CHAPTER 7

A COMPLEX SYSTEM WITH CORRELATED FAILURES

7.1 INTRODUCTION

Most reliability models assume the continuous operation of the unit (or system) until a failure occurs. However, situations may arise where the unit (or system) needs rest after its operation for some time [Muller, 2005]. Very few attempts have been made in this direction. Murari and Muruthachalan (1981), Sarma (1982), Botha (2002), Hargreaves (2003) considered a two-unit system with a provision for rest for the system. The working and the rest are assumed to be random variables with negative exponential distributions. However, the idea of preparation time for the system may prove expensive as no output is obtained from the system during rest. This situation can be avoided in a two-unit cold standby system by providing rest to each unit alternately and operating the other unit when one requires rest. Further, in repairable systems, the dependence of repair time on the failure time of unit is a common experience of systems engineers, but this fact has also been ignored so far by reliability researchers. Keeping these factors in view, we analyse in this chapter a two-unit cold standby system with independent failure and repair times, with provision for the rest of the operative unit.

7.2 SYSTEM DESCRIPTION

1. The system consists of two identical units; initially, one is operative and the other is kept as a cold standby.

2. After operating for a random amount of time, the operating unit may require rest and again become fit for operation. The operating time and rest periods are independent random variables which are distributed exponentially.

3. As soon as the operative unit goes to rest, the standby unit starts operation.

4. There is a single repair facility
5. The repair facility is available instantaneously to repair the failed unit. The failure and repair are distributed according to bivariate exponential law.
6. Both units cannot go for rest simultaneously.
7. If the operative unit fails (after operating for time $X = x$) while the other unit is under repair. The unit failed later is repaired first and its repair time Y follows the bivariate exponential density jointly with X. The repair time already spent in the repair of the earlier failed unit is wasted and the further repair time Y' of this unit need not depend on x. It is assumed to have an independent negative exponential distribution with parameter θ.

7.3 NOTATION

Let O, S, R, F_r, F_{rc} and F_{wr}, denote respectively the operative, standby, under rest, under repair, under repair from previous state, and waiting for repair states of the unit. With these notation, the possible states of the system are:

Up states		Down states	
S_0	(O, S)	S_3	(R, F_r)
S_1	(O, R)	S_4	(F_{rc}, R)
S_2	(F_r, O)		
		S_6	(F_{wr}, F_r)
S_5	$(\mathrm{F_{rc}}, O)$		
S_7	(O, F_r)		

The possible transitions together with the corresponding transition probability density functions are shown in Figure 7.1

X, Y :random variables representing respectively the failure and repair times of a unit.

$f(x, y)$:joint pdf of (X, Y)

$$f(x, y) = \lambda\, \mu\, (1 - r) e^{-\lambda x - \mu y}\, I_0\left(2\sqrt{\lambda\, \mu\, x\, y}\right); \qquad x, y, \lambda, \mu > 0; |r| < 1 \tag{7.1}$$

with

$$I_0\left(2\sqrt{\lambda\, \mu\, r\, x\, y}\right) = \sum_{k=0}^{\infty} \frac{(\lambda\, \mu\, r\, x\, y)^k}{(k!)^2}$$

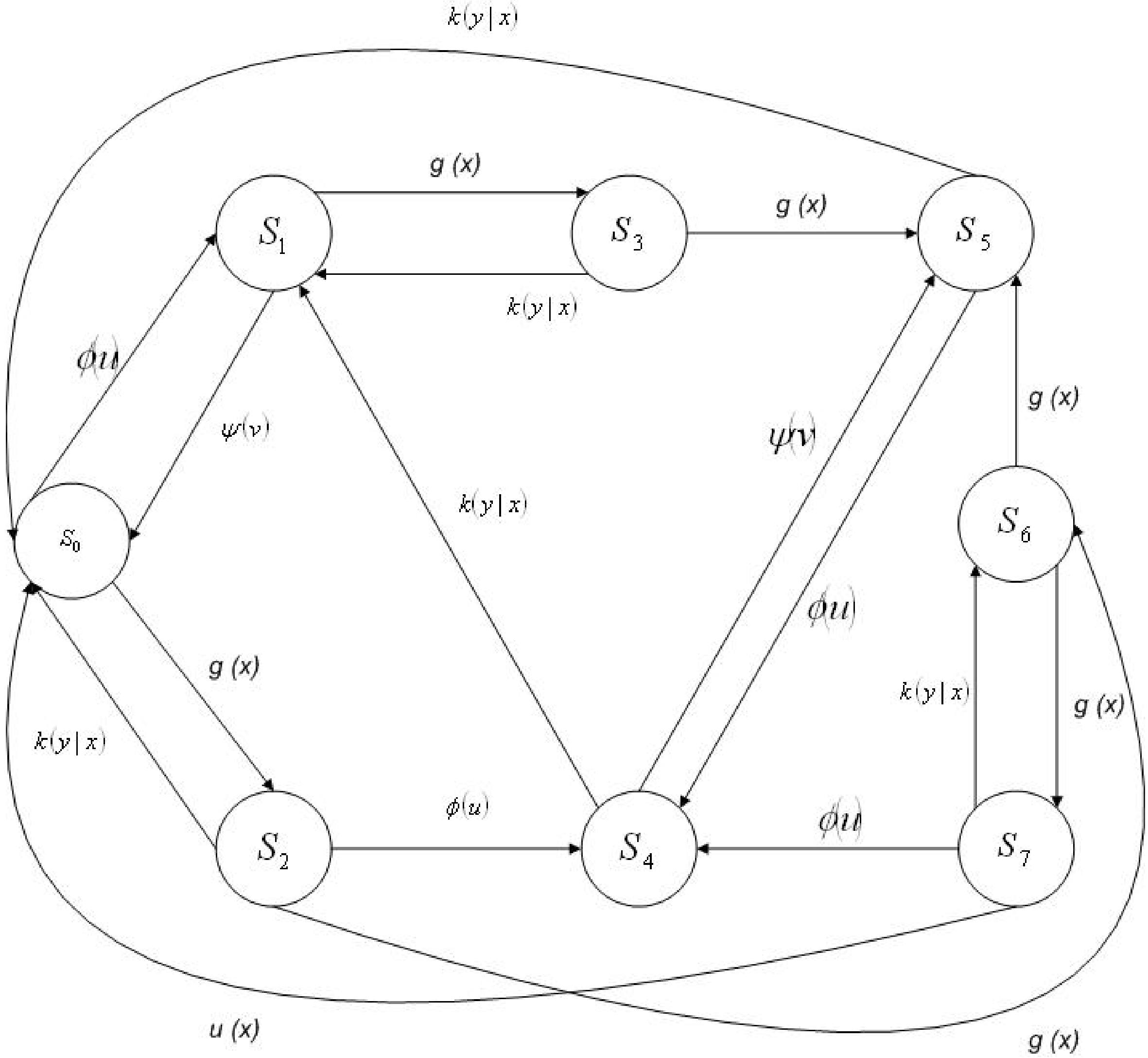
$k(y|x)$
$g\ (x)$
S_1
S_3
$g\ (x)$
S_5
$k(y|x)$
$\phi(u)$
$g\ (x)$
$\psi(v)$
$\psi(v)$
$k(y|x)$
S_0
S_6
$\phi(u)$
$g\ (x)$
$k(y|x)$
$g\ (x)$
$k(y|x)$
$\phi(u)$
$\phi(u)$
S_2
S_4
S_7
$u\ (x)$
$g\ (x)$

FIGURE 7.1

$g(x), G(x)$:pdf and cdf of X

$g(x) = \lambda(1-r)e^{-\lambda(1-r)x\lambda}$; $x > 0$

$G(x)=1-e^{-\lambda(1-r)x}$; $|r|<1.$

$h(y)=\mu(1-r)e^{-\mu(1-r)y}$; $y > 0$

$H(y)=1-e^{-\mu(1-r)y}$; $|r|<1.$

$k(y/x), k(y/x)$: :conditional pdf and cdf of y given x

$$k(y/x)=\mu\, e^{-\mu y-\lambda r x}\, I_0\left(2\sqrt{\lambda\,\mu\, r\, x\, y}\right)$$

$$k(y/x)=\int_0^y k(t/x)\,dt \qquad ;x, y, \lambda, \mu > 0, |r|<1.$$

$u(y'), U(y')$: pdf and cdf of Y', the random variable representing the repair time of a unit whose repair was interrupted.

$u(y')=\theta\, e^{-\theta y'}$; $\theta, y'>0$

$U(y')=1-e^{-\theta y'}$

$\phi(u), \Phi(u)$:pdf and cdf of the working period of a unit

$\phi(u)=\alpha\, e^{-\alpha\,\mu}\;;\;\alpha\,,u>0$

$\Phi(v)=1-e^{-\alpha\,u}$

$\varphi(v),\psi(v)$: pdf and cdf of the rest period of a unit.

$\varphi(v)=\beta\, e^{-\beta\,v};\;\; \beta,r>0$

$\psi(v)=1-e^{-\beta\,v}$

$q_{ij}(k,l,....)$:pdf of transition time from state $S_i\; to\, S_j$ (both regenerative) passing through $S_k, S_l,.....$

$Q_{ij}^{(k,l,....)}$:cdf of transition time from state $S_i\;\; to\, S_j$ passing through $S_k, S_l,....$

$p_{ij/x}^{(k,l,...)}$:steady state probability of transformation from state $S_i\;\; to\, S_j$ (or first return to state $S_i\;\; if\;\; j=i$) through states $S_k, S_l,....$ given that the system entered S_i after a sojourn for time x in the preceeding state.

$p_{ij}^{(k,l,\ldots)}$:steady state probability of transition from state $S_i \; to \, S_j$ (both regenerative) passing through $S_k, S_l, \ldots$.

p_{ij} : steady-state probability of direct transition from state $S_i \; to \, S_j$ given that the system entered states S_i after a sojourn time x in the preceeding state.

$v_i(t)$: cdf of sojourn time in state S_i

7.4 TRANSITION PROBABILITIES AND SOJOURN TIMES

We know that $p_{67} = 1$.

We first obtain the steady-state conditional probabilities as follows:

$$p_{20/x} = \int \mu \, e^{-\mu y - \lambda r x} I_0\left(2\sqrt{\lambda \, \mu \, r \, x \, y}\right) e^{-[\lambda(1-r)+\alpha]y} dy$$

$$= \frac{\mu}{\mu'} \exp\left[-\lambda \, r \, x\left(1 - \frac{\mu}{\mu'}\right)\right]$$

where

$$\mu' = \mu + \alpha + \lambda(1-r)$$

$$= p_{50/x}$$

$$p_{24/x} = \alpha \int e^{-[\lambda(1-r)+\alpha]y} dy \int_y^{\infty} \mu\, e^{-\mu \Im - \lambda r x} I\left(2\sqrt{\lambda\, \mu\, r\, x\, y}\right) dz$$

$$= \frac{\alpha}{\lambda(1-r)+\alpha}\left\{1 - \frac{\mu}{\mu'}\exp\left\{-\lambda\, r\, x\left(1-\frac{\mu}{\mu'}\right)\right\}\right\}$$

$$= p_{54/x}$$

$$p_{26/x} = \int \lambda(1-r) e^{-[\lambda(1-r)+\alpha]x} dy \left(\int_y^{\infty} \mu\, e^{-\mu r - \lambda r x} I_0\left(2\sqrt{\lambda\, \mu\, r\, x\, \Im}\right) dr\right)$$

$$= \frac{\lambda(1-r)}{\lambda(1-r)+\alpha}\left\{1 - \frac{\mu}{\mu'}\exp\left\{-\lambda\, r\, x\left(1-\frac{\mu}{\mu'}\right)\right\}\right\}$$

$$= p_{56/x}$$

$$p_{34/x} = \int e^{-\beta y} \mu\, e^{-\mu y - \lambda r x}\ I_0\left(2\sqrt{\lambda\, \mu\, r\, x\, y}\right) dy$$

$$= \frac{\mu}{\mu+\beta} e^{-\lambda r x\left(1-\frac{\mu}{\mu'}+\beta\right)}$$

$$= p_{41/x}$$

$$p_{35/x} = \int \beta\, e^{-\beta y} dy \int_y^{\infty} \mu\, e^{-\mu z - \lambda r x} I_0\left(2\sqrt{\lambda\, \mu\, r\, x\, z}\right) dz$$

$$= \int \beta\, e^{-\beta y}\, \overline{K}\,(\mathrm{y} \mid \mathrm{x})$$

$$= 1 - \frac{\mu}{\mu+\beta} e^{-\lambda r x\left(1-\frac{\mu}{\mu'}+\beta\right)}$$

$$= p_{45/x}$$

$$p_{67/x} = \int \mu\, e^{-\mu y - \lambda r x} I_0\left(2\sqrt{\lambda\,\mu\, r\, x\, y}\right) dy$$

Using these conditional probabilities we obtain the following unconditional probabilities:

$$p_{20} = \int p_{20/x}\, g(x)\, dx$$

$$= \int \left(\frac{\mu}{\mu'}\right) \exp\left\{-\lambda\, r\, x\left(1 - \frac{\mu'}{\mu}\right)\right\}\left[(1-r)e^{-(1-r)x}\, dx\right]$$

$$= \frac{\mu(1-r)}{(\lambda+\mu)(1-r)+\alpha}$$

$$= p_{50} \quad = \quad (A, \ \ say)$$

$$p_{24} = \int \frac{\alpha}{\lambda(1-r)+\alpha}\left[1 - \mu\mu'^{-1} \exp\left\{-\lambda\, r\, x\left(1 - \mu\mu'^{-1}\right)\right\}\right]\lambda(1-r)\exp\left[-\lambda(1-r)x\right]dx$$

$$= \frac{\alpha}{(\lambda+\mu)(1-r)+\alpha}$$

$$= p_{54} \quad = \quad (A_4, \ \ say)$$

Similarly,

$$p_{26} = p_{56} = \frac{(\lambda+\mu)(1-r)}{(\lambda+\mu)(1-r)+\alpha} \quad (= A_3,\ say)$$

$$p_{31} = p_{41} = \frac{\mu(1-r)}{\mu(1-r)+\beta} \quad (= \mathbf{B},\ say)$$

$$p_{35} = p_{45} = \frac{\beta}{\mu(1-r)+\beta} \quad \left(\overline{B} = 1 - B\right)$$

$$p_{67} = 1$$

The other unconditional transition probabilities are

$$p_{70} = \frac{\theta}{\theta + \alpha + \lambda(1-r)}; (= c_1; say)$$

$$p_{74} = \frac{\alpha}{\theta + \alpha + \lambda(1-r)}; (= c_2, say)$$

$$p_{76} = \frac{\lambda(1-r)}{\theta + \alpha + \lambda(1-r)}; (= c_3, \ say)$$

$$p_{01} = \frac{\alpha}{\lambda(1-r)+\alpha}, \ (= D, \ say)$$

$$p_{02} = \frac{\lambda(1-r)}{\lambda(1-r)+\alpha}, \left(= \overline{D} = 1 - D, \quad say\right)$$

$$p_{10} = \frac{\beta}{\lambda(1-r)+\beta}, (= E, \ say)$$

$$p_{13} = \frac{\lambda(1-r)}{\lambda(1-r)+\beta}, \left(= \overline{E} = 1 - E, \quad say\right)$$

Hence the non-zero elements of the transition probability matrix

$$p = \left[p_{ij}^{(k,l,\ldots)}\right] = \left[Q_{ij}^{(k,l,\ldots)}(\infty)\right]$$

are

$$p_{01} = D, \quad p_{01}^{(2,4)} = \overline{D}\, A_2\, B$$

$$p_{01}^{(2,4,5,4)} = \overline{D}\, A_2\, \overline{B}\, A_2$$

$$p_{00}^{(2)} = \overline{D}\, A_1$$

$$p_{00}^{(2,4,5)} = \overline{D}\, A_2\, \overline{B}\, A_1$$

$$p_{07}^{(2,6)} = p_{04}^{(2)} = \overline{D}\, A_3$$

$$p_{07}^{(2,4,5,6)} = p_{04}^{(2,4,5)} = \overline{D}\, A_2\, \overline{B}\, A_3$$

$$p_{10} = E \;\; ; \; p_{10}^{(3,5)} = \overline{E}\, \overline{B}\, A_1$$

$$p_{11}^{(3)} = \overline{E}.B; \quad p_{11}^{(3,5,4)} = \overline{E}\, \overline{B}\, A_2\, \overline{B}\, A_3$$

$$p_{10}^{(3,5,4,5,6)} = p_{16}^{(3,5,4,5)} = \overline{E}\, \overline{B}\;\, A_2\, \overline{B}\, A_1$$

$$p_{11}^{(3,5,4,5,4)} = \overline{E}\, \overline{B}\, A_2\, \overline{B}\, A_2$$

$$p_{70} = C_1, \qquad p_{70}^{(4,5)} = C_2\, \overline{B}\, A_1$$

$$p_{71}^{(4)} = C_2 B, \quad p_{71}^{(4,5,4)} = C_2\, \overline{B}\, A_2$$

$$p_{77}^{(6)} = C_3, \quad p_{77}^{(4,5,6)} = C_2\, \overline{B}\, A_3$$

These transition probabilities are seen to satisfy the following relations.

$$p_{01} + p_{01}^{(2,4)} + p_{01}^{(3,4,5,4)} + p_{00}^{(2)} + p_{00}^{(2,4,5)} + p_{07}^{(2,6)} + p_{07}^{(2,4,5,6)} = 1 \quad (7.2)$$

$$p_{10} + p_{10}^{(3,5)} + p_{10}^{(3,5,4,5)} + p_{11}^{(3)} + p_{11}^{(3,5,4)} + p_{11}^{(3,5,4,5,4)} + p_{17}^{(3,5,6)} + p_{17}^{(3,5,4,5,6)} = 1 \quad (7.3)$$

$$p_{70} + p_{70}^{(4,5)} + p_{71}^{(4)} + p_{71}^{(4,5,4)} + p_{77}^{(6)} + p_{77}^{(4,5,6)} = 1 \quad (7.4)$$

The sojourn times in various regenerative states are

$$\mu_0 = [\alpha + \lambda(1-r)]^{-1} \quad (7.5)$$

$$\mu_1 = [\beta + \lambda(1-r)]^{-1} \quad (7.6)$$

$$\mu_2 = [\alpha + \theta + \lambda(1-r)]^{-1} \quad (7.7)$$

7.5 MEANTIME TO SYSTEM FAILURE

Time to system failure can be regarded as the first passage time to the failed states $S_i \ (i = 3,4,6)$. Considering the states as absorbing we have, by simple probabilistic reasoning

$$\pi\, dt = Q_{00}^{(2)}(t) \circledS \pi_0(t) + Q_{01}(t) \circledS \pi_1(t) + Q_{04}^{(2)}(t) + Q_{06}^{(2)}(t) \quad (7.8)$$

$$\pi_1(t) = Q_{01}(t) \circledS \pi_0(t) + Q_{13}(t) \tag{7.9}$$

Taking Laplace-Stieltjes transform and solve for $\tilde{\pi}_0(s)$, we get

$$\tilde{\pi}_0 = \frac{\left[\tilde{Q}_{04}^{(2)} + \tilde{Q}_{06}^{(2)} + \tilde{Q}_{01}\tilde{Q}_{13}\right]}{1 - \tilde{Q}_{00}^{(2)} - \tilde{Q}_{01}\tilde{Q}_{10}} \tag{7.10}$$

which gives

$$MTSF = \frac{m_0 + p_{01}m_1}{1 - p_{00}^{(2)} - p_{01}p_{10}}$$

where

$$m_0 = m_{01} + m_{00}^{(2)} + m_{04}^{(2)} + m_{06}^{(2)}$$

$$m_1 = m_{10} + m_{13}$$

and

$m_{ij}^{(k,j,....)}$ have their usual meaning.

7.6 AVAILABILITY ANALYSIS

Let $A_i(t)$= P [the system is up at any time t | S_i at t = 0]

From the arguments used in the theory of regenerative processes,

$$A_0(t) = \{q_{00}^{(2)}(t) + q_{00}^{(2,4,5)}(t)\} © A_0(t)$$
$$+ \{q_{01}(t) + q_{01}^{(2,4)}(t) + q_{01}^{(2,4,5,4)}(t)\} © A_1(t)$$
$$+ \{q_{07}^{(2,6)}(t) + q_{07}^{(2,4,5,6)}(t)\} © A_7(t)$$
$$+ e^{-\{\lambda(1-r)+\alpha\}t} + Q_{02}(t) © e^{-\{(\lambda+\mu)(1-r)+\alpha\}t} \tag{7.11}$$

$$A_1(t) = \{q_{10}(t) + q_{10}^{(3,5)}(t) + q_{10}^{(3,5,4,5)}(t)\} © A_0(t)$$
$$+ \{q_{11}^{(3)}(t) + q_{11}^{(3,5,4)}(t) + q_{11}^{(3,5,4,5,4)}(t)\} © A_1(t)$$
$$+ \{q_{17}^{(3,5,6)}(t) + q_{17}^{(3,5,4,5,6)}(t)\} © A_7(t) + e^{-\{\lambda(1-r)+\beta\}t} \tag{7.12}$$

$$A_7(t) = \{q_{70}(t) + q_{70}^{(4,5)}(t)\} © A_0(t)$$
$$+ \{q_{71}^{(4)}(t) + q_{71}^{(4,5,4)}(t)\} © A_1(t)$$
$$+ \{q_{77}^{(6)}(t) + q_{77}^{(4,5,6)}\} © A_7(t) + e^{-\{\theta+\alpha+\lambda(1-r)\}t} \tag{7.13}$$

Taking Laplace transforms for (7.11) – (7.13) and solving for $A_0^*(s)$, we get

$$A_0^*(s) = \frac{N_1(s)}{D_1(s)}$$

where

$N_1(s)$

$$= \frac{1}{[\lambda(1-r)+\alpha+s]}\left[\begin{array}{l}\left(1 - q^{(3)*}{}_{11} - q^{(3,5,4)*}{}_{11} - q^{(3,5,4,5,4)*}{}_{11}\right) \\ \times\left(1 - q^{(6)*}{}_{77} - q^{(4,5,4)*}{}_{77}\right) - \left(q_{71}^{(4)*} + q_{71}^{(4,5,4)*}\right) \times \left(q_{17}^{(3,5,4,5,6)*} + q_{71}^{(3,5,6)*}\right.\end{array}\right.$$

$$+ \frac{1}{\lambda(1-r)+\beta+s}\left[\left(q_{01}^* + q_{01}^{(2,4)*}\right) \times \left(1 - q_{77}^{(6)*} - q_{77}^{(4,5,6)*}\right)\right] + q_{01}^{(2,4,5,4)*}$$
$$+ \left(q_{07}^{(2,6)*} + q_{07}^{(2,4,5,6)*}\right)\left(q_{71}^{(4)*} + q_{71}^{(4,5,4)*}\right)$$

$$+\frac{1}{\theta+\alpha+\lambda(1-r)+s}\begin{bmatrix}\left(q_{01}^{*}+q_{01}^{(2,4)}+q_{01}^{(2,4,5,4)*}\right)\times\left(q_{17}^{(3,5,6)*}+q_{17}^{(3,5,4,5,6)*}\right)+\left(q_{07}^{(2,6)*}+q_{07}^{(2,\right.\\ \times\left(1-q_{11}^{(3)*}-q_{11}^{(3,5,4)*}-q_{11}^{(3,5,4,5,4)*}\right)\end{bmatrix}$$

and

$$D_1(s)=\left(1-q_{00}^{(2)*}-q_{00}^{(2,4,5)}\right)\begin{bmatrix}\left(1-q_{11}^{(3)*}-q_{11}^{(3,5,4)*}-q_{11}^{(3,5,4,5,4)*}\right)\left(1-q_{77}^{(6)*}-q_{77}^{(4,5,6)*}\right)\\ -\left(q_{71}^{(4)*}+q_{71}^{(4,5,4)*}\right)\left(q_{17}^{(3,5,4,5,6)*}+q_{17}^{(3,5,6)*}\right)\end{bmatrix}-$$

$$\left(q_{01}^{*}+q_{01}^{(2,4)*}+q_{01}^{(2,4,5,4)*}\right)\begin{bmatrix}\left(q_{10}^{*}+q_{10}^{(3,5)*}+q_{10}^{(3,5,4,5)*}\right)\left(1-q_{77}^{(6)*}-q_{77}^{(4,5,6)*}\right)+\\ \left(q_{70}^{*}+q_{70}^{(4,5)*}\right)\left(q_{17}^{(3,5,6)*}+q_{17}^{(3,5,4,5,6)*}\right)\end{bmatrix}$$

$$-\left(q_{07}^{(2,6)*}+q_{07}^{(2,4,5,6)*}\right)\begin{bmatrix}\left(q_{10}^{*}+q_{10}^{(3,5)*}+q_{10}^{(3,5,4,6)*}\right)\left(q_{71}^{(4)}+q_{71}^{(4,5,4)*}\right)+\left(q_{70}^{*}+q_{70}^{(4,5)*}\right)\\ \left(1-q_{11}^{(3)*}-q_{11}^{(3,5,4)*}-q_{11}^{(3,5,4,5,4)*}\right)\end{bmatrix}$$

The steady state availability of the system is

$$A_{\infty}=\frac{\frac{1}{\lambda(1-r)+\alpha}.U_1+\frac{1}{\lambda(1-r)+\beta}.U_2+\frac{1}{\theta+\alpha+\lambda(1-r)}.U_3}{n_0U_1+n_1U_2+n_7U_3}$$

where

$$U_1=\begin{bmatrix}\left(1-p_{11}^{(3)}-p_{11}^{(3,5,4)}-p_{11}^{(3,5,4,5,4)}\right)\left(1-p_{77}^{(6)}-p_{77}^{(4,5,6)}\right)-\left(p_{17}^{(3,5,6)}+p_{17}^{(3,5,4,5,6)}\right)\\ \left(p_{71}^{(4)}+p_{71}^{(4,5,4)}\right)\end{bmatrix}$$

$$U_2=\left[\left(1-p_{00}^{(2)}-p_{00}^{(2,4,5)}\right)\left(1-p_{77}^{(6)}-p_{77}^{(4,5,6)}\right)-\left(p_{07}^{(2,6)}+p_{07}^{(2,4,5,6)}\right)\left(p_{70}+p_{70}^{(4,5)}\right)\right]$$

$$U_3 = \begin{bmatrix} \left(1 - p_{00}^{(2)} - p_{00}^{(2,4,5)}\right)\left(1 - p_{11}^{(3)} - p_{11}^{(3,5,4)} - p_{11}^{(3,5,4,5,4)}\right) \\ -\left(p_{01} + p_{01}^{(2,4)} + p_{01}^{(2,4,5,4)}\right)\left(p_{10} + p_{10}^{(3,5)} + p_{10}^{(3,5,4,5)}\right) \end{bmatrix}$$

$$n_0 = \sum_j m_{0j}^{(k,l,....)}$$

$$n_1 = \sum_j m_{0j}^{(k,l,....)}$$

$$n_7 = \sum_j m_{7j}^{(k,l,....)}; \qquad k,\ l = 2,\ 3,\ 4,\ 5,\ 6.$$

Therefore the interval availability (Sarma, 1982), for the interval $(0,\ t)$ is

$$A_0(t) = \frac{1}{t}\int_0^t A_0(u)\,du \tag{7.14}$$

so that

$$A_0^*(s) = \int_0^\infty \frac{A_0^*(u)}{u}\,du \tag{7.15}$$

The inherent (limiting interval) availability of the system is

$$A_0(\infty) = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0} s^2\, L\left[\int_0^t A_0(u)\,du\right]$$

$$= \lim_{s\to 0} s\, A_0^*(s) = A_\infty$$

7.6 BUSY PERIOD ANALYSIS

By probabilistic arguments, we obtain the following equations for $\beta_i(t)$.

$\beta_i(t) = P$[the repairman is busy at t | S_i at t = 0]

$$\beta_0(t) = \left\{q_{00}^{(2)}(t) + q_{00}^{(2,4,5)}(t)\right\} © \beta_0(t)$$
$$+ \left\{q_{01}(t) + q_{01}^{(2,4)}(t) + q_{00}^{(2,4,5,4)}(t)\right\} © \beta_1(t)$$
$$+ \left\{q_{07}^{(2,6)}(t) + q_{07}^{(2,4,5,4)}(t)\right\} © \beta_7(t) \tag{7.16}$$

$$\beta_1(t) = \left\{q_{10}(t) + q_{10}^{(3,5)}(t) + q_{10}^{(3,5,4,5)}(t)\right\} © \beta_0(t)$$
$$+ \left\{q_{11}^{(3)}(t) + q_{11}^{(3,5,4)}(t) + q_{11}^{(3,5,4,5,4)}(t)\right\} © \beta_1(t)$$
$$+ \left\{q_{17}^{(3,5,6)}(t) + q_{17}^{(3,5,4,56)}(t)\right\} © \beta_7(t) \tag{7.17}$$

$$\beta_7(t) = \left\{q_{70}(t) + q_{70}^{(4,5)}\right\} © \beta_0(t)$$
$$+ \left\{q_{71}^{(4)}(t) + q_{71}^{(4,5,4)}(t)\right\} © \beta_1(t)$$
$$+ \left\{q_{77}^{(6)}(t) + q_{77}^{(4,5,6)}(t)\right\} © \beta_7(t) + e^{-\theta t} \tag{7.18}$$

Taking the Laplace transforms for (4.16) – (4.18) and solve for $\beta_0^*(s)$, *we get*

$$\beta_0(s) = \frac{N_3(s)}{D_2(s)} \qquad (7.19$$

where

$$N_3(s)=\frac{1}{\theta+s}\left[\left(q_{01}^{*}+q_{01}^{(3,5,4)*}+q_{01}^{(3,5,4,5,4)*}\right)\left(q_{17}^{(3,5,6)*}+q_{17}^{(3,5,4,5,6)*}\right)+\left(1-q_{11}^{(3)*}-q_{11}^{(3,5,4)*}-\right.\right.$$

$$\times\left(q_{07}^{(2,4)*}+q_{07}^{(2,4,5,6)*}\right)$$

and $D_2(s)$ is same as $D_1(s)$.

Then the steady state probability that the repairman will be busy is

$$\beta_\infty=\lim_{t\to\infty}\beta_o(t)=\lim_{s\to 0}s\beta_0^*(s)=\frac{N_2^*(s)}{D_2^*(s)}$$

$$N_3^*(0)=\frac{U_3}{\theta}$$

The expected busy period of the repairman in $(0,\ t]$ is

$$\mu_b(t)=\int_0^t \beta_0(u)\,du \tag{7.20}$$

so that

$$U_b^*(s)=\frac{\beta_0^*(s)}{s} \tag{7.21}$$

and the expected idle period of the repairman in $(0,t]$ is

$$\mu_I(t)=t-\mu_b(t) \tag{7.22}$$

so that

$$\mu_I^*(s) = \frac{1}{s^2} - \mu_b^*(s) \tag{7.23}$$

As $\beta_0^*(s)$ is known explicitly, these quantities can easily be calculated.

7.7 PROFIT ANALYSIS

The expected up-time of the system in $(0,t]$ can be calculated from the pointwise availability as

$$\mu_u(t) = \int_0^t A_0(v)\,dv$$

so that

$$\mu^*{}_u(s) = \frac{A_0^*(s)}{s} \tag{7.24}$$

Let k_0 represent the expected revenue per unit up-time and k_1, the expected repair cost per unit time, then the expected profit in $(0,t]$ is

$$G(t) = k_0\,\mu_u(t) - k_1\,\mu_b(t) \tag{7.25}$$

The expected net profit per unit time in the long run is

$$G = \lim_{t \to \infty} \frac{G(t)}{t} = k_0\ A_\infty - k_1 \beta_\infty$$

7.8 SPECIAL CASES

1. When the failure and repair times are independent; i.e. $r = 0$

$$MTSF = \frac{\mu_0 + p_{01}\mu_1 + p_{02}\mu_2}{1 - \left(p_{02}p_{20} + p_{01}p_{10}\right)} \tag{7.26}$$

$$A_\infty = \frac{N_1}{D_1} \qquad ; \beta_\infty = \frac{N_2}{D_2}$$

$$; D_1 = D_2$$

$$N_1 = \mu_0\left[(1 - p_{26}) - (p_{23}p_{32})\right]$$
$$+ \mu_1\left[(1 - p_{26})(p_{01} + p_{31}) + p_{23}(p_{02}p_{31} - p_{01}p_{32})\right]$$
$$+ \mu_2\left[p_{02} + p_{13}(p_{01}p_{32} - p_{02}p_{31})\right]$$

$$D_1 = \mu_0\left[p_{20}(1 - p_{13}p_{31}) - p_{10}p_{23}p_{32}\right] + \mu_1\left[p_{01}(1 - p_{26}) - p_{23}p_{32}\right]$$
$$+ \mu_2\left[p_{02}(1 - p_{13}p_{31}) + p_{01}p_{13}p_{32}\right]$$
$$+ \mu_3\left[p_{13}(1 - p_{26}) - p_{13}p_{02}p_{20}\right] + \mu_6\left[p_{26}(1 - p_{13}p_{31} - p_{01}p_{10})\right]$$

$$N_2 = (\mu_2 + p_{26}\mu_6)\left[p_{01}p_{13}p_{32} + p_{02}(1 - p_{13}p_{31})\right] + \mu_3\left[p_{01}p_{13}(1 - p_{26}) - p_{02}p_{23}\right]$$

2. When $\phi(u) = \psi(v) = 0$ and then the states $S_1, S_3, S_4\ and\ S_5$ do not exist. Then

$$MTSF = \frac{\mu + 2\lambda}{\lambda^2(1-r)} \tag{7.27}$$

$$A_\infty = \left[\frac{1 + 2\lambda^2(1-r)}{\theta(\mu + 2\lambda)}\right]^{-1} \tag{7.28}$$

$$\beta_\infty = \frac{\lambda^2}{\theta(\mu + 2\lambda) + 2\lambda^2(1-r)} \tag{7.29}$$

3. When there is no provision for rest and failure and repair times are independent

$$MTSF = \frac{\mu + 2\lambda}{\lambda^2} \tag{7.30}$$

$$A_\infty = \frac{\mu(2\lambda + \mu)}{\mu(2\lambda + \mu) + 2\lambda^2} \tag{7.31}$$

$$\beta_\infty = \frac{\lambda^2}{\mu(2\lambda + \mu) + \lambda^2} \tag{7.32}$$

7.10 NUMERICAL ILLUSTRATION

When $\mu = \theta = 10$; $\alpha = \beta = 0$.

Table 7.1

Profit

λ	r =-0.5	r=0	r=0.5
0	100.0010	100.0911	100.1101
2	92.9110	93.1525	94.6616
4	79.1502	81.5612	85.0315
6	66.8816	73.6116	80.1506
8	54.1606	61.4441	69.7012
10	43.2806	53.3315	62.1111
12	40.0015	47.6106	56.0152
14	36.1585	42.6150	49.1566
16	32.6617	39.9915	45.8106

Table 7.2

Profit

λ	$\alpha = 4, \beta = 3$	$\alpha = \beta = 0$
0	100.0911	100.0911
2	90.9106	93.1525
4	77.5505	81.5612
6	67.8819	73.6116
8	62.5531	61.4441
10	53.3316	53.3315
12	49.1606	47.6106
14	44.1629	42.6150
16	42.8718	39.4915

Table 7.2

CONCLUSIONS:

From **Table 7.1** we conclude that as failure rate increases the mean time to system failure (MTSF) decreases. For both models as the failure rate increases the MTSF of the system decreases but as the failure rate continues increasing MTSF goes on decreasing.

From **Table 7.2** we conclude that for both models as the failure rate increases the profit of the system decreases but comparatively less when the failure rate increases less; model 2 is more beneficial than model 1 and as the failure rate continues increase the profit difference goes on decreasing. As cost per visit of the repairman increases, the profit of the system decreases.

To observe the effect of correlation and rest on the profit (in the steady state), we plot the profit function against λ,
setting $\mu = \theta = 10, k_0 = 100 \ \ and \ \ k_1 = 20.$ The curves so obtained are shown in Table 7.1 and 7.2 respectively. In Table 7.1, in addition, we set $\alpha = \beta = 0$ and obtain three different curves for profit function vs λ .Taking r=-0.5, 0.0 and 0.5 respectively. In table 7.2, we put r=0 in addition to the values of μ, θ, k_0, k_1 and obtain two different values of profit function against λ ,one
with $\alpha = 4, \ \beta = 3$ (i.e. when there is provision for rest) and the other with $\alpha = \beta = 0$ (i.e. when there is no provision for rest).

These values reveal two important facts:

1. The profit/unit time (in steady state) decreases with respect to the increase in λ. However, for the same λ the profit increases with increases in r. Thus a high positive correlation between failure and repair times tends to increase the profit earned by the system in steady state.

The effect of providing rest for the operative unit depends on the proportion of values of λ and μ. Although in both cases (i.e. when $\alpha = \beta = 0$ or when $\alpha = 4, \beta = 3$) the profit decreases with increase in failure rate, a favourable effect of providing rest is observed when only when $\lambda > \mu$, i.e. when the failure rate is higher than the repair rate. As long as $\lambda < \mu$, the provision of rest is nothing but a costly burden on the systems manager, and when $\lambda = \mu$, the profit with or without rest is the same, so there is no advantage in providing rest. Thus, one must avoid providing rest as long as $\lambda \leq \mu$. But since, in practice, most of the time, the failure rate is much higher than the repair rate, a considerable increase in profit can be obtained by providing rest to the operative unit and taking output from the standby unit during the rest time of the operating unit.