at hand is a known situation in ordinary Functional Analysis: see, for instance, Hilbert spaces and structures having similar properties; so we do have the so-called *complemented topological algebras, Hilbert algebras* and the likes with the aforementioned property for *ideals* (: *modules*), and also analogous examples in *infinite-dimensional Hamiltonian mechanics.* (I am indebted to A. Mallios for *this comment* on convenient $\mathcal{A}$-modules.)

Definition 4.8 *Let ϕ and ψ be $\mathcal{A}$-bilinear morphisms on free $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ respectively. Then ϕ and ψ are $\mathcal{A}$-**isometric** if for every open subset U of X there is an $\mathcal{A}(U)$-module isomorphism $\sigma_U : \mathcal{E}(U) \longrightarrow \mathcal{F}(U)$ with*

$$\psi_U(\sigma_U(s), \sigma_U(t)) = \phi_U(s,t) \ for\ all\ s,t \in \mathcal{E}(U). \tag{4.8}$$

*The family of maps $\sigma \equiv (\sigma_U)_{U\in\mathcal{T}}$ is called an $\mathcal{A}(U)$-**isometry** and we will often say that $\mathcal{E}(U)$ and $\mathcal{F}(U)$ are $\mathcal{A}(U)$-isometric.*

Note that if ϕ_U is non-degenerate for every open subset U of X and $\sigma_U(s) = 0$, then $\psi_U(\sigma_U(s), \sigma_U(t)) = 0$ for every $t \in \mathcal{E}$ and 4.8 implies that $\phi_U(s,t) = 0$, so s is an element of the kernel of $\mathcal{E}$, hence, $s = 0$ and σ_U is injective. Also, if $\sigma_U : \mathcal{E}(U) \longrightarrow \mathcal{F}(U)$ is an $\mathcal{A}(U)$-isometry,

$$\phi_U(e_i, e_j) = \alpha_{i,j} = \phi_U(\sigma_U(e_i), \sigma_U(e_j)) = \alpha_{i',j'}$$

where $\{e_i\}$ is the basis and $e_{i'} = \sigma_U(e_i)$, so $(\det_U \sigma_U)^2 = 1$. The $\mathcal{A}(U)$-isometry for which $(\det_U \sigma_U) = 1$ are called **rotations** and those for which $(\det_U \sigma_U) = -1$ are called **reversions**. In certain cases, all $\mathcal{A}(U)$-isometries are called rotations.

Lemmas 4.9, 4.10, and 4.11, below, are essential for the Cartan-Dieudonné theorem, which will be proved in the next section. Proofs of these lemmas are found in [52].

Lemma 4.9 *Let $(\mathcal{E}, \phi)$ be a free $\mathcal{A}$-module of rank 2, endowed with a non-degenerate symmetric or antisymmetric $\mathcal{A}$-bilinear form ϕ. For an* *open subset $U \subseteq X$, the non-isotropic $\mathcal{A}(U)$-plane $\mathcal{E}(U)$ is hyperbolic if it contains a* nowhere-zero isotropic section *r.*

Lemma 4.10 *Let $(\mathcal{E}, \phi)$ be a* non-isotropic convenient $\mathcal{A}$-module, *and $\mathcal{F}$ any free sub-$\mathcal{A}$-module of $\mathcal{E}$. Moreover, let U be an open subset of X, let the sections $s_1, s_2, \ldots, s_k \in \mathcal{F}(U)$ form a basis of $(rad\, \mathcal{F})(U) = rad\, \mathcal{F}(U)$, and $\mathcal{G}$ be a free sub-$\mathcal{A}$-module of $\mathcal{F}$ such that $\mathcal{F} = rad\, \mathcal{F} \perp \mathcal{G}$. Then, there are* isotropic sections *$t_1, t_2, \ldots, t_k \in \mathcal{E}(U)$ such that the planes $P_i := [s_i, t_i]$ are* hyperbolic, pairwise orthogonal *and also* orthogonal to $\mathcal{G}(U)$. *The $\mathcal{A}(U)$-module*

$$P_1 \perp P_2 \perp \ldots \perp P_k \perp \mathcal{G}(U)$$

contains $\mathcal{F}(U)$.

For the sake of Lemma 4.11, we recall that the sheaf of germs of $\mathcal{A}$-automorphisms of $\mathcal{E}$ is denoted $\mathcal{A}ut\ \mathcal{E}$, cf. [50, p. 294].

Lemma 4.11 *Let $\mathcal{H}_i$, $1 \leq i \leq k$, be hyperbolic $\mathcal{A}$-planes, $\mathcal{E} = \mathcal{H}_1 \perp \mathcal{H}_2 \perp \cdots \perp \mathcal{H}_k$ within an $\mathcal{A}$-isomorphism, and $\sigma \in \mathcal{A}ut\ \mathcal{E}$ an $\mathcal{A}$-isometry such that if $\mathcal{H}_i(U) = [r_i, s_i]$, then $\sigma_U(r_i) = r_i$ for every open $U \subseteq X$. Then σ is a rotation.*

Proof. For every $1 \leq i \leq k$ and open U in X, the relations

$$\begin{aligned} \sigma_U(s_i) &= \textstyle\sum_i^j r_j + \sum_i^j s_j \\ \phi_U(\sigma_U(r_i), \sigma_U(s_i)) &= \phi_U(r_i, s_i) \\ \phi_U(\sigma_U(s_i), \sigma_U(s_j)) &= \phi_U(s_i, s_j) \end{aligned}$$

immediately corroborate the claim. ■

We are now ready to state the sheaf-theoretical version of Cartan-Dieudonné. To this end, the following notion is required.

Definition 4.12 Let $\mathcal{E}$ be a non-isotropic convenient $\mathcal{A}$-module of rank n, and $\mathcal{H}$ a non-isotropic free sub-$\mathcal{A}$-module of $\mathcal{E}$ of rank 1. The **orthogonal symmetry** of $\mathcal{E}$ with respect to $\mathcal{H}^{\perp}$ is the element $u^{\mathcal{H}} \in \mathrm{E}nd_{\mathcal{A}}\mathcal{E}$ such that

$$u^{\mathcal{H}}|_{\mathcal{H}^{\perp}} = 1_{\mathcal{H}^{\perp}}$$

and

$$u^{\mathcal{H}}|_{\mathcal{H}} = -1_{\mathcal{H}}$$

where U is any open subset of X. The free sub-$\mathcal{A}$-module $\mathcal{H}^{\perp}$ (whose rank is $n-1$) is called a **hyperplane** of $\mathcal{E}$.

Theorem 4.13 *Let $\mathcal{A}$ be a PID $\mathbb{C}$-algebra sheaf and ϕ a non-degenerate $\mathcal{A}$-bilinear form on a convenient $\mathcal{A}$-module $\mathcal{E}$ of rank n, having nowhere-zero (local) isotropic sections. Then, every $\mathcal{A}$-isometry $\sigma \equiv (\sigma_U) \in \mathrm{E}nd_{\mathcal{A}}\mathcal{E}$ is a product of at most n orthogonal symmetries with respect to (local) non-isotropic hyperplanes.*

Proof. The proof is by induction on n. If $n = 1$, then $\sigma = \pm 1$. In fact, for $n = 1$, every $\mathcal{A}(U)$-module $\mathcal{E}(U)$ has the form $\mathcal{E}(U) = [s_U]$ where $[s_U]$ stands for the $\mathcal{A}(U)$-module generated by the (nowhere-zero) section s_U, such that if V is an open set contained in U, then $s_V = \rho^U_V(s_U)$, with the $\{\rho^U_V\}$ being the restriction maps for the (complete) presheaf of sections of $\mathcal{E}$. So suppose now, for some fixed open $U \subseteq X$, that $\sigma_U(s_U) = a_U s_U$, with $a_U \in \mathcal{A}(U)$, then since $\phi_U(\sigma_U(s_U), \sigma_U(s_U)) = \phi_U(s_U, s_U)$ and

$$\tau^U_V(\phi_U(s_U, s_U)) \equiv \phi_U(s_U, s_U)|_V := \phi_V(s_V, s_V) \neq 0$$

for every open $V \subseteq U$, $a^2_U = 1$ (the τ^U_V are the restriction maps for $\Gamma\mathcal{A}$). It is clear that if $a_U = 1$ (resp. $a_U = -1$) i.e. $\sigma_U(s_U) = s_U$ (resp. $\sigma_U(s_U) = -s_U$), then $\sigma_V(s_V) = s_V$ (resp. $\sigma_V(s_V) = -s_V$) for any open V contained in U. Therefore, if $\sigma_X(s_X) = s_X$ (resp. $\sigma_X(s_X) = -s_X$), then $\sigma_U(s_U) = s_U$ (resp. $\sigma_U(s_U) = -s_U$) for every open $U \subseteq X$. Since 1 is a product of 0 $\mathcal{A}$-symmetries and -1 is the unique $\mathcal{A}$-symmetry, the result clearly follows for $n = 1$.

Now, we shall consider four cases.

Case 1. There exists a non-isotropic section $s \in \mathcal{E}(X)$ such that $\sigma_X s = s$. Let $F = [s]$, i.e. the sub-$\mathcal{A}(X)$-module of $\mathcal{E}(X)$ of rank 1, generated by s. Clearly, $F^\perp$ is a sub-$\mathcal{A}(X)$-module of $\mathcal{E}(X)$; but $\mathcal{A}(X)$ is a PID $\mathbb{C}$-algebra, therefore $F^\perp$ is free. For any $t \in F^\perp$, we have

$$\phi_X(\sigma_X(t), s) = \phi_X(\sigma_X(t), \sigma_X(s)) = \phi_X(t, s) = 0;$$

which implies that $\sigma_X(t) \in F^\perp$. Thus, we have the inclusion $\sigma_X(F^\perp) \subseteq F^\perp$. Since $\sigma_X : \mathcal{E}(X) \longrightarrow \mathcal{E}(X)$ is an $\mathcal{A}(X)$-isomorphism, the restriction $\sigma_X|_{F^\perp} : F^\perp \longrightarrow \sigma_X(F^\perp) \subseteq F^\perp$ is an $\mathcal{A}(X)$-isomorphism; so $\sigma_X(F^\perp) = F^\perp$. Let ψ be the restriction of σ_X to $F^\perp$, a free $\mathcal{A}(X)$-module of rank $n-1$ (because $\mathcal{E}(X) = F \perp F^\perp$ with both F and $F^\perp$ free); we may by the induction hypothesis write

$$\psi = \tau_1 \tau_2 \cdots \tau_k$$

with $k \leq n-1$, where every τ_i is a symmetry of the $\mathcal{A}(X)$-module $F^\perp$ with respect to some hyperplane H_i of $F^\perp$. Now, let G_i be the free sub-$\mathcal{A}(X)$-module of $F^\perp$ of rank 1 which determines the symmetry τ_i, so $F^\perp = G_i \perp H_i$, and let $\overline{\tau}_i : \mathcal{E}(X) \longrightarrow \mathcal{E}(X)$ be an $\mathcal{A}(X)$-morphism given by the prescription

$$\overline{\tau}_i = -1_{G_i}, \quad \overline{\tau}_i|_{F \perp H_i} = 1_{F \perp H_i}$$

(obviously, $\overline{\tau}_i$ is an extension of the (corresponding) τ_i to all the $\mathcal{A}(X)$-module $\mathcal{E}(X)$). By its very definition, $\overline{\tau}_i$ is a symmetry of $\mathcal{E}(X)$ with respect to $F \perp H_i$. A tedious calculation shows that

$$\overline{\tau}_1 \overline{\tau}_2 \cdots \overline{\tau}_k = 1_F \perp \psi.$$

But

$$1_F \perp \psi = \sigma_X,$$

so we have expressed σ_X by at most $n-1$ symmetries.

Case 2. There exists a non-isotropic section $s \in \mathcal{E}(X)$ such that $\sigma_X s - s$ is non-isotropic. Let $H := [\sigma_X s - s]^{\perp}$, and let τ be the symmetry of $\mathcal{E}(X)$ with respect to H. Since

$$\phi_X(\sigma_X s + s, \sigma_X s - s) = \phi_X(\sigma_X s, \sigma_X s) - \phi_X(s, s) = 0$$

we have $\sigma_X s + s \in H$. Therefore

$$\tau(\sigma_X s + s) = \sigma_X s + s, \;\; \tau(\sigma_X s - s) = s - \sigma_X s,$$

whence $\tau\sigma_X(s) = s$; so we are led back to **Case 1** because $\tau\sigma_X$ is an isometry $\mathcal{E}(X) \longrightarrow \mathcal{E}(X)$ which leaves s fixed. By **Case 1**, $\tau\sigma_X = \tau_1\tau_2\cdots\tau_k$ with $k \leq n-1$, where τ_i is a symmetry of $\mathcal{E}(X)$ with respect to some hyperplane of $\mathcal{E}(X)$. Since τ is an involution (so $\tau^2 = 1$), it is easy to see that

$$\sigma_X = \tau\tau_1\tau_2\cdots\tau_k,$$

which is a product of $k+1$ ($k+1 \leq n$) symmetries.

Case 3. Suppose $n = 2$, and let r be a nowhere-zero isotropic section in $\mathcal{E}(X)$. By Lemma 4.9, we may assume that $\mathcal{E}(X) = [r, s]$, where (r, s) is a hyperbolic pair. Then, there exist $a_i \in \mathcal{A}(X)$, $1 \leq i \leq 4$, such that

$$\sigma_X r = a_1 r + a_2 s, \;\; \sigma_X s = a_3 r + a_4 s.$$

Since (r, s) is hyperbolic, there are only two possibilities:

(a) $\sigma_X r = \alpha s$, $\sigma_X s = \alpha^{-1} r$, $\alpha \in \mathcal{A}(X)$. Therefore, $\sigma_X(r + \alpha s) = r + \alpha s$ is a fixed non-isotropic section. Hence, **Case 1** applies.

(b) $\sigma_X r = \alpha r$, $\sigma_X s = \alpha^{-1} s$, $\alpha \in \mathcal{A}(X)$. We may assume that $\alpha \neq 1$, since $\alpha = 1$ means that σ_X is the identity $1_{\mathcal{E}(X)}$. Put $t := r + s$; t and $\sigma_X t - t = (\alpha - 1)r + (\alpha^{-1} - 1)s$ are both non-isotropic, and this evokes **Case 2**.

Case 4. We can assume the following: $n \geq 3$ (we are not in **Case 3**), if $s \in \mathcal{E}(X)$ is non-isotropic, then $\sigma_X s - s$ is isotropic (we are not in **Case 2**) and not zero (we are not in **Case 1**).

Let t be a nowhere-zero isotropic section in $\mathcal{E}(X)$. The sub-$\mathcal{A}(X)$-module $[t]^\perp \subseteq \mathcal{E}(X)$ is free and such that rank $[t]^\perp \geq 2$ and rad $[t]^\perp = [t]$. On another hand, we observe that $[t]^\perp$ contains a non-isotropic section r (otherwise $[t]^\perp$ would be totally isotropic, and as rad $[t]^\perp = [t]$, $[t]$ would be equal to $[t]^\perp$).

We have $\phi_X(r, r) \neq 0$ and for $a \in \mathcal{A}(X)$, $\phi_X(r + at, r + at) = \phi_X(r, r) \neq 0$. We deduce by the hypothesis of **Case 4** that $\sigma_X r - r$ as well as the sections

$$\sigma_X(r + at) - (r + at) = (\sigma_X r - r) + a(\sigma_X t - t)$$

are isotropic. Therefore,

$$2a\phi_X(\sigma_X r - r, \sigma_X t - t) + a^2\phi_X(\sigma_X t - t, \sigma_X t - t) = 0.$$

Putting first $a = 1$, then $a = -1$, and adding the results thus obtained, we have

$$\phi_X(\sigma_X t - t, \sigma_X t - t) = 0,$$

i.e. $\sigma_X t - t$ is isotropic.

We have established that $\sigma_X s - s$ will be isotropic whether s is isotropic or not. The set G consisting of all $\sigma_X s - s$, where $s \in \mathcal{E}(X)$, has the structure of an $\mathcal{A}(X)$-module as $G = \text{im } (\sigma_X - 1_{\mathcal{E}(X)}) \equiv \text{im } (\sigma_X - 1) := (\sigma_X - 1)\mathcal{E}(X)$. G contains only isotropic sections and is consequently an isotropic free sub-$\mathcal{A}(X)$-module of $\mathcal{E}(X)$. It is easily seen that for all $s, t \in \mathcal{E}(X)$, $\phi_X(\sigma_X s - s, \sigma_X t - t) = 0$.

Let now $s \in \mathcal{E}(X)$ and $u \in G^\perp$. One has

$$\phi_X(\sigma_X s - s, \sigma_X u - u) = \phi_X(\sigma_X s, \sigma_X u) - \phi_X(s, \sigma_X u) - \phi_X(\sigma_X s - s, u) = 0.$$

Since $\sigma_X s - s \in G$, $u \in G^\perp$ and σ_X an isometry, it follows that

$$\phi_X(s, u - \sigma_X u) = 0.$$

This equality is true for all $s \in \mathcal{E}(X)$, so that $u - \sigma_X u \in \operatorname{rad} \mathcal{E}(X) = 0$, i.e. $u = \sigma_X u$.

We have established that every section in $G^\perp$ is left fixed. By the assumption which defines **Case 4**, every section in $G^\perp$ is isotropic; therefore $G^\perp$ is totally isotropic. On account of Theorem 3.20, we have $\operatorname{rank} G + \operatorname{rank} G^\perp = n$; since both G and $G^\perp$ are isotropic, it follows, by virtue of Lemma 4.10, that $\operatorname{rank} G \leq n/2$ and $\operatorname{rank} G^\perp \leq n/2$. We deduce that $\operatorname{rank} G = \operatorname{rank} G^\perp = n/2$. The $\mathcal{A}(X)$-module $\mathcal{E}(X)$ is, therefore, a hyperbolic $\mathcal{A}(X)$-module H_{2l}, $n = 2l$ and $G^\perp$ a maximal isotropic sub-$\mathcal{A}(X)$-module of H_{2l}. The isometry σ_X leaves every element of $G^\perp$ fixed, and Lemma 4.11 shows that σ_X is a rotation. Therefore **Case 4** cannot occur if σ_X is a reflection, and the proof is complete for a reflection. If σ_X is a rotation in H_{2l}, and s any symmetry, then $s\sigma_X$ is a reflection. Hence, $s\sigma_X = \tau_1\tau_2\cdots\tau_k$, $k \leq 2l$ with k an odd number, so $\sigma_X = s\tau_1\tau_2\cdots\tau_k$ with at most $2l = n$ symmetries. ■

But for a *Riemannian convenient $\mathcal{A}$-module $\mathcal{E}$* of finite rank, equipped with a non-degenerate $\mathcal{A}$-bilinear form ϕ, the condition that for every open $U \subseteq X$, $\mathcal{E}(U)$ have nowhere-zero isotropic sections does not play any role, and may thus be dropped. Hence,

Theorem 4.14 *Let $\mathcal{A}$ be a PID $\mathbb{C}$-algebra sheaf and $\mathcal{E}$ a convenient Riemannian $\mathcal{A}$-module (i.e., still equipped with a Riemannian $\mathcal{A}$-metric ϕ) of rank n. Then, every $\mathcal{A}$-isometry of $\mathcal{E}$ is a product of at most n orthogonal symmetries with respect to (local) non-isotropic hyperplanes.*

Proof. Here as well, we distinguish 4 cases, of which **Case 1** is the same as **Case 1** of Theorem 4.13.

Case II. If there exists a *non-isotropic section* $r \in \mathcal{E}(X)$ such that $\sigma_X(r) = -r$, and that τ_r is the orthogonal symmetry defined by r, one has: $(\tau_r \circ \sigma_X)(r) = r$; so we are back to **Case I** since $\tau_r \circ \sigma_X$ is an $\mathcal{A}(X)$-isometry.

If σ_X has no eigen-section in $\mathcal{E}(X)$, let us consider a non-isotropic section r and its image $s := \sigma_X(r)$. Since $\phi_X(r,r) = \phi_X(s,s)$,

$$\phi_X(r+s, r-s) = 0,$$

i.e. $r+s$ and $r-s$ are orthogonal, and

$$\phi_X(r+s, r+s) + \phi_X(r-s, r-s) = 4\phi_X(r,r) \neq 0.$$

Consequently at least one of the sections $r+s$ and $r-s$ is non-isotropic. We will distinguish the following cases:

Case III (of which **Case II** is a particular instance.) There exists a *non-isotropic section* r such that $t \equiv r - \sigma_X(r) := r - s$ *is non-isotropic.* Therefore, $(\tau_t \circ \sigma_X)(r) = r$. In fact,

$$\tau_t(\sigma_X(r) + r) := \tau_t(s+r) = r+s,$$

$$\tau_t(-r + \sigma_X(r)) := \tau_t(-r+s) = r-s,$$

whence the expected result. Here too, we are back to **Case I.**

Case IV For any *non-isotropic section* $r \in \mathcal{E}(X)$, the section $r - \sigma_X(r)$ *is isotropic* (we are not in **Case III**) and *non-zero* (we are not in **Case I**). In other words, the kernel $\ker(\sigma_X - I_X)$ is a totally isotropic (free) sub-$\mathcal{A}(X)$-module of $\mathcal{E}(X)$, and the image $\mathrm{im}(\sigma_X - I_X)$ is such that any non-isotropic section $r \in \mathcal{E}(X)$ has as image an isotropic section. Let us consider a section $u \equiv r + \sigma_X(r) := r + s$. Based on **Case II** and since $r - s$ is isotropic, $r + s$ is non-isotropic. The orthogonal symmetry τ_u swaps r and $-s$. In fact, this follows from

$$\tau_u(r+s) = -r-s, \quad \tau_u(r-s) = r-s.$$

Furthermore,

$$(\tau_u \circ \sigma_X)(r) = -r \quad \text{and} \quad \sigma'_X(r) \equiv (\tau_r \circ \tau_u \circ \sigma_X)(r) = r.$$

By virtue of Theorem 3.20, $\mathcal{E}(X) = [r] \perp [r]^{\perp}$; if σ'_H, the restriction of σ'_X to $H \equiv [r]^{\perp}$, is a product of p orthogonal symmetries, then σ_X is a product of $(p+2)$ orthogonal symmetries.

Now, we need to show that σ_X is a product of at most n orthogonal symmetries. For the sake of brevity, the property:

for any non-isotropic section $r \in \mathcal{E}(X)$, $\sigma_X(r) - r$ is isotropic and non-zero

is called Property (P). We have

$$\phi_X(\sigma_X(r) - r, \sigma_X(r) - r) = 2[\phi_X(r,r) - \phi_X(r, \sigma_X(r))].$$

Condition (P) implies that $\phi_X(r,r) = \phi_X(r, \sigma_X(r))$ for any non-isotropic section r.

If $r, s \in \mathcal{E}(X)$ are non-isotropic, then for some $\lambda, \mu \in \mathcal{E}(X)$, $\lambda r + \mu s$ is non-isotropic. For this purpose, note that

$$\phi_X(r+s, r+s) + \phi_X(r-s, r-s) = 2\phi_X(r,r) + 2\phi_X(s,s).$$

If $\phi_X(r,r) = -\phi_X(s,s)$, then clearly $\phi_X(r+s, r+s) + \phi_X(r-s, r-s) = 0$. Assume further that $\phi_X(r-s, r-s) = 0$ (then obviously $\phi_X(r+s, r=s) = 0$). Then, $\phi_X(r-2s, r-2s) = -3\phi_X(s,s) \neq 0$, so that $r-2s$ is non-isotropic. Since

$$\phi_X(\lambda r + \mu s, \lambda r + \mu s) = \phi_X(\lambda r + \mu s, \sigma_X(\lambda r + \mu s)),$$

we deduce that

$$2\phi_X(r,s) = \phi_X(r, \sigma_X(s)) + \phi_X(s, \sigma_X(r))$$

for any pair of non-isotropic sections $r, s \in \mathcal{E}(X)$.

As there is associated with ϕ_X an orthogonal basis of non-isotropic sections (cf. Mallios [50, pp.335-340]), it follows that $\phi_X(r,r) = \phi_X(r, \sigma_X(r))$ for any $r \in \mathcal{E}(X)$, and
$2\phi_X(r,s) = \phi_X(r, \sigma_X(s)) + \phi_X(s, \sigma_X(r))$ for all $r, s \in \mathcal{E}(X)$. Thus,

$$\begin{aligned} &\phi_X(\sigma_X(r) - r, \sigma_X(s) - s) = \\ &\qquad \phi_X(\sigma_X(r), \sigma_X(s)) + \phi_X(r,s) - \phi_X(r, \sigma_X(s)) = 0 \end{aligned} \tag{4.9}$$

for all $r, s \in \mathcal{E}(X)$. Put $\theta := \sigma - I$, where $I := \mathrm{Id}_{\mathcal{E}}$;
$\mathrm{im}\,\theta \equiv \theta(\mathcal{E}) := (\theta(\mathcal{E}), \rho|_{\theta(\mathcal{E})}, X)$ is a (free) sub-$\mathcal{A}$-module of $\mathcal{E}$. Based on (4.9), the image $\theta_X(\mathcal{E}(X))$ is a totally isotropic (free) sub-$\mathcal{A}(X)$-module of $\mathcal{E}(X)$.

But by Condition (P), we deduce that $\ker\theta_X$ is a totally isotropic (free) sub-$\mathcal{A}(X)$-module of $\mathcal{E}(X)$. The kernel $\ker\theta_X$ is a maximal totally isotropic (free) sub-$\mathcal{A}(X)$-module of $\mathcal{E}(X)$. One has

$$r \in \ker\theta_X \quad \text{iff} \quad \theta_X(r) = r$$

and

$$\phi_X(r, \sigma_X(s) - s) = \phi_X(\sigma_X(r), \sigma_X(s)) - \phi_X(r,s) = 0$$

for any $r \in \ker\theta_X$ and $s \in \mathcal{E}(X)$; every $\sigma_X(s) - s$ is therefore an element of $(\ker\theta_X)^{\perp}$. In other words, one has

$$\mathrm{im}\,\theta_X := \{\sigma_X(s) - s;\ s \in \mathcal{E}(X)\} \subseteq (\ker\theta_X)^{\perp}.$$

But $\mathcal{E}(X) = \ker\theta_X \perp (\ker\theta_X)^{\perp}$, so $(\ker\theta_X)^{\perp}$ is totally isotropic, and

$$\text{rank } \mathcal{E}(X) = \text{rank } \ker\theta_X + \text{rank } (\ker\theta_X)^{\perp}.$$

On the other hand, since $\ker\theta_X$ and $(\ker\theta_X)^{\perp}$ are both totally isotropic, rank $\ker\theta_X \leq \frac{1}{2}$rank $\mathcal{E}(X)$ and rank $(\ker\theta_X)^{\perp} \leq \frac{1}{2}$rank $\mathcal{E}(X)$, it follows that

$$\ker\theta_X = \text{im } \theta_X = (\ker\theta_X)^{\perp}.$$

If S is a maximal totally isotropic (free) sub-$\mathcal{A}(X)$-module of $\mathcal{E}(X)$ such that $\mathcal{E}(X) = \ker\theta_X \oplus S$, the union of any basis of $\ker\theta_X$ and of any basis of S yields a basis of $\mathcal{E}(X)$ with respect to which the matrix representing the $\mathcal{A}(X)$-morphism θ_X is triangular and has zeros on the main diagonal. In fact, $\sigma_X(s) - s \in \ker\theta_X$, for any $s \in S$. Therefore,

$$\det{}_X \sigma_X = \det{}_X (I_X + \theta_X) = 1_X,$$

so that σ_X can only be a product of an even number of symmetries.

Let τ be an arbitrary orthogonal symmetry of $\mathcal{E}(X)$; then

$$\det{}_X (\tau \circ \sigma_X) = -1_X$$

and consequently $\tau \circ \sigma_X$ cannot satisfy Condition (P). Therefore **Case I** or **II** or **III** applies: $\tau \circ \sigma_X$ is a product of at most n symmetries, and σ_X a product of at most $n+1$ symmetries. Since n is even and σ_X is a product of an even number of symmetries, it follows that σ_X is a product of at most n symmetries, and the proof is finished. ■

Chapter 5

$\mathcal{A}$-transvections

Building on prior joint work by Mallios and Ntumba, we study *transvections* (Dieudonné) in the realm of *Abstract Geometric Algebra,* referring herewith to *symplectic $\mathcal{A}$-modules.* A characterization of *$\mathcal{A}$-transvections* in terms of *$\mathcal{A}$-hyperplanes* is given together with the associated *matrix* definition. By taking the domain of coefficients $\mathcal{A}$ to be a PID algebra sheaf, we also consider the analogue of a form of the classical *Witt's extension theorem,* concerning $\mathcal{A}$-symplectomorphisms defined on appropriate *Lagrangian sub-$\mathcal{A}$-modules.* The chapter ends with a counterpart of the classical factorization theorem of symplectomorphisms of symplectic vector spaces of finite dimension into symplectic transvections, cf. [16] and [18]; more accurately, this counterpart concerns *$\mathcal{A}$-symplectomorphisms* of *symplectic orthogonally convenient $\mathcal{A}$-modules of finite rank*, where $\mathcal{A}$ is a *torsion-free PID* $\mathbb{C}$-*algebra sheaf,* having the usual additional property that *all its nowhere-zero sections are invertible.*

5.1 Symplectic $\mathcal{A}$-transvections

We notice that if $\mathcal{E}$ is a free $\mathcal{A}$-module and $\mathcal{F}$ an $\mathcal{A}$-hyperplane of $\mathcal{E}$, then every $\mathcal{A}$-endomorphism ϕ of $\mathcal{E}$ that leaves $\mathcal{F}$ stable induces on the *line $\mathcal{A}$-module* $\mathcal{E}/\mathcal{F}$ an *$\mathcal{A}$-homothecy*, which we denote by $\widetilde{\phi}$. More explicitly, if U is open in X and s a section of $\mathcal{E}/\mathcal{F}$ over U, then

$$\widetilde{\phi}(s) \equiv \widetilde{\phi}_U(s) = a_U s \equiv as$$

for some $a_U \equiv a \in \mathcal{A}(U)$. The coefficient sections a_U are such that $a_V = a_U|_V$ whenever V is contained in U. The global section $a_X \equiv a$ is called the *ratio* of the $\mathcal{A}$-homothecy $\widetilde{\phi}$.

Lemma 5.1 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, and $\mathcal{F}$ a proper free sub-$\mathcal{A}$-module of $\mathcal{E}$. Then, the following assertions are equivalent.*

(1) *$\mathcal{F}$ is an $\mathcal{A}$-hyperplane of $\mathcal{E}$.*

(2) *For every (local) section $s \in \mathcal{E}(U)$ such that $s|_V \notin \mathcal{F}(V)$ for every open $V \subseteq U$,*

$$\mathcal{E}(U) = \mathcal{A}(U)s \oplus \mathcal{F}(U).$$

(3) *For every open $U \subseteq X$, there exists a section $s \in \mathcal{E}(U)$ with $s|_V \notin \mathcal{F}(V)$, where V is any open subset contained in U, such that*

$$\mathcal{E}(U) = \mathcal{A}(U)s \oplus \mathcal{F}(U).$$

(4) *The free sub-$\mathcal{A}$-module $\mathcal{F}$ is a maximal sub-$\mathcal{A}$-module in the inclusion-ordered set of proper free sub-$\mathcal{A}$-modules of $\mathcal{E}$.*

Proof. (1) $\Rightarrow$ (2): For every open $U \subseteq X$ and section $s \in \mathcal{E}(U)$ such that $s|_V \notin \mathcal{F}(V)$ for any open $V \subseteq U$, it is clear that $\mathcal{A}(U)s +$

$\mathcal{F}(U)$ is a direct sum. On the other hand, the equivalence class containing s is a nowhere-zero section of $\mathcal{E}/\mathcal{F}$; it spans $\mathcal{E}(U)/\mathcal{F}(U)$ since $\mathcal{E}(U)/\mathcal{F}(U)$ has rank 1. It thus follows that $\mathcal{E}(U) = \mathcal{A}(U)s + \mathcal{F}(U)$.

(2) $\Rightarrow$ (3): Evident.

(3) $\Rightarrow$ (1): Since $\text{rank}(\mathcal{E}/\mathcal{F})(U) = \text{rank}(\mathcal{E}(U)/\mathcal{F}(U)) = \text{rank}(\mathcal{A}(U)s) = 1$ for every open $U \subseteq X$ and $s \in \mathcal{E}(U)$ with $s|_V \notin \mathcal{F}(V)$, where V is any open subset contained in U.

(2) $\Rightarrow$ (4): Let $\mathcal{F}'$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ containing $\mathcal{F}$ and such that $\text{rank}\,\mathcal{F}' > \text{rank}\,\mathcal{F}$. For every open U there exists a section $s \in \mathcal{F}'(U)$ such that $s|_V \notin \mathcal{F}(V)$ for every open $V \subseteq U$. By (2), for every open $U \subseteq X$, $\mathcal{E}(U) = \mathcal{A}(U)s \oplus \mathcal{F}(U)$; but $\mathcal{A}(U)s \oplus \mathcal{F}(U)$ is contained in $\mathcal{F}'(U)$, therefore $\mathcal{F}' = \mathcal{E}$.

(4) $\Rightarrow$ (2): Let U be an open set in X. There exists a section $s \in \mathcal{E}(U)$ with $s|_V \notin \mathcal{F}(V)$ for any open $V \subseteq U$; then $\mathcal{A}(U)s \oplus \mathcal{F}(U)$ contains strictly $\mathcal{F}(U)$, thus $\mathcal{A}(U)s \oplus \mathcal{F}(U) = \mathcal{E}(U)$, since $\mathcal{F}$ is maximal. ■

Lemma 5.1 will be referred to in the proof of Theorem 5.4, which characterizes the kind of $\mathcal{A}$-*transvections* dealt with in the course of this paper. For the classical notion of transvection, see [3], [15, p. 152, Proposition 12.9], [16], [18, p. 419 ff], [20], [43, p. 542- 544]. To this end, we require some preparations.

Definition 5.2 (*Mallios*) Let $\mathcal{E}$ be an $\mathcal{A}$-module. An element $\phi \in \mathcal{E}nd\,\mathcal{E} \equiv \mathcal{E}nd_{\mathcal{A}}\mathcal{E} := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$ is called a **homothecy** of **ratio** $\alpha \in \mathcal{E}nd_{\mathcal{A}}\mathcal{A} =: \mathcal{A}^*(X) \simeq \mathcal{A}(X)$ if

$$\phi = \alpha \cdot I, \tag{5.1}$$

where I stands for the identity of the group $\mathcal{E}nd\,\mathcal{E} := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$.

Section-wise, Equation (5.1) means that, given any section $s \in \mathcal{E}(U)$, one has

$$\phi_U(s) = \alpha|_U \cdot s \equiv \alpha \cdot s.$$

Now, suppose we have a *free* $\mathcal{A}$*-module* $\mathcal{E}$ and $\mathcal{F}$ an $\mathcal{A}$*-hyperplane* of $\mathcal{E}$, so that one has

$$\mathcal{E}/\mathcal{F} \simeq \mathcal{A}, \tag{5.2}$$

cf. [51]. Moreover, let $\phi \in \mathcal{E}nd\, \mathcal{E}$ such that

$$\phi(\mathcal{F}) \subseteq \mathcal{F}; \tag{5.3}$$

then, ϕ gives rise to an element, say $\widetilde{\phi}$, of $\mathcal{E}nd\, (\mathcal{E}/\mathcal{F})$; viz.

$$\widetilde{\phi} \in \mathcal{E}nd(\mathcal{E}/\mathcal{F}),$$

such that, in view of (5.3),

$$\widetilde{\phi} \circ q = q \circ \phi,$$

where $q : \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{F}$ is the *canonical* $\mathcal{A}$-epimorphism. However, due to (5.2), one has

$$\widetilde{\phi} \in \mathcal{E}nd(\mathcal{E}/\mathcal{F}) \simeq \mathcal{E}nd\, \mathcal{A} =: \mathcal{A}^*(X) \simeq \mathcal{A}(X),$$

viz. one obtains

$$\widetilde{\phi} = \alpha \in \mathcal{A}(X) \simeq \mathcal{E}nd\, \mathcal{A},$$

thus

$$\widetilde{\phi} = \alpha \cdot I,$$

so that α is the *ratio* of $\widetilde{\phi}$. Hence, ϕ *induces a homothecy of* $\mathcal{E}/\mathcal{F}(\simeq \mathcal{A})$ *of ratio* α. Furthermore, by the rank formula (cf. [64]), viz.

$$\mathrm{rank}(\mathrm{Im}\widetilde{\phi}) + \mathrm{rank}(\ker \widetilde{\phi}) = \mathrm{rank}(\mathcal{E}/\mathcal{F}) = \mathrm{rank}\ \mathcal{A} = 1,$$

one sees that α *is either zero or nowhere zero* on X.

Definition 5.3 (*Mallios*) Let $\mathcal{E}$ be an $\mathcal{A}$-module. An element $\phi \in \mathcal{E}nd\ \mathcal{E}$ is called an **$\mathcal{A}$-transvection** if:

(i) There exists a sub-$\mathcal{A}$-module $\mathcal{H}$ in $\mathcal{E}$ such that $\mathcal{E}/\mathcal{H} \simeq \mathcal{A}$.

(ii) $\phi|_{\mathcal{H}} = I$.

(iii) $\mathrm{Im}(\phi - I) \subseteq \mathcal{H}$.

More accurately, we say that ϕ is an $\mathcal{A}$-transvection with respect to the sub-$\mathcal{A}$-module $\mathcal{H}$.

Clearly, an element $\phi \in \mathcal{E}nd\ \mathcal{E}$, where $\mathcal{E}$ is an $\mathcal{A}$-module, *is an $\mathcal{A}$-transvection if and only if it is locally so.*

In the light of [3, p. 160, Definition 4.1], Definition 5.3 can be rephrased as follows. An *$\mathcal{A}$-transvection* (with respect to an *$\mathcal{A}$-hyperplane* $\mathcal{H}$, par abus de language) of an $\mathcal{A}$-module $\mathcal{E}$ is an $\mathcal{A}$-endomorphism of $\mathcal{E}$, *which keeps every section of $\mathcal{H}$ fixed and moves any other section $s \in \mathcal{E}(U)$ by some section of $\mathcal{H}(U)$*, namely $\phi(s) - s \in \mathcal{H}(U)$.

Theorem 5.4 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, $\mathcal{H}$ an $\mathcal{A}$-hyperplane of $\mathcal{E}$, ϕ an $\mathcal{A}$-endomorphism of $\mathcal{E}$ that fixes every section of $\mathcal{H}$, and $\widetilde{\phi}$ the $\mathcal{A}$-homothecy, of ratio α, induced by ϕ on the line $\mathcal{A}$-module $\mathcal{E}/\mathcal{H}$. Then,*

(1) *If α is nowhere 1, there exists a unique line $\mathcal{A}$-module $\mathcal{L} \subseteq \mathcal{E}$ such that $\mathcal{E} = \mathcal{H} \oplus \mathcal{L}$ and $\mathcal{L}$ is stable by ϕ, i.e. $\phi(\mathcal{L}) \cong \mathcal{L}$.*

(2) *If $\alpha = 1$, then for every $\mathcal{A}$-morphism $\theta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ with $\ker\theta \cong \mathcal{H}$, there exists, for every open subset $U \subseteq X$, a unique*

section $r \in \mathcal{H}(U)$ *such that*

$$\phi(s) = s + \theta(s)r \tag{5.4}$$

for every $s \in \mathcal{E}(U)$.

Proof. Assertion (1). *Uniqueness.* Let $\mathcal{L}$ be a line $\mathcal{A}$-module satisfying the hypotheses of the assertion, and s a nowhere-zero global section of $\mathcal{L}$ (such a section s does exist because $\mathcal{L} \cong \mathcal{A}$ and $\mathcal{A}$ is unital). Therefore, there exists $b \in \mathcal{A}(X)$ such that $\phi(s) = \beta s$. Next, assume that q is the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto $\mathcal{E}/\mathcal{H}$. It is clear that $\widetilde{\phi}_X(q_X(s)) = \beta q_X(s) \equiv \beta q(s)$; thus $\widetilde{\phi}_X$ is a homothecy of ratio $\alpha = b$, hence, by hypothesis, β is nowhere 1. Now, let u be an element of $\mathcal{E}(X)$ such that $u \notin \mathcal{H}(X)$; then there exists a non-zero $\lambda \in \mathcal{A}(X)$ and an element $t \in \mathcal{H}(X)$ such that

$$u = \lambda s + t.$$

It follows that

$$\phi(u) = \lambda\beta s + t.$$

Of course, $\phi(u)$ and u are colinear if and only if $t = 0$. Thus, we have proved that every section $u \in \mathcal{E}(X)$ which is colinear with its image $\phi(u)$ belongs to $\mathcal{L}(X)$. A similar argument holds should we consider the decomposition $\mathcal{E}(U) = \mathcal{H}(U) \oplus \mathcal{L}(U)$, where U is any other open subset U of X. Hence, $\mathcal{L}$ is the unique complement of $\mathcal{H}$ in $\mathcal{E}$, up to $\mathcal{A}$-isomorphism, and stable by ϕ.

Existence. Since α is nowhere 1 on X, there exists a nowhere-zero section $s \in \mathcal{E}(X)$ such that

$$\widetilde{\phi}_U(q_U(s|_U)) := \widetilde{\phi}_U(q_U(s_U)) \neq q_U(s_U) =: q_U(s|_U)$$

for any open $U \subseteq X$. As $\widetilde{\phi} \circ q = q \circ \phi$, it follows that $r_U := \phi_U(s_U) - s_U$ does not belong to $\mathcal{H}(U)$, for any open $U \subseteq X$. The line $\mathcal{A}$-module

$\mathcal{L} := [r_U]_{X \supseteq U,\ open}$ clearly complements $\mathcal{H}$. It remains to show that $\mathcal{L}$ is stable by ϕ: To this end, we first observe that every s_U does not belong to the corresponding $\mathcal{H}(U)$, and, by Lemma 5.1, $\mathcal{E}(U) \cong \mathcal{A}(U)s_U \oplus \mathcal{H}(U)$. So, since $r_U \notin \mathcal{H}(U)$ for every open $U \subseteq X$, there exists for every r_U sections $\alpha_U \in \mathcal{A}(U)$ and $t_U \in \mathcal{H}(U)$ such that

$$r_U = \alpha_U s_U + t_U. \tag{5.5}$$

We deduce from (5.5) that

$$\phi_U(r_U) = (\alpha_U + 1)r_U,$$

and the proof is complete.

Assertion 2. <u>*Uniqueness.*</u> Let us fix an open set U in X. The uniqueness of r such that (5.4) holds is immediate, as $\theta_U(s) \equiv \theta(s) \neq 0$ for some $s \in \mathcal{E}(U)$. Relation (5.4) also shows that if $s \in \mathcal{E}(U)$ and $\theta(s)$ is nowhere zero, then necessarily

$$r = (\theta(s))^{-1}(\phi(s) - s).$$

<u>*Existence.*</u> Suppose given a section $s_0 \in \mathcal{E}(U)$ such that $s_0|_V \notin \mathcal{H}(V)$ for any open $V \subseteq U$. Let us consider the section
$r = (\theta(s_0))^{-1}(\phi(s_0) - s_0)$. Clearly, $r \in \mathcal{H}(U)$; indeed

$$(q \circ \phi)(s_0) - q(s_0) = (\widetilde{\phi} \circ q)(s_0) - q(s_0) = 0.$$

The two $\mathcal{A}(U)$-morphisms $s \longmapsto \phi(s)$ and $s \longmapsto s + \theta(s)r$ are equal, since they take on, on one hand, the same value at s_0, and, on the other hand, the same value at every $s \in \mathcal{H}(U)$. ∎

In the course of this chapter, we are interested in $\mathcal{A}$-transvections of free $\mathcal{A}$-modules of finite rank $\mathcal{E}$ such that locally for Condition *(iii)* of Definition 5.3, one has one and only one section $s_0^U \in \mathcal{H}(U)$ such that

$$\phi_U(s) := s + \theta_U(s)s_0^U,$$

for every $s \in \mathcal{E}(U)$, and where $\theta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ is such that $\ker\theta$ *is* $\mathcal{A}$-*isomorphic to* $\mathcal{H}$. Such $\mathcal{A}$-transvections shall be called **$\mathcal{A}$-transvections** of **classical type**.

So, assume $\mathcal{E}$ is free and of rank n and $(e_1, \ldots, e_n)$ a basis for $\mathcal{E}(U)$, where U is a fixed open subset of X, such that $(e_1, \ldots, e_{n-1})$ is a basis for $\mathcal{H}(U)$. The matrix representing ϕ_U is given by

$$(\phi_U^{ij}) := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \lambda s_0^1 \\ 0 & 1 & \cdots & 0 & 0 & \lambda s_0^2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \lambda s_0^{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \in M_n(\mathcal{A}(U)) \simeq M_n(\mathcal{A})(U),$$

where $\lambda := \theta_U(e_n) \in \mathcal{A}(U)$ and $s_0^U \equiv s_0 := s_0^1 e_1 + \cdots + s_0^{n-1} e_{n-1}$. If we consider the determinant $\mathcal{A}$-morphism $\partial et : M_n(\mathcal{A}) \longrightarrow \mathcal{A}$ (cf. [50, p. 294]), it follows that

$$\overline{\partial et_U}(\phi_U^{ij}) \equiv \partial et_U(\phi_U^{ij}) =: \det{}_U(\phi_U^{ij}) = 1$$

(we have assumed that $\overline{\partial et} : \Gamma(M_n(\mathcal{A})) \longrightarrow \Gamma(\mathcal{A})$ is the $\Gamma(\mathcal{A})$-morphism of complete presheaves of sections of sheaves $M_n(\mathcal{A})$ and $\mathcal{A}$ that corresponds to ∂et); hence, $\mathcal{A}$-transvections are invertible.

Keeping with the notations above, the inverse of an $\mathcal{A}$-transvection ϕ is the $\mathcal{A}$-transvection ϕ^{-1} such that

$$\phi_U^{-1}(s) := s - \theta_U(s)s_0^U$$

for every open $U \subseteq X$ and section $s \in \mathcal{E}(U)$.

Fix an open subset U of X and let $\phi \in \mathcal{E}nd_{\mathcal{A}}\mathcal{E}$ be an $\mathcal{A}$-transvection. If $s_0 \equiv s_0^U \in \mathcal{E}(U)$ is nowhere zero, we may assume it to be one of the basis elements of $\mathcal{H}(U)$; therefore the matrix of ϕ_U will just be the *identity natrix with one non-zero element off the main diagonal.*

Conversely, any $\mathcal{A}$-endomorphism ϕ of a free $\mathcal{A}$-module of finite rank $\mathcal{E}$ such that, for every open $U \subseteq X$, the matrix representing ϕ_U with respect to some basis is the identity matrix with one non-zero entry off the main diagonal is an $\mathcal{A}$-transvection.

We formalize the above argument in the following

Corollary 5.5 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of rank n, $\mathcal{H}$ an $\mathcal{A}$-hyperplane of $\mathcal{E}$, and ϕ an $\mathcal{A}$-transvection of classical type. Then, for every open $U \subseteq X$, there exists a basis of $\mathcal{E}(U)$ such that the matrix (ϕ_U) of ϕ_U in this basis is of the form*

$$(\phi_U) = I_n + \lambda M^{ij}, \quad i \neq j, \tag{5.6}$$

where $\lambda \in \mathcal{A}(U)$ and $(M^{ij})_{1\leq i,j\leq n}$ represents a canonical basis of $M_n(\mathcal{A}(U))$. Matrix (5.6) *is called an $\mathcal{A}(U)$**-transvection matrix.***

For the need of what follows, we make the following important observation (cf. Lemma 3.19), concerning symplectic $\mathcal{A}$-modules of finite rank, the proof of which is based on the following concept.

Definition 5.6 Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, ϕ and ϕ' non-degenerate $\mathcal{A}$-bilinear forms on $\mathcal{E}$ and $\mathcal{E}'$, respectively. Moreover, let ψ be an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{E}'$. An $\mathcal{A}$-morphism $\theta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}', \mathcal{E})$ such that

$$\phi' \circ (\psi, \mathrm{Id}) = \phi \circ (\mathrm{Id}, \theta). \tag{5.7}$$

is called an **adjoint** of ψ, and is denoted ψ^*.

Section-wise, Equation (5.7) means that for every open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$, $t \in \mathcal{E}'(U)$,

$$\phi'(\psi(s), t) \equiv \phi'_U(\psi_U(s), t) = \phi_U(s, \theta_U(t)) \equiv \phi(s, \theta(t)).$$

Keeping with the notations of Definition 5.6 above, we have

Proposition 5.7 *θ is unique whenever it exists.*

Proof. Suppose that θ_1 and θ_2 are adjoint of ψ, so given any open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$, $t \in \mathcal{E}'(U)$,

$$\phi^L_U(\theta_{1,U}(t))(s) = \phi^L_U(\theta_{2,U}(t))(s),$$

where $\phi^L \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}'^*)$ is given by

$$\phi^L_U(u)(v) \equiv (\phi^L)_U(u)(v) := \phi_V(u|_V, v)$$

for sections $u \in \mathcal{E}(U)$ and $v \in \mathcal{E}'(V)$. Since s is arbitrary in $\mathcal{E}(U)$,

$$\phi^L_U(\theta_{1,U}(t)) = \phi^L_U(\theta_{2,U}(t)).$$

But ϕ^L is injective, therefore

$$\theta_{1,U} = \theta_{2,U}.$$

Finally, since U is arbitrary, $\theta_1 = \theta_2$. ■

Let us now enquire on the existence of the adjoint of an $\mathcal{A}$-morphism $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$, where $\mathcal{E}$ and $\mathcal{E}'$ are $\mathcal{A}$-modules equipped with $\mathcal{A}$-bilinear forms ϕ and ϕ', respectively.

Proposition 5.8 *Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, equipped with non-degenerate $\mathcal{A}$-bilinear forms ϕ and ϕ', respectively. If $\mathcal{E}$ is* free *and* of finite rank, *then for every $\mathcal{A}$-morphism $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$ there exists an adjoint, denoted ψ^*, which is given by*

$$\psi^* = (\phi^L)^{-1} \circ {}^t\psi \circ \phi'^L,$$

where ${}^t\psi : (\mathcal{E}')^* \longrightarrow \mathcal{E}^*$ is the transpose of ψ.

Proof. Let U be an open subset of X, $s \in \mathcal{E}(U)$ and $t \in \mathcal{E}'(U)$. Using the right insertion $\mathcal{A}$-morphism ϕ'^L, one has

$$\phi'_U(\psi_U(s), t) = \phi'^L_U(t)(\psi_U(s)) = ({}^t\psi)_U(\phi'^L_U(t))(s). \tag{5.8}$$

Since $\mathcal{E}$ has finite rank and ϕ is non-degenerate, ϕ^L is an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{E}^*$; so ${}^t\psi \circ \phi'^L$ may be written

$$ {}^t\psi \circ \phi'^L = \phi^L \circ ((\phi^L)^{-1} \circ {}^t\psi \circ \phi'^L). $$

It follows from (5.8) that

$$ \begin{aligned} \phi'_U(\psi_U(s), t) &= [\phi^L_U(((\phi^L_U)^{-1} \circ ({}^t\psi)_U \circ \phi'^L_U)(t))](s) \\ &= \phi_U(s, ((\phi^L_U)^{-1} \circ ({}^t\psi)_U \circ \phi'^L_U)(t)), \end{aligned} $$

which ends the proof. ■

Corollary 5.9 *Adjoints commute with restrictions.*

Proof. Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, ϕ and ϕ' non-degenerate $\mathcal{A}$-bilinear forms on $\mathcal{E}$ and $\mathcal{E}'$, respectively. Assume that $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$. Let U be an open subset of X, and s, t be sections of $\mathcal{E}$ and $\mathcal{E}'$ on U, respectively. By Definition 5.6, we have

$$ \phi'_U(\psi_U(s), t) = \phi_U(s, (\psi^*)_U(t)). $$

On the other hand, since ϕ_U and ϕ'_U are non-degenerate and

$$ \psi_U \in \mathrm{Hom}_{\mathcal{A}(U)}(\mathcal{E}(U), \mathcal{E}'(U)), $$

then by virtue of [15, pp. 385, 386], we have

$$ \phi'_U(\psi_U(s), t) = \phi_U(s, (\psi_U)^*(t)). $$

On account of uniqueness of adjoints, we have

$$ (\psi^*)_U = (\psi_U)^*, $$

as desired. ■

Definition 5.10 An $\mathcal{A}$-transvection of classical type is called a **nowhere-identity $\mathcal{A}$-transvection** if locally it differs from the identity map.

There is an interesting relationship between nowhere-identity symplectic $\mathcal{A}$-transvections and $\mathcal{A}$-symplectomorphisms as is the case with the respective counterparts in the classical theory, see, for instance, [18, pp. 418-420].

For this purpose we need the following

Lemma 5.11 *Let $(\mathcal{E}, \omega)$ be a symplectic $\mathcal{A}$-module of finite rank, and f an $\mathcal{A}$-endomorphism of $\mathcal{E}$. Then, if f satisfies two of the three following conditions, it satisfies all of them three, and* $\mathrm{Id} + f$ *is called a* ***singular $\mathcal{A}$-symplectomorphism of*** $(\mathcal{E}, \omega)$*:*

(1) $\mathrm{Id} + f$ *is an $\mathcal{A}$-automorphism of $\mathcal{E}$;*

(2) *f is ω-skewsymmetric, i.e., for any open $U \subseteq X$ and sections $s, t \in \mathcal{E}(U)$,*
$$\omega_U(f_U(s), t) + \omega_U(s, f_U(t)) = 0;$$

(3) *$Im\ f \equiv f(\mathcal{E})$ is totally isotropic, i.e.,*
$$\omega|_{f(\mathcal{E})} = 0.$$

Proof. Using the equality

$$\omega_U(s+f_U(s), t+f_U(t)) - \omega_U(s,t) = \omega_U((f_U+f_U^*)(s), t) + \omega_U(f_U(s), f_U(t)),$$

where U is any open subset of X, s and t sections of $\mathcal{E}$ over U, one easily checks the implications: (1), (2) $\Rightarrow$ (3); (1), (3) $\Rightarrow$ (2); and (2), (3) $\Rightarrow$ (1). ■

Theorem 5.12 *Let $(\mathcal{E}, \omega)$ be a symplectic orthogonally convenient $\mathcal{A}$-module of rank $2n$, and f a $\mathcal{A}$-endomorphism of $\mathcal{E}$. If f is skewsymmetric and* $\mathrm{Id} + f$ *an $\mathcal{A}$-automorphism of $\mathcal{E}$, then*

(1) $f^2 = 0$;

(2) $\ker f \simeq (Im\ f)^\perp$;

(3) *For every open subset $U \subseteq X$, there exists a symplectic basis of $\mathcal{E}(U)$, whose first k elements (sections), $k \leq n$, form a basis of $(Im\ f)(U) := Im\ f_U \equiv f_U(\mathcal{E}(U))$, with respect to which the $\mathcal{A}(U)$-morphism*

$$(\mathrm{Id} + f)_U := \mathrm{Id}_U + f_U$$

is represented by the matrix

$$\begin{pmatrix} I_n & H \\ 0 & I_n \end{pmatrix}$$

with ${}^tH = H$.

Proof. (**1**) From Lemma 5.11, Imf is totally isotropic. Therefore, for any open subset U of X and sections $s, t \in \mathcal{E}(U)$,

$$\omega_U(f_U(s), f_U(t)) = 0.$$

Since

$$\begin{aligned} \omega_U((f^*)_U f_U(s), t) &= \omega_U((f_U)^* f_U(s), t) \\ &= \omega_U(f_U(s), f_U(t)) \\ &= 0 \end{aligned}$$

and ω is symplectic, it follows that

$$(f^*)_U f_U = (f_U^*) f_U = 0.$$

Thus,

$$f^* f = 0;$$

since $f^* = -f$, one reaches the desired property that $f^2 = 0$.

(**2**) Fix an open set U in X and $s \in (\ker f)(U) = \ker f_U$, see [70, p. 37, Definition 3.1]. Moreover, let $t \in \mathcal{E}(U)$; then

$$\omega_U(s, f_U(t)) = -\omega_U(f_U(s), t) = 0.$$

Thus,

$$s \in (\mathrm{Im} f)(U)^{\perp} \equiv f_U(\mathcal{E}(U))^{\perp}$$

and hence

$$(\ker f)(U) = \ker f_U \subseteq (\mathrm{Im} f)(U)^{\perp} = (\mathrm{Im} f)^{\perp}(U)$$

or

$$\ker f \subseteq (\mathrm{Im} f)^{\perp} \equiv f(\mathcal{E})^{\perp}.$$

Conversely, let $t \in (\mathrm{Im} f)^{\perp}(U) = (\mathrm{Im} f)(U)^{\perp}$. Then, for any $s \in (\mathrm{Im} f)(U) \equiv \mathrm{Im} f_U := f_U(\mathcal{E}(U)) \equiv f(\mathcal{E})(U)$, one has

$$\omega_U(t, s) = 0.$$

But $s = f_U(r)$ for some $r \in \mathcal{E}(U)$, therefore

$$\omega_U(t, f_U(r)) = -\omega_U(f_U(t), r) = 0. \tag{5.9}$$

Since (5.9) is true for any $r \in \mathcal{E}(U)$,

$$f_U(t) = 0,$$

i.e.

$$t \in (\ker f)(U) := \ker f_U.$$

Hence,

$$(\mathrm{Im} f)^{\perp}(U) \subseteq (\ker f)(U)$$

or

$$(\mathrm{Im} f)^{\perp} \subseteq \ker f.$$

(**3**) As $\mathrm{Im} f \subseteq \ker f = (\mathrm{Im} f)^{\perp}$, so the sub-$\mathcal{A}$-module $\mathrm{Im} f$ is totally isotropic. Therefore, for any open $U \subseteq X$,

$$\mathrm{rank}(\mathrm{Im} f)(U) := \mathrm{rank}\ \mathrm{Im} f_U \leq n.$$

Now, let us fix an open set U in X and consider a basis $(s_1, \ldots, s_k)$, $k \leq n$, of $(\mathrm{Im}f)(U) \equiv \mathrm{Im}f_U$. By [52, Lemma 7], there exists a totally isotropic sub-$\mathcal{A}(U)$-module S of $\mathcal{E}(U)$, equipped with a basis, which we denote

$$(s_{k+1}, \ldots, s_{n+k})$$

such that

$$\omega_U(s_i, s_{n+j}) = \delta_{ij}, \qquad \text{for } i, j = 1, \ldots, k.$$

Clearly,

$$S \cap (\mathrm{Im}f)^{\perp}(U) = S \cap (\ker f)(U) = 0. \tag{5.10}$$

As a result of (5.10), the sum $S + \mathrm{Im}f_U$ is direct and $S \oplus \mathrm{Im}f_U$ is non-isotropic; therefore, one has

$$\mathcal{E}(U) = (S \oplus \mathrm{Im}f_U) \perp F$$

for some sub-$\mathcal{A}(U)$-module F of $\mathcal{E}(U)$, (cf. [52, Theorem 1]). Since $F = (S \oplus \mathrm{Im}f_U)^{\perp}$, F is contained in $(\mathrm{Im}f_U)^{\perp} = (\mathrm{Im}f)^{\perp}(U) = (\ker f)(U)$ and

$$F^{\perp} = (\mathrm{Im}f)(U) := \mathrm{Im}f_U;$$

i.e. F is an orthogonal supplementary of $(\mathrm{Im}f)(U)$ in $(\ker f)(U)$. Since F is free, non-isotropic and of rank $2n - 2k$, it can be equipped with a symplectic basis, say $(s_{k+1}, \ldots, s_n, s_{n+k+1}, \ldots, s_{2n})$, see [62]. As $s_1, \ldots, s_n \in (\ker f)(U)$, it follows that

$$(\mathrm{Id}_U + f_U)(s_j) = s_j, \qquad j = 1, \ldots, n.$$

Therefore, if H is the matrix representing f_U, $\mathrm{Id}_U + f_U$ is represented by the matrix

$$\begin{pmatrix} \mathrm{I}_n & \mathrm{H} \\ 0 & \mathrm{I}_n \end{pmatrix},$$

and this is a sympletic matrix if and only if ${}^tH = H$, ie. H is symmetric. ■

On account of Theorem 5.12, we have the following. Let $(\mathcal{E}, \omega)$ be a *symplectic orthogonally convenient $\mathcal{A}$-module of finite rank,* and $\phi \in \mathcal{E}nd\, \mathcal{E}$ a (symplectic) $\mathcal{A}$-transvection of $(\mathcal{E}, \omega)$. Suppose that

$$\phi = I + \psi,$$

where $I = \mathrm{Id}_{\mathcal{E}}$ and $\psi \in \mathcal{E}nd\, \mathcal{E}$. Then, necessarily, if $\mathrm{Im}\ \psi$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, then rank $\psi :=$ rank $\mathrm{Im}\ \psi = 1$, i.e., ϕ is a nowhere-identity $\mathcal{A}$-transvection. This necessary condition for nowhere-identity $\mathcal{A}$-transvections is not sufficient, for if $\mathcal{H}$ is the sub-$\mathcal{A}$-module of $\mathcal{E}$ defining ϕ, one must have

$$\psi(\mathcal{E}/\mathcal{H}) = 0,$$

i.e.

$$\psi^2 = 0.$$

Using Theorem 5.12, we thus obtain

Corollary 5.13 *Let $(\mathcal{E}, \omega)$ be a symplectic orthogonally convenient $\mathcal{A}$-module of finite rank. There is a bijection between $\mathcal{A}$-symplectomorphisms of the form $I + \psi$ such that Im ψ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$ and the nowhere-identity symplectic $\mathcal{A}$-transvections.*

Let us describe precisely nowhere-identity symplectic $\mathcal{A}$-transvections. For this purpose, we fix an open subset U of X and suppose that rank $\psi_U = 1$: it follows that, for every $s \in \mathcal{E}(U)$,

$$\psi_U(s) = \alpha_U(s)s_0^U,$$

where s_0^U is nowhere zero and $\alpha \in \mathcal{E}^*(U)$. Based on Theorem 5.12, the necessary and sufficient condition for $I_{\mathcal{E}(U)} + \psi_U$ to be symplectic is that

$$\omega_U(\psi_U(s), t) + \omega_U(s, \psi_U(t)) = \omega_U(s_0^U, \alpha_U(s)t - \alpha_U(t)s) = 0, \quad (5.11)$$

for all sections $s, t \in \mathcal{E}(U)$. Since ω is non-degenerate, we may associate with α_U a *nowhere-zero section* $r \in \mathcal{E}(U)$ such that

$$\alpha_U(s) = \omega_U(r, s), \quad s \in \mathcal{E}(U);$$

therefore (5.10) becomes

$$\omega_U(s_0^U, \omega_U(r, s)t - \omega_U(r, t)s) = 0. \tag{5.12}$$

Since $\mathcal{A}$ is a PID algebra sheaf and $(\mathcal{E}, \omega)$ is orthogonally convenient, hence, by [64, Corollary 3.2(2)], if $\mathcal{G}$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, then

$$\text{rank } \mathcal{G} + \text{rank } \mathcal{G}^{\perp} = \text{rank } \mathcal{E}.$$

Consequently, $r^{\perp}$ is a hyperplane of $\mathcal{E}(U)$; whence if we take t in $r^{\perp}$, (5.12) reduces to

$$\omega_U(s_0^U, t) = 0.$$

Applying [64, Corollary 3.2(2)], here as well, and since ω is non-degenerate, one has

$$s_0^U \in (r^{\perp})^{\perp};$$

therefore, there exists $\lambda_U \in \mathcal{A}(U)$ such that

$$r = \lambda_U s_0^U.$$

Thus, the symplectic $\mathcal{A}(U)$-transvection $\phi_U := I_{\mathcal{E}(U)} + \psi_U$ is of the form:

$$\phi_U(s) = s + \lambda_U \omega_U(s_0^U, s)s_0^U$$

for every $s \in \mathcal{E}(U)$.

We have thus proved:

Theorem 5.14 *Let $(\mathcal{E}, \omega)$ be a symplectic $\mathcal{A}$-module of finite rank, where $\mathcal{A}$ is a PID algebra sheaf, and ϕ a symplectic $\mathcal{A}$-transvection of $\mathcal{E}$. Then, for every open subset U of X and $s \in \mathcal{E}(U)$,*

$$\phi_U(s) = s + \lambda_U \omega_U(s_0^U, s)s_0^U$$

for some $s_0^U \in \mathcal{E}(U)$. s_0^U *is called the* ***base section of*** ϕ_U.

5.2 $\mathcal{A}$-symplectomorphisms as products of symplectic $\mathcal{A}$-transvections

The results below generalize their classical counterparts; see for instance [16, pp. 18-20] and [18, pp. 422-424]. We will assume for each of these results that $(X, \mathcal{A}, \mathcal{P})$ is an ordered algebraized space (cf. [50, pp. 316-318]), with $\mathcal{A}$ a *torsion-free PID algebra sheaf* such that all *its nowhere-zero sections are invertible,* and $(\mathcal{E}, \omega)$ a *symplectic (free) $\mathcal{A}$-module of finite rank.* (Sheaves of continuous, smooth and holomorphic functions have the property that every nowhere-zero section is invertible.)

Lemma 5.15 *If $s, t \in \mathcal{E}(U)$ are sections of $\mathcal{E}$, everywhere, on U, non orthogonal, then, there exists a symplectic transvection τ on $\mathcal{E}(U)$ such that $\tau(s) = t$. On the other hand, if $\omega_U(s,t)|_V = \omega_V(s|_V, t|_V) = 0$ and $s|_V \neq t|_V$ for some open $V \subseteq U$, there is no nowhere-identity symplectic transvection on $\mathcal{E}(U)$ carrying s onto t.*

Proof. Since $\omega_U(s,t)$ is nowhere zero, and every nowhere-zero section of $\mathcal{A}$ is invertible, it suffices to take

$$\tau(u) = u - \frac{1}{\omega_U(s,t)}\omega_U(t-s, u)(t-s).$$

As for the second part of the lemma, suppose there exists a symplectic transvection τ, of base section $a \in \mathcal{E}(U)$, mapping s onto t. Then, on V, we have

$$c|_V \omega_V(a|_V, s|_V)^2 = 0$$

or, since $\mathcal{A}$ *is torsion-free* and c is everywhere no zero,

$$\omega_V(a|_V, s|_V) = 0$$

which implies that

$$\tau_V(s|_V) = s|_V = t|_V.$$

■

Lemma 5.16 *Let τ be a symplectomorphism of $\mathcal{E}(U)$ and F_τ the free sub-$\mathcal{A}(U)$-module of $\mathcal{E}(U)$ consisting of all fixed sections in $\mathcal{E}(U)$ (by τ). Then, the necessary and sufficient condition for the existence of a nowhere-identity transvection α such that the free sub-$\mathcal{A}(U)$-module of fixed sections of $\phi \equiv \alpha \circ \tau$, F_ϕ, with the property that rank $F_\phi >$ rank F_τ, is that there exists a section s_1 such that $\omega_U(\tau(s_1), s_1)$ is nowhere zero, (which implies that $\tau(s_1)|_V \neq s_1|_V$ for any open $V \subseteq U$, which, in turn, implies that $s_1 \notin F_\tau$).*

Proof. Suppose there exists s_1 with $\omega_U(\tau(s_1), s_1)$ nowhere zero; it follows that

$$\omega_U(\tau(s_1) - s_1, s_1)|_V \neq 0$$

for any open $V \subseteq U$. Clearly, $(\tau(s_1) - s_1)^\perp$ is an $\mathcal{A}(U)$-hyperplane and contains F_τ, for, if $\tau(s) = s$,

$$\begin{aligned} \omega_U(\tau(s_1) - s_1, s) &= \omega_U(\tau(s_1), s) - \omega_U(s_1, s) \\ &= \omega_U(\tau(s_1), \tau(s)) - \omega_U(s_1, s) = 0. \end{aligned}$$

Considering the transvection

$$\alpha(s) = s - \frac{1}{\omega_U(\tau(s_1), s_1)}\omega_U(s_1 - \tau(s_1), s)(s_1 - \tau(s_1)),$$

one has

$$\alpha(\tau(s_1)) = s_1.$$

Its fixed sections yield the $\mathcal{A}(U)$-hyperplane $(s_1 - \tau(s_1))^\perp$, which contains F_τ. Since

$$\phi(s_1) = (\alpha \circ \tau)(s_1) = s_1$$

and s_1 is nowhere zero and is contained in F_ϕ, so rank $F_\phi >$ rank F_τ.

Conversely, if $\phi = \alpha \circ \tau$ for some *nowhere-identity transvection* α and rank $F_\phi >$ rank F_τ, it follows that there exists a nowhere-zero section $s_1 \in \mathcal{E}(U)$ such that $\phi(s_1) = (\alpha \circ \tau)(s_1) = s_1$ and $\tau(s_1)|_V \neq s_1|_V$ for any open $V \subseteq U$. Assume that the transvection α is given by

$$\alpha(s) = s + c\omega_U(a, s)a,$$

where $c \in \mathcal{A}(U)$ and $a \in \mathcal{E}(U)$ (since α is a nowhere-identity transvection, therefore, of necessity, $\omega_U(a, s)$ is nowhere zero). Then,

$$(\alpha \circ \tau)(s_1) = \tau(s_1) + c\omega_U(a, \tau(s_1))a = s_1,$$

which implies that

$$\tau(s_1) - s_1 = c\omega_U(a, \tau(s_1))a.$$

Since $(\tau(s_1) - s_1)|_V \neq 0$ for any open $V \subseteq U$, $\omega_U(a, \tau(s_1))|_V \neq 0$ for any open $V \subseteq U$. But, for any open $V \subseteq U$,

$$\omega_U(a, \tau(s_1))|_V = \omega_U(\alpha(a), \alpha(\tau(s_1)))|_V = \omega_U(a, s_1)|_V \neq 0$$

and

$$\begin{aligned} \omega_U(s_1, \tau(s_1)) &= \omega_U(\alpha(s_1), \alpha(\tau(s_1))) \\ &= \omega_U(s_1 + c\omega_U(a, s_1)a, s_1) = c\omega_U(a, s_1)^2, \end{aligned}$$

therefore

$$\omega_U(s_1, \tau(s_1))|_V \neq 0$$

for any open $V \subseteq U$. ■

By Lemma 5.15, for open sets U and V such that $V \subseteq U$, there exists no nowhere-identity symplectic transvection mapping s_1 to $\tau(s_1)$

and such that $\omega_U(\tau(s_1), s_1)|_V = 0$ and $s_1|_V \neq \tau(s_1)|_V$. Therefore, the *only case of exception of Lemma 5.16*, that is, when $\omega_U(\tau(s), s) = 0$ for any $s \in \mathcal{E}(U)$, is *to be ruled out* as, by Lemma 5.15, if α is a symplectic transvection of $\mathcal{E}(U)$, mapping s to $\tau(s)$ and $\omega_U(\tau(s), s) = 0$ then $\tau(s) = \alpha(s) = s$. In this instance α is the identity, a *contradiction* to the wish that α should be a nowhere-identity symplectic transvection.

When put together, Lemmas 5.15 and 5.16 are a proof of the following theorem.

Theorem 5.17 *Let* $(\mathcal{E}, \omega)$ *be a symplectic* $\mathcal{A}$*-module of rank* $2n$*, and* σ *an* $\mathcal{A}$*-symplectomorphism of* $\mathcal{E}$*. Then,* σ *is a product of at most* $(4n-2)$ *nowhere-identity symplectic* $\mathcal{A}$*-transvections.*

Proof. The proof is done by induction on the rank of the free sub-$\mathcal{A}(X)$-module F_{σ_X} of fixed sections of σ_X, by showing that if σ_X is not the identity map, there exist at most two $\mathcal{A}(X)$-transvections α and β of $\mathcal{E}(X)$ such that, if $\phi = \alpha \circ \beta \circ \sigma_X$, rank $F_\phi >$ rank F_{σ_X}.

Beside, if rank $F_{\sigma_X} = 2n - 1$, by Corollary 5.13, σ_X is an $\mathcal{A}$-transvection. Otherwise, there exist m $\mathcal{A}$-transvections, $m \leq (4n-2)$, such that

$$\alpha_m \circ \alpha_{m-1} \circ \cdots \circ \alpha_1 \circ \sigma = I.$$

■

5.3 Witt's theorem and symplectic orthogonally convenient $\mathcal{A}$-modules

As suggested in the title of this section, our first aim is to find an analogue of the Witt's theorem (cf. [68, pp. 46-48]) for symplectic $\mathcal{A}$-

modules. For this purpose, we refer the reader to [53], [62] and Chapter 4 for useful details regarding symplectic $\mathcal{A}$-modules and symplectic bases (of sections). Sheaves of symplectic groups arise in a natural way when one considers $\mathcal{A}$-isomorphisms between symplectic $\mathcal{A}$-modules which respect the symplectic structures involved, see [53]. For some other versions of the Witt's theorem, see [52] and [65]. Finally, the section ends with a characterization of singular $\mathcal{A}$-symplectomorphisms of symplectic orthogonally convenient $\mathcal{A}$-modules of finite rank. Orthogonally convenient $\mathcal{A}$-modules were introduced in Chapter 2.

For the classical Witt's theorem, see [1, pp. 368-387], [3, pp. 121, 122], [5, p. 21], [68], [16, pp. 11, 12], [18, pp. 148- 152], [43, pp. 591, 592], [66, p. 9]. But, first we need the following definition (cf. [65]).

Definition 5.18 Let $(\mathcal{E}, \omega)$ be symplectic orthogonally convenient $\mathcal{A}$-module of finite rank.

(i) A free sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$ with $\omega|_{\mathcal{F}}$ non-degenerate is called a **symplectic orthogonally convenient sub-$\mathcal{A}$-module** of $\mathcal{E}$.

(ii) A free sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$ with $\mathcal{F}^{\perp}$ is isotropic is called **coisotropic.**

(iii) A free sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$ which is both isotropic and coisotropic is called a **Lagrangian sub-$\mathcal{A}$-module.**

From [64, Corollary 4.1], if $\mathcal{F}$ is Lagrangian, then

$$\text{rank } \mathcal{F} = \text{rank } \mathcal{F}^{\perp}.$$

Theorem 5.19 *Let $\mathcal{A}$ be a PID algebra sheaf, $\mathcal{E}$ a symplectic free $\mathcal{A}$-module of rank $2n$ (ω is the symplectic structure on $\mathcal{E}$), $\mathcal{F}$ a Lagrangian* *(free) sub-$\mathcal{A}$-module of $\mathcal{E}$ and $\mathcal{G}$ any sub-$\mathcal{A}$-module of $\mathcal{E}$ such that $\mathcal{F}$ and $\mathcal{G}$ are supplementary. Then, using $\mathcal{G}$ we can construct a Lagrangian sub-$\mathcal{A}$-module $\mathcal{H}$ of $\mathcal{E}$ such that $\mathcal{E} \simeq \mathcal{F} \oplus \mathcal{H}$.*

Proof. The restriction ω' of ω to $\mathcal{F} \oplus \mathcal{G} \subseteq \mathcal{E} \oplus \mathcal{E}$ is also non-degenerate. In fact, let $\mathcal{F}^{\perp}_{\omega'}$ and $\mathcal{G}^{\perp}_{\omega'}$ denote the kernels of $\mathcal{F}$ and $\mathcal{G}$ respectively. More precisely, for every open $U \subseteq X$,

$$\mathcal{F}^{\perp}_{\omega'}(U) = \{r \in \mathcal{G}(U)|\ \omega'(\mathcal{F}(V), r|_V) = 0 \text{ for any open } V \subseteq U\}$$

and similarly

$$\mathcal{G}^{\perp}_{\omega'}(U) = \{r \in \mathcal{F}(U)|\ \omega'(r|_V, \mathcal{G}(V)) = 0 \text{ for any open } V \subseteq U\}.$$

Analogously we denote by $\mathcal{F}^{\perp}_{\omega}$ and $\mathcal{G}^{\perp}_{\omega}$ the kernels of $\mathcal{F}$ and $\mathcal{G}$ respectively with respect to the $\mathcal{A}$-bilinear morphism $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$, i.e. for every open $U \subseteq X$,

$$\mathcal{F}^{\perp}_{\omega}(U) = \{r \in \mathcal{E}(U)|\ \omega(\mathcal{F}(V), r|_V) = 0 \text{ for any open } V \subseteq U\}$$

and

$$\mathcal{G}^{\perp}_{\omega}(U) = \{r \in \mathcal{E}(U)|\ \omega(\mathcal{G}(V), r|_V) = 0 \text{ for any open } V \subseteq U\}.$$

It is obvious that $\mathcal{F}^{\perp}_{\omega} = \mathcal{F}^{\top}_{\omega}$ and $\mathcal{G}^{\perp}_{\omega} = \mathcal{G}^{\top}_{\omega}$. By hypothesis, we are given that $\mathcal{F} = \mathcal{F}^{\perp}_{\omega}$. Clearly, for every open $U \subseteq X$, $\mathcal{F}^{\perp}_{\omega'}(U) \subseteq \mathcal{F}^{\perp}_{\omega}(U)$ and $\mathcal{G}^{\perp}_{\omega'}(U) \subseteq \mathcal{G}^{\perp}_{\omega}(U)$. But since $\mathcal{F}^{\perp}_{\omega}(U) = \mathcal{F}(U)$ and $\mathcal{F}(U) \cap \mathcal{G}(U) = 0$, $\mathcal{F}^{\perp}_{\omega'}(U) = 0$. Thus, $\mathcal{F}^{\perp}_{\omega'} = 0$. On the other hand, let $r \in \mathcal{G}^{\perp}_{\omega'}(U) \subseteq \mathcal{F}(U) \cap \mathcal{G}^{\perp}_{\omega}(U)$. As $\mathcal{E}(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$, we deduce that $r \in \text{rad}\ \mathcal{E}(U) = 0$, therefore $r = 0$. Hence, $\mathcal{G}^{\perp}_{\omega'} = 0$. Since $\omega' : \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{A}$ is non-degenerate, the $\mathcal{A}$-morphism $\widetilde{\omega'} : \mathcal{F} \longrightarrow \mathcal{G}^*$ such that for every open $U \subseteq X$, and sections $r \in \mathcal{F}(U)$ and $s \in \mathcal{G}(U)$, $\widetilde{\omega'}(r)(s) := \omega'(r, s)$ is bijective.

Let us construct the sought Lagrangian complement $\mathcal{H}$ of $\mathcal{F}$ in $\mathcal{E}$. For every open $U \subseteq X$, we let

$$\mathcal{H}(U) := \{r + \phi(r)|\ r \in \mathcal{G}(U)\},$$

where $\phi : \mathcal{G} \longrightarrow \mathcal{F}$ is some $\mathcal{A}$-morphism. It is clear that $\mathcal{H}$ is a sub-$\mathcal{A}$-module of $\mathcal{E}$. For $\mathcal{H}$ to be Lagrangian, it takes the following: For every open $U \subseteq X$ and sections $r, s \in \mathcal{G}(U)$

$$\omega(r + \phi(r), s + \phi(s)) = 0$$

i.e.

$$\omega(r, s) = \widetilde{\omega}'(\phi(s))(r) - \widetilde{\omega}'(\phi(r))(s). \tag{5.10}$$

Let $\phi' := \widetilde{\omega}' \circ \phi : \mathcal{G} \longrightarrow \mathcal{G}^*$, so that (5.10) becomes

$$\omega(r, s) = \phi'(s)(r) - \phi'(r)(s). \tag{5.11}$$

Clearly, by taking $\phi'(r) = -\frac{1}{2}\omega(r, -)$ for every $r \in \mathcal{G}(U)$, (5.11) is satisfied. By setting $\phi := (\widetilde{\omega}')^{-1} \circ \phi'$, we contend that the claim holds. In fact, fix an open subset U of X, and suppose that $(r_1, \ldots, r_n)$ is a basis of $\mathcal{G}(U)$. If $a_1, \ldots, a_n \in \mathcal{A}(U)$ such that

$$a_1(r_1 + \phi(r_1)) + \ldots + a_n(r_n + \phi(r_n)) = 0,$$

one has that

$$\underbrace{a_1 r_1 + \ldots + a_n r_n}_{\in \mathcal{G}(U)} = \underbrace{-\phi(a_1 r_1 + \ldots + a_n r_n)}_{\in \mathcal{F}(U)}.$$

Since $\mathcal{F}(U) \cap \mathcal{G}(U) = 0$, it follows that

$$\phi(a_1 r_1 + \ldots + a_n r_n) = 0.$$

As the chosen ϕ' is injective and $\widetilde{\omega}'$ is an $\mathcal{A}$-isomorphism, ϕ is injective; thence

$$a_1 r_1 + \ldots + a_n r_n = 0;$$

so that $a_1 = \cdots = a_n = 0$. Now, let us show that $\mathcal{F}(U) \cap \mathcal{H}(U) = 0$. For this purpose, suppose that $r \in \mathcal{F}(U) \cap \mathcal{H}(U)$. Then for some $s \in \mathcal{G}(U)$

$$r = s + \phi(s).$$

It follows that

$$\underbrace{r - \phi(s)}_{\in \mathcal{F}(U)} = \underbrace{s}_{\in \mathcal{G}(U)}$$

from which we deduce that $s = 0$, and hence $r = 0$. That $\mathcal{E}(U) \cong \mathcal{F}(U) \oplus \mathcal{H}(U)$ is now clear. Since U is arbitrary, $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{H}$ as desired.
■

Theorem 5.20 (***Witt's Theorem***) *Let $\mathcal{A}$ be a PID algebra sheaf, let $\mathcal{E}$ be a free $\mathcal{A}$-module of rank $2n$, equipped with two symplectic $\mathcal{A}$-morphisms ω_0 and ω_1, and finally let $\mathcal{F}$ be a sub-$\mathcal{A}$-module of $\mathcal{E}$, Lagrangian with respect to both ω_0 and ω_1. Then, there exists an $\mathcal{A}$-symplectomorphism $\phi : (\mathcal{E}, \omega_0) \longrightarrow (\mathcal{E}, \omega_1)$ such that $\phi|_{\mathcal{F}} = Id_{\mathcal{F}}$.*

Proof. Let $\mathcal{G}$ be any complement of $\mathcal{F}$ in $\mathcal{E}$. By Theorem 5.19, given symplectic $\mathcal{A}$-morphisms ω_0 and ω_1, there exist Lagrangian complements $\mathcal{G}_0$ and $\mathcal{G}_1$ of $\mathcal{F}$ respectively. Again by the proof of Theorem 5.19, the restrictions ω_0', ω_1' of ω_0, ω_1 to $\mathcal{G}_0 \oplus \mathcal{F}$ and $\mathcal{G}_1 \oplus \mathcal{F}$ respectively are nondegenerate and yield $\mathcal{A}$-isomorphisms $\widetilde{\omega_0'} : \mathcal{G}_0 \longrightarrow \mathcal{F}^*$ and $\widetilde{\omega_1'} : \mathcal{G}_1 \longrightarrow \mathcal{F}^*$ respectively. Since $\mathcal{G}_0$ and $\mathcal{G}_1$ are free and of the same finite rank, there exists an $\mathcal{A}$-isomorphism $\psi : \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ such that $\widetilde{\omega_1'} \circ \psi = \widetilde{\omega_0'}$, i.e. for any sections $r \in \mathcal{G}_0(U)$ and $s \in \mathcal{F}(U)$

$$\omega_0(r, s) = \omega_1(\psi(r), s).$$

Let us extend ψ to the rest of $\mathcal{E}$ by setting it to be the identity on $\mathcal{F}$:

$$\phi := \mathrm{Id}_{\mathcal{F}} \oplus \psi : \mathcal{F} \oplus \mathcal{G}_0 \longrightarrow \mathcal{F} \oplus \mathcal{G}_1$$

and we have for any sections $r, r' \in \mathcal{G}_0(U)$ and $s, s' \in \mathcal{F}(U)$

$$\begin{aligned} \omega_1(\phi(s+r), \phi(s'+r')) &= \omega_1(s + \psi(r), s' + \psi(r')) \\ &= \omega_1(s, \psi(r')) + \omega_1(\psi(r), s') \\ &= \omega_0(s, r') + \omega_0(r, s') \\ &= \omega_0(s + r, s' + r'). \end{aligned}$$

■

5.4 Conclusion

Our attempt to obtain a complete abstract analog of the book "*Geometric Algebra*" by [3] was not completely possible because some of the geometric approach employed therein were not easily obtainable in our context. Therefore, the following selected problems present a research interest, and complement some of the ideas expounded in this work.

- The geometry of quadratic forms and the structure of the general linear group sheaf.
- Geometry over ordered algebraized space – Sylvester's theorem.
- The Structure of symplectic and orthogonal $\mathcal{A}$-modules, (see [3, Chapter V]): Investigate questions similar to those of the preceding chapters where we consider an $\mathcal{A}$-module with either a symplectic or an orthogonal geometry and try to determine invariant sub-$\mathcal{A}$-modules of the symplectic (respectively, orthogonal) $\mathcal{A}$-module. Furthermore, for deeper results on orthogonal $\mathcal{A}$-modules we shall need to construct a certain $\mathbb{C}$-algebraized space called the Clifford algebra $C(\mathcal{A})$ of $\mathcal{A}$.
- In CDG, the sheaf $\mathcal{A} \equiv \mathcal{C}^{\infty}_{X}$ is not a PID sheaf do not hold for manifolds. It would be very interesting to see how these results are formulated, for arbitrary algebra sheaf – (good comment from M. H. Papatriantafillou, external examiner).
- In Chapter 2 and the later, sheaves of free $\mathcal{A}$-modules are studied. What can we say about the case when we consider finitely generated $\mathcal{A}$-modules or generalized locally free $\mathcal{A}$-modules instead of free $\mathcal{A}$-modules? Would it become impossible to obtain similar results if we drop the demand of freeness and replace it

with finite generation – (good comment from Mart Abel, external examiner).

- It will be useful to compare the results not only with their classical counterparts in the classical linear and geometric algebra of finite-dimensional vector spaces, but also with the results of linear algebra of free modules over general commutative rings (some of which are given e.g. in Bourbaki's "Commutative Algebra") and especially over von Neumann regular commutative rings. Sheaves of modules naturally occur there with the ground topological space being the Zariski spectrum of the ground ring. Moreover, one can move further towards algebraic geometry by considering "nice" modules over schemes – expecting better properties than one can get for modules over arbitrary sheaves of rings – (good comment from G. Janelidze, external examiner).

- It would also be interesting to analyze the obtained results from the point of view of mathematical logic: in principal, as we know from topos theory, they should be the same as what one would get with "choice-free arguments" intuitively – G. Janelidze.

Further investigation of these problems would be a worthwhile contribution towards the same direction of research. The interested readers are kindly invited to explore the new territory opened by Abstract Geometric Algebra.

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