CHAPTER 2

A STUDY OF A TWO UNIT PARALLEL SYSTEM WITH ERLANGIAN REPAIR TIME

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2.1 INTRODUCTION

In order to improve the reliability, availability, quality and safety operational systems the well known tools to be used are redundancy, repair and preventive maintenance, etc. (Birollini et al, (1994)). Most of the past studies of reliability systems are confined to obtaining expressions for various measures of system performance and do not consider the associated inference problems. Chandrasekhar and Natarajan (1994), Yadavalli et al (2001), (2002) have considered a two unit parallel system and obtained exact confidence limits for the steady state availability of the system, when the failure rate of an operative unit is constant and the repair time of the failed unit is a two stage Erlang distribution. The Bayesian methods for these problems were subsequently studied by Yadavalli et al (2003).

In general, the failure- free time and repair time are independent random variables. Thus there is need to study a model by relaxing this imposed condition. An attempt is made in this paper to study a two-unit parallel system, wherein the failure rate of a unit is constant and the repair time distribution is a two Erlangian distribution under the assumption that an operative unit has a zero failure rate if a failed unit is in the second stage of repair. Apart from expressions for the system reliability, MTBF, availability and steady state availability, we obtain a CAN estimator and an asymptotic confidence interval for the steady state availability of the system and the MLE of the system reliability.

2.2 MODEL AND ASSUMPTIONS

The system under consideration is a two unit parallel system with a single repair facility, subjected to the following assumptions:

(i) The units are similar and statistically independent. Each unit has a constant failure rate λ.

(ii) There is only one repair facility and the repair time distribution is a two stage Erlangian distribution with probability density function (p.d.f) given by

$$g(y) = (2\mu)^2 e^{-2\mu y} y,\ 0 < y < \infty,\ \mu > 0 \qquad (2.1)$$

(iii) Each unit is new after repair.

(iv) Switch is perfect and the switchover is instantaneous.

(v) An operative unit has a zero failure rate if a failed unit is in the second stage of repair.

2.3 ANALYSIS OF THE SYSTEM

To analyze the behaviour of the system, we note that at any time t, the system may be in any one of the mutually exclusive and exhaustive states

So:	Both units are operating
S1:	One unit is operating and the other is in the first stage of repair.

S2:	One unit is operating and the other is in the second stage of repair.
S3:	One unit is in the first stage of repair and the other is waiting for repair
S4:	One unit is in the second stage of repair and the other is waiting for repair.

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Since an Erlang distribution is the distribution of the sum of two independent and identically distributed exponential random variables, the stochastic

process describing the behaviour of the system is a Markov process. Let $p_i(t)$, i = 0, 1, 2, 3, 4 be the probability that the system is in state at S_i time t. Clearly, the infinitesimal generator of the Markov process is given by:

$$Q = \begin{matrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{matrix} \begin{bmatrix} -2\lambda & 2\lambda & 0 & 0 & 0 \\ 0 & -(\lambda+2\mu) & 2\mu & \lambda & 0 \\ 2\mu & 0 & -2\mu & 0 & 0 \\ 0 & 0 & 0 & -2\mu & 2\mu \\ 0 & 2\mu & 0 & 0 & -2\mu \end{bmatrix} \quad (2.2)$$

It should be noted that states S_0, S_1 and S_2 are up-states, whereas S_3 and S_4 are down states. We assume that initially, both the units are operative.

2.3.1. Reliability

The system reliability R (t) is the probability of failure free operation of the system in (0, t]. To derive an expression for the reliability of the system, we restrict the transitions of the Markov process to the system, we restrict the transitions of the Markov process to the system up-states namely S_0, S_1 and

S_2.Using the infinitesimal generator given in (2.2) pertaining to these up-states and standard probabilities arguments, we obtain the following system of differential-difference equations

$$p_0'(t) = -2\lambda p_o(t) + 2\mu p_2(t)$$

$$p_1'(t) = 2\lambda p_0(t) - (\lambda + 2\mu) p_1(t)$$

$$p_2'(t) = 2\mu p_1(t) - 2\mu p_2(t).$$

With the condition $p_0(0) = 1$ and $p_i(0) = 0$ for $i = 1, 2$. Thus,

$$\sum_{i=0}^{2} p_i(t) = 1.$$

Let $L_i(s)$ be the Laplace transform of p_i (t), i = 0, 1, 2. Taking Laplace transforms for $p_i(t)$, we get

$$(s + 2\lambda) p_0^*(s) - 2\mu p_2^*(s) = 1$$

$$(s + \lambda + 2\mu) p_1^*(s) - 2\lambda p_0^*(s) = 0$$

$$(s + 2\mu) p_2^*(s) - 2\mu p_1^*(s) = 0$$

$$R(t) = \sum_{i=1}^{3} \frac{[(\alpha_i + 2\mu)(\alpha_i + 3\lambda + 2\mu) + 4\lambda\mu]}{\prod_{j=1}^{3} (\alpha_i - \alpha_j)} e^{\alpha_i t} \qquad (2.3)$$

where α_1, α_2, and α_3 are the roots of the cubic equation:

$$s^3 + (3\lambda + 4\mu)s^2 + (2\lambda^2 + 10\lambda\mu + 4\mu^2)s + 4\lambda^2\mu = 0$$

2.3.2 Mean Time Before Failure (MTBF)

The system mean time before failure is given by

$$MTBF = L_0(0) + L_1(0) + L_2(0) = \frac{5\lambda + 2\mu}{2\lambda^2}$$

2.3.3 System Availability

The system availability A (t) is the probability that the system operates (within the tolerances) at a given instant of time t.

Using the infinitesimal generator given in (2.2), we obtain the following system of differential-difference equations:

$$p_0'(t) = -2\lambda p_0(t) + 2\mu p_2(t) \tag{2.4}$$

$$p_1'(t) = 2\lambda p_0(t) - (\lambda + 2\mu)p_1(t) + 2\mu p_4(t) \tag{2.5}$$

$$p_2'(t) = 2\mu p_1(t) - 2\mu p_2(t) \tag{2.6}$$

$$p_3'(t) = \lambda p_1(t) - 2\mu p_3(t) \tag{2.7}$$

$$p_4'(t) = 2\mu p_3(t) - 2\mu p_4(t) \tag{2.8}$$

with the condition $p_o(0) = 1$ and

$$\sum_{i=0}^{4} p_i(t) = 1 \tag{2.9}$$

Taking the Laplace transforms for the equations (2.4) – (2.8), we get

$$(s + 2\lambda)p_0^*(s) - 2\mu p_2^*(s) = 1 \tag{2.10}$$

$$(s + \lambda + 2\mu)p_1^*(s) - 2\lambda p_0^*(s) - 2\mu p_4^*(s) = 0 \tag{2.11}$$

$$(s + 2\mu)p_2^*(s) - 2\mu p_1^*(s) = 0 \tag{2.12}$$

$$(s + 2\mu)p_3^*(s) - \lambda p_1^*(s) = 0 \tag{2.13}$$

$$(s + 2\mu)p_4^*(s) - 2\mu p_3^*(s) = 0 \tag{2.14}$$

solving the equations (2.10) – (2.14) using the relation (2.9), we get

$$p_i^*(s),\ i = 0, 1, 2, \ldots, 4.$$

Inverting $p_i^*(s)$, we get

$$p_0(t) = \frac{\mu^2}{(\lambda + \mu)^2} + 8\lambda\mu^2 \sum_{i=1}^{3} \frac{(\alpha_i + 2\mu)}{\alpha_i(\alpha_i + 2\lambda)\prod_{\substack{i=1 \\ j\approx 1}}^{3}(\alpha_i - \alpha_j)} e^{-\alpha_i t} \tag{2.15}$$

$$p_1(t) = \frac{\lambda\mu}{(\lambda + \mu)^2} + 2\lambda \sum_{i=1}^{3} \frac{(\alpha_i + 2\mu)^2}{\alpha_i \prod_{\substack{i=1 \\ j\approx 1}}^{3}(\alpha_i - \alpha_j)} e^{-\alpha_i t} \tag{2.16}$$

$$p_2(t)=\frac{\lambda\mu}{(\lambda+\mu)^2}+4\lambda\mu\sum_{i=1}^{3}\frac{1}{\alpha_i}\frac{(\alpha_i+2\mu)}{\prod_{\substack{i=1\\ j\approx 1}}^{3}(\alpha_i-\alpha j)}e^{-\alpha_i t} \quad (2.17)$$

$$p_3(t)=\frac{\lambda^2}{2(\lambda+\mu)^2}+2\lambda^2\sum_{i=1}^{3}\frac{1}{\alpha_i}\frac{(\alpha_i+2\mu)}{\prod_{\substack{i=1\\ j\approx 1}}^{3}(\alpha_i-\alpha_j)}e^{-\alpha_i t} \quad (2.18)$$

$$p_4(t)=\frac{\lambda^2}{2(\lambda+\mu)^2}+4\lambda^2\mu\sum_{i=1}^{3}\frac{1}{\alpha_i\prod_{\substack{i=1\\ j\approx 1}}^{3}(\alpha_i-\alpha_j)}e^{-\alpha t t} \quad (2.19)$$

where α_1, α_2 and α_3 are the roots of the cubic equation.

$$s^3+3(\lambda+2\mu)s^2+2(\lambda^2+8\lambda\mu+6\mu^2)s+8\mu(\lambda+\mu)^2=0 \quad (2.20)$$

Since S_0, S_1 and S_2 are the up-states, the availability of the system is given by:

$$A(t)=p_0(t)+p_1(t)+p_2(t) \quad (2.21)$$

$$A(t)=\frac{\mu(2\lambda+\mu)}{(\lambda+\mu)^2}+8\lambda\mu\sum_{i=1}^{3}\frac{(\alpha_i+2\mu)e^{-\alpha t\, t_i}}{\alpha_i(\alpha_i+2\lambda)\prod_{j=1}^{3}(\alpha_i-\alpha_j)}+2\lambda\sum_{i=1}^{3}\frac{(\alpha_i+2\mu)^2}{\alpha_i\prod_{j=1}^{3}(\alpha_i-\alpha_j)}e^{-\alpha_i t}$$
$$+4\mu\sum_{i=1}^{3}\frac{(\alpha_i+2\mu)}{\alpha_i\prod_{j=1}^{3}(\alpha_i-\alpha_j)}$$

2.3.4 Steady State Availability

The system steady state availability is given by:

$$A_\infty = \lim_{t\to\infty} A(t) = \frac{\mu(2\lambda+\mu)}{(\lambda+\mu)^2} \tag{2.22}$$

which is in agreement with Mohammed Abu-Salih *et al.* (1990).

In the following sections, we obtain a CAN estimator, a 100(1- α) % asymptotic confidence interval for steady state availability of the system and the MLE of the system reliability.

2.4 CONFIDENCE INTERVAL FOR STEADY-STATE AVAILABILITY OF THE SYSTEM

Let $X_1, X_2,\ldots,X_n$ be a random sample of failure free-times of a unit with probability density function (p.d.f) given by

$$f(x) = \lambda e^{-\lambda x}; 0 < x < \infty, \lambda > 0 \tag{2.23}$$

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of the repair times with the p.d.f given by $g(y) = \mu e^{-\mu y}$.It is clear that $E(\overline{X}) = \frac{1}{\lambda}$ and $E\left(\frac{\overline{Y}}{2}\right) = \frac{1}{\mu}$, where $\overline{X}$ and $\overline{Y}$ are respectively the sample means of the failure-free times and the repair times of s unit. It can be shown that $\overline{X}$ and $\frac{\overline{Y}}{2}$ are respectively the maximum likehood estimators (MLE's) of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$.

Let $\theta_1 = \frac{1}{\lambda}$ and $\theta_2 = \frac{1}{\mu}$. Clearly, the steady state availability, given by (2.16), reduces to

$$A_\infty = \frac{\theta_1(\theta_1 + 2\theta_2)}{(\theta_1 + \theta_2)^2}$$

Hence, the MLE of A_∞ is given by

$$A_\infty = \frac{4\overline{X}(\overline{X} + \overline{Y})}{(2\overline{X} + \overline{Y})^2} \quad (2.24)$$

It should be noted that A_∞ is real valued differential function in X and Y. Now consider the following application of the multiplicative central limit theorem (Rao, 1974).

Suppose that $T_1^{'}, T_2^{'}, T_3^{'}, \ldots$ are independent and identically distributed k-dimensional random variables such that:

$$T_n' = (T_{1n}, T_{2n}, \ldots, T_{kn})$$

has first and second order moments

$$E(T_n) = \mu \text{ and } D(T_n) = \Sigma .$$

Define the sequence of random variables $\overline{T}_n = (\overline{T}_{1n}, \overline{T}_{2n}, \ldots, \overline{T}_{kn})$, $n = 1, 2$ where:

$$\overline{T}_{in} = \frac{1}{n}\sum_{j=1}^{n} T_{ij}, \qquad i = 1, 2, \ldots, k \quad and \quad j = 1, 2, \ldots, n$$

Then, $\sqrt{n}\left(\overline{T}_n - \mu\right) \xrightarrow{d} N(0, \Sigma)$ as n $\rightarrow$ ∞. Hence, the applying the Multivariate Central Limit theorem, it follows that:

$$\sqrt{n}\left[\left(\overline{X}, \ \frac{\overline{Y}}{2}\right) - \left(\theta_1, \ \theta_2\right)\right] \xrightarrow{d} N(0, \Sigma) \quad as \ \ n \rightarrow \infty$$

where the dispersion matrix $\sum = \left(\left(\sigma_{ij}\right)\right)_{2x2}$ is given by $\sum = diag\left(\theta_1^2, \frac{\theta_2^2}{2}\right)_{2x2}$

Again from Rao (1974), we have:

$$\sqrt{n}\left(\hat{A}_\infty - A_\infty\right) \xrightarrow{d} N\left(0, \sigma^2(\theta)\right) \quad as \quad n \rightarrow \infty,$$

where $\theta = \left(\theta_1, \ \theta_2\right)$ and

$$\sigma^2(\theta) = \sum_{i=1}^{2}\left(\frac{\partial A_\infty}{\partial \theta_i}\right)^2 \sigma_{ii} = \frac{6\theta_1^2\theta_2^4}{\left(\theta_1 + \theta_2\right)^6}$$

Consequently $\hat{A}_\infty$ is a CAN estimator of A_∞:

Let $\sigma^2\left(\hat{\theta}\right)$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator $\hat{\theta}$ namely:

$$\hat{\theta} = \left(\overline{X},\ \frac{\overline{Y}}{2} \right).$$

Moreover, let $\hat{\theta}^2 = \sigma^2\left(\hat{\theta}\right)$. Since $\sigma^2(\theta)$ is a continuous function of θ , $\hat{\sigma}_2^2$ is a consistent estimator of $\sigma^2(\theta)$, i.e. $\hat{\sigma}_2^2 \xrightarrow{p} \sigma^2(\theta)$ as n→∞.

By Slutskey's theorem

$$\frac{\sqrt{n}\left(\hat{A}_\infty - A_\infty\right)}{\hat{\sigma}} \xrightarrow{d} N(0,\ 1)$$

That is,

$$\mathrm{P}\left[-k_{\frac{\alpha}{2}} < \frac{\sqrt{n\left(\hat{A}_\infty - A_\infty\right)}}{\hat{\sigma}} < k_{\frac{\alpha}{2}} \right] = (1-\alpha),$$

where $k_{\frac{\alpha}{2}}$ is obtainable from normal tables. Hence, a 100 (1 – α) % asymptotic confidence interval for A_∞ is given by $\hat{A}_\infty \pm k_{\frac{\alpha}{2}} . \frac{\sigma}{\sqrt{n}}$

2.5 MLE OF SYSTEM RELIABILITY

Since $\overline{X}$ and $\frac{\overline{Y}}{2}$ are the MLE's of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively, we obtain by applying a method given in Zacks (1972), the MLE of system reliability as

$$\hat{R}(t) = \sum_{i=1}^{3} \frac{\left[\left(\hat{\alpha}_i \overline{Y} + 4\overline{X} + 3\overline{Y}\right) + 8\overline{Y}\right]}{\overline{X}\,\overline{Y}^2 \prod_{j=1}^{3} \left(\hat{\alpha}_{i-} \hat{\alpha} j\right)} e^{\alpha \hat{i}} t,$$

where $\hat{\alpha}_{1,} \hat{\alpha}_2 and \hat{\alpha}_3$ are the roots of the cubic equation

$$\left(\overline{X}\,\overline{Y}\right)^2 s^3 + \overline{X}\,\overline{Y}\left(8\overline{X} + 3\overline{Y}\right)s^2 + \left(16\overline{X}^2 + 20\overline{X}\,\overline{Y} + 2\overline{Y}^2\right)s + 8\overline{Y} = 0.$$

2.6 NUMERICAL ILLUSTRATION

For $A_\infty = \dfrac{\mu(2\lambda + \mu)}{(\lambda + \mu)^2}$

When λ = 0.01, 0.015, 0.02, 0.025 and μ = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35.

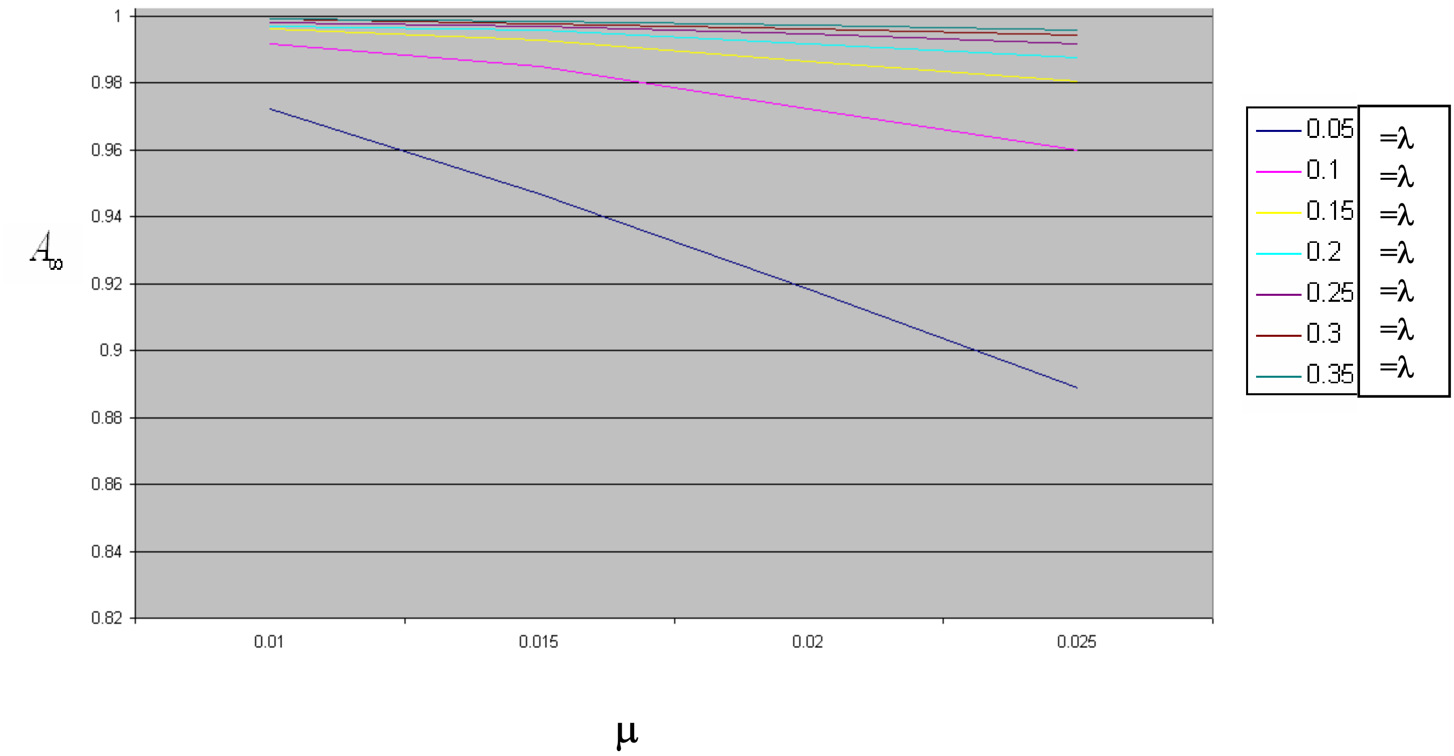

Figure 2.1

The λ and μ values are chosen from an exponential data available (Yadavalli et al, 2005)

From Figure 2.1, it is observed that as repair time increases, the steady state availability decreases.

Table 2.1 : CONFIDENCE INTERVALS FOR THE MODEL

For

λ=0.10

n	μ	95% CI	99% CI
100	0.01	(0.8101,0.9991)	(0.7811,0.9999)
	0.015	(0.7006,0.8192)	(0.6933,0.8399)
	0.02	(0.5218,0.6368)	(0.6066,0.6566)
	0.025	(0.5006,0.6019)	(0.4888,0.5771)
200	0.01	(0.8332,0.9673)	(0.8206,0,9709)
	0.015	(0.7161,0.8006)	(0.7988,0.8113)
	0.02	(0.6314,0.7091)	(0.6111,0.7108)
	0.025	(0.5822,0.6641)	(0.5316,0.5669)
2000	0.01	(0.8608,0.8992)	(0.8541,0.9053)
	0.015	0.6879,0.7227)	(0.6790,0.7306)
	0.02	0.6041,0.6330)	(0.5991,0.6376)
	0.025	(0.5911,0.6130)	(0.5444,0.5619)

Table 2.1 presents the 95% and 99% confidence intervals for different simulated samples. It can be observed that, as n increases, the steady state availability decreases.

CONCLUSION:

A two-unit system with Erlangian repair time is studied in this chapter. The system of simultaneously differential equations is developed to obtain the availabilities analytically. The asymptotic confidence limits for steady state availability are studied at the end of this chapter. A numerical example illustrated the results. The results show that, as n increases A_∞ decreases.