Appendix A

Examples concerning ergodicity

A.1 On the definition of ergodicity

This section is devoted to the construction of a $*$-dynamical system $(\mathfrak{A}, \varphi, \tau)$ with the property that if $\|\tau(A) - A\|_\varphi = 0$, then $\|A - \alpha\|_\varphi = 0$ for some $\alpha \in \mathbb{C}$, but for which the fixed points of the operator U defined in Proposition 2.3.3 in terms of some cyclic representation, form a vector subspace of $\mathfrak{H}$ with dimension greater than one. This will prove the necessity of a sequence, rather than a single element, in Definition 2.3.2, in order for Proposition 2.3.3 to hold.

First some general considerations. Consider a dense vector subspace $\mathfrak{G}$ of a Hilbert space $\mathfrak{H}$, and let $\mathfrak{L}(\mathfrak{H})$ be the bounded linear operators $\mathfrak{H} \to \mathfrak{H}$. Set

$$\mathfrak{A} := \{A|_{\mathfrak{G}} : A \in \mathfrak{L}(\mathfrak{H}),\, A\mathfrak{G} \subset \mathfrak{G} \text{ and } A^*\mathfrak{G} \subset \mathfrak{G}\}$$

where $A|_{\mathfrak{G}}$ denotes the restriction of A to $\mathfrak{G}$, then $\mathfrak{A}$ is clearly a vector subspace of $\mathfrak{L}(\mathfrak{G})$. For any $A \in \mathfrak{A}$, denote by $\overline{A}$ the (unique) bounded linear extension of A to $\mathfrak{H}$. Now define an involution on $\mathfrak{A}$ by

$$A^* := \overline{A}^*|_{\mathfrak{G}}$$

for all $A \in \mathfrak{A}$, then it is easily verified that $\mathfrak{A}$ becomes a unital $*$-algebra. (For $A, B \in \mathfrak{A}$ it is clear that AB is a bounded linear operator $\mathfrak{G} \to \mathfrak{G}$ which therefore has the extension $\overline{A}.\overline{B} \in \mathfrak{L}(\mathfrak{H})$ for which $\overline{A}.\overline{B}\mathfrak{G} \subset \mathfrak{G}$ and $\left(\overline{A}.\overline{B}\right)^* \mathfrak{G} = \overline{B}^*\overline{A}^*\mathfrak{G} \subset \mathfrak{G}$ by the definition of $\mathfrak{A}$. Hence $AB \in \mathfrak{A}$, which means that $\mathfrak{A}$ is a subalgebra of $\mathfrak{L}(\mathfrak{G})$. Also, $(AB)^* = \left(\overline{A}.\overline{B}\right)^*|_{\mathfrak{G}} = \left(\overline{B}^*\overline{A}^*\right)|_{\mathfrak{G}} = \overline{B}^*\left(\overline{A}^*|_{\mathfrak{G}}\right) = \overline{B}^*A^* = B^*A^*$. Similarly for the other defining properties of an involution.) Note that for $A \in \mathfrak{A}$ and $x, y \in \mathfrak{G}$ we have

$$\langle x, Ay\rangle = \left\langle x, \overline{A}y\right\rangle = \left\langle \overline{A}^*x, y\right\rangle = \langle A^*x, y\rangle .$$

For a given norm one vector $\Omega \in \mathfrak{G}$ we define a state φ on $\mathfrak{A}$ by

$$\varphi(A) = \langle \Omega, A\Omega \rangle .$$

Next we construct a cyclic representation of $(\mathfrak{A}, \varphi)$. Let

$$\pi : \mathfrak{A} \to L(\mathfrak{G}) : A \mapsto A$$

then clearly π is linear with $\pi(1) = 1$ and $\pi(AB) = \pi(A)\pi(B)$. Note that for any $x, y \in \mathfrak{G}$ we have $(x \otimes y)^* = y \otimes x$, hence $(x \otimes y)\mathfrak{G} \subset \mathfrak{G}$ and $(x \otimes y)^*\mathfrak{G} \subset \mathfrak{G}$, so $(x \otimes y)|_{\mathfrak{G}} \in \mathfrak{A}$. Now, $\pi\left((x \otimes \Omega)|_{\mathfrak{G}}\right)\Omega = x \langle \Omega, \Omega \rangle = x$, hence $\pi(\mathfrak{A})\Omega = \mathfrak{G}$. Furthermore, $\langle \pi(A)\Omega, \pi(B)\Omega \rangle = \langle A\Omega, B\Omega \rangle = \langle \Omega, A^*B\Omega \rangle = \varphi(A^*B)$. Thus $(\mathfrak{G}, \pi, \Omega)$ is a cyclic representation of $(\mathfrak{A}, \varphi)$.

Suppose we have a unitary operator $U : \mathfrak{H} \to \mathfrak{H}$ such that $U\mathfrak{G} = \mathfrak{G}$ and $U\Omega = \Omega$. Then $U^*\mathfrak{G} = U^{-1}\mathfrak{G} = \mathfrak{G}$, so $V := U|_{\mathfrak{G}} \in \mathfrak{A}$, and $V^* = U^*|_{\mathfrak{G}}$. It follows that $VAV^* \in \mathfrak{A}$ for all $A \in \mathfrak{A}$, hence we can define a linear function $\tau : \mathfrak{A} \to \mathfrak{A}$ by

$$\tau(A) = VAV^*.$$

Clearly $V^*V = 1 = VV^*$, so $\tau(1) = 1$ and $\varphi(\tau(A)^*\tau(A)) = \varphi(VA^*AV^*) = \langle U^*\Omega, A^*AU^*\Omega \rangle = \varphi(A^*A)$, since $U^*\Omega = U^{-1}\Omega = \Omega$. Therefore $(\mathfrak{A}, \varphi, \tau)$ is a $*$-dynamical system. Note that $U|_{\mathfrak{G}}$ satisfies equation (3.1) of Section 2.3, namely $U\pi(A)\Omega = UA\Omega = UAU^*\Omega = \tau(A)\Omega = \pi(\tau(A))\Omega$, hence U is the operator which appears in Proposition 2.3.3.

Assume $\{x \in \mathfrak{G} : Ux = x\} = \mathbb{C}\Omega$. If $\|\tau(A) - A\|_\varphi = 0$, it then follows for $x = \iota(A)$, with ι given by equation (2.1) of Section 2.2, that $\|Ux - x\| = \|\iota(\tau(A) - A)\| = \|\tau(A) - A\|_\varphi = 0$, so $x = \alpha\Omega$ for some $\alpha \in \mathbb{C}$. Therefore $\|A - \alpha\|_\varphi = \|\iota(A - \alpha)\| = \|x - \alpha\Omega\| = 0$.

In other words, assuming that the fixed points of U in $\mathfrak{G}$ form the one-dimensional subspace $\mathbb{C}\Omega$, it follows that $\|\tau(A) - A\|_\varphi = 0$ implies that $\|A - \alpha\|_\varphi = 0$ for some $\alpha \in \mathbb{C}$.

It remains to construct an example of a U with all the properties mentioned above, whose fixed point space in $\mathfrak{H}$ has dimension greater than one. The following example was constructed by L. Zsidó:

Let $\mathfrak{H}$ be a separable Hilbert space with an orthonormal basis of the form

$$\{\Omega\, , \, y\} \cup \{u_k \, : \, k \in \mathbb{Z}\}$$

(that is to say, this is a total orthonormal set in $\mathfrak{H}$) and define the linear operator $U : \mathfrak{H} \longrightarrow \mathfrak{H}$ via bounded linear extension by

$$\begin{aligned} U\Omega &= \Omega\, , \\ Uy &= y\, , \\ Uu_k &= u_{k+1}\, , \quad k \in \mathbb{Z}. \end{aligned}$$

Clearly U is isometric, while $U\mathfrak{H}$ is dense in $\mathfrak{H}$, hence U is surjective, since $\mathfrak{H}$ is complete. Since U is a surjective isometry, it is unitary. Let $\mathfrak{G}$ be the linear span of

$$\{\Omega\} \cup \{y + u_k : k \in \mathbb{Z}\}.$$

Then $U\mathfrak{G} = \mathfrak{G}$. Furthermore, $\mathfrak{G}$ is dense in $\mathfrak{H}$. Indeed,

$$\|y - \frac{1}{n}\sum_{k=1}^{n}(y + u_k)\| = \frac{1}{n}\|\sum_{k=1}^{n} u_k\| = \frac{1}{\sqrt{n}} \longrightarrow 0$$

implies that $y \in \overline{\mathfrak{G}}$, the closure of $\mathfrak{G}$, hence also

$$u_k = (y + u_k) - y \in \overline{\mathfrak{G}}$$

for $k \in \mathbb{Z}$.

Next we show that

$$\{x \in \mathfrak{G} : Ux = x\} = \mathbb{C}\Omega. \tag{1.1}$$

If $\alpha\Omega + \sum_{k=-n}^{n} \beta_k(y + u_k) \in \mathfrak{G}$ is left fixed by U, then

$$\alpha\Omega + \sum_{k=-n}^{n} \beta_k y + \sum_{k=-n}^{n} \beta_k u_{k+1} = \alpha\Omega + \sum_{k=-n}^{n} \beta_k y + \sum_{k=-n}^{n} \beta_k u_k$$

and it follows that $\beta_{-n} = 0$, and that $\beta_{k+1} = \beta_k$ for $k = -n, ..., n-1$. Thus

$$\alpha\Omega + \sum_{k=-n}^{n} \beta_k(y + u_k) = \alpha\Omega$$

proving (1.1).

On the other hand,

$$\{x \in \mathfrak{H} : Ux = x\}$$

clearly contains the two-dimensional vector space spanned by Ω and y.

A.2 An example of an ergodic system

Here we give the proof that Example 2.5.7 is indeed ergodic. It is clear that τ is linear and that $\tau(1) = 1$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be complex matrices. Then

$$\tau(A)^* = \begin{pmatrix} \overline{a_{22}} & \overline{c_2 a_{21}} \\ \overline{c_1 a_{12}} & \overline{a_{11}} \end{pmatrix}$$

and

$$\tau(A)^*\tau(A) = \begin{pmatrix} |a_{22}|^2 + |c_2 a_{21}|^2 & \overline{a_{22}} c_1 a_{12} + \overline{c_2 a_{21}} a_{11} \\ \overline{c_1 a_{12}} a_{22} + \overline{a_{11}} c_2 a_{21} & |c_1 a_{12}|^2 + |a_{11}|^2 \end{pmatrix}$$

while

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}$$

and

$$A^*A = \begin{pmatrix} |a_{11}|^2 + |a_{21}|^2 & \overline{a_{11}} a_{12} + \overline{a_{21}} a_{22} \\ \overline{a_{12}} a_{11} + \overline{a_{22}} a_{21} & |a_{12}|^2 + |a_{22}|^2 \end{pmatrix}$$

so

$$\begin{aligned} \varphi(\tau(A)^*\tau(A)) &= \frac{1}{2}\left(|a_{22}|^2 + |c_2 a_{21}|^2 + |c_1 a_{12}|^2 + |a_{11}|^2\right) \\ &\leq \frac{1}{2}\left(|a_{22}|^2 + |a_{21}|^2 + |a_{12}|^2 + |a_{11}|^2\right) \\ &= \varphi(A^*A) \end{aligned}$$

for all A, meaning that $(\mathfrak{A}, \varphi, \tau)$ is a $*$-dynamical system, if and only if $|c_1| \leq 1$ and $|c_2| \leq 1$, which is what we will assume.

Next we prove that it is ergodic. For even $k \geq 0$ we have

$$\tau^k(B) = \begin{pmatrix} b_{11} & c_1^k b_{12} \\ c_2^k b_{21} & b_{22} \end{pmatrix}$$

and therefore

$$A\tau^k(B) = \begin{pmatrix} a_{11}b_{11} + a_{12}c_2^k b_{21} & a_{11}c_1^k b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}c_2^k b_{21} & a_{21}c_1^k b_{12} + a_{22}b_{22} \end{pmatrix}$$

which means

$$\varphi\left(A\tau^k(B)\right) = \frac{1}{2}\left(a_{11}b_{11} + a_{12}c_2^k b_{21} + a_{21}c_1^k b_{12} + a_{22}b_{22}\right).$$

For odd $k > 0$ we then get

$$\varphi\left(A\tau^k(B)\right) = \frac{1}{2}\left(a_{11}b_{22} + a_{12}c_2^k b_{21} + a_{21}c_1^k b_{12} + a_{22}b_{11}\right)$$

by switching b_{11} and b_{22}. For $c \in \mathbb{C}$ it is clear that $U : \mathbb{C} \to \mathbb{C} : x \mapsto cx$ is a linear operator with $\|U\| \leq 1$ if and only if $|c| \leq 1$, and for $c \neq 1$ the only fixed point of U is 0, in which case

$$\frac{1}{n}\sum_{k=0}^{n-1} c^k x = \frac{1}{n}\sum_{k=0}^{n-1} U^k x \longrightarrow 0$$

for all $x \in \mathbb{C}$ as $n \to \infty$, by the Mean Ergodic Theorem 2.4.1. Hence, for $c_1 \neq 1$ and $c_2 \neq 1$ it follows that

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi\left(A\tau^k(B)\right) \\ = & \lim_{n\to\infty} \frac{1}{n}\left\{ \begin{array}{c} \frac{n}{2}\left[\frac{1}{2}(a_{11}b_{11} + a_{22}b_{22}) + \frac{1}{2}(a_{11}b_{22} + a_{22}b_{11})\right] \text{ for } n \text{ even} \\ \frac{n-1}{2}\left[\frac{1}{2}(a_{11}b_{11} + a_{22}b_{22}) + \frac{1}{2}(a_{11}b_{22} + a_{22}b_{11})\right] + \frac{1}{2}(a_{11}b_{11} + a_{22}b_{22}) \text{ for } n \text{ odd} \end{array} \right\} \\ = & \left(\frac{a_{11} + a_{22}}{2}\right)\left(\frac{b_{11} + b_{22}}{2}\right) \\ = & \varphi(A)\varphi(B) \end{aligned}$$

which means that $(\mathfrak{A}, \varphi, \tau)$ is ergodic, by Proposition 2.5.6(ii).

On the other hand, if $c_1 = 1$ and $c_2 \neq 1$, then we have by a similar calculation that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi\left(A\tau^k(B)\right) = \varphi(A)\varphi(B) + \frac{a_{21}b_{12}}{2}.$$

Likewise for the other cases where either c_1 or c_2 or both are equal to 1. So $(\mathfrak{A}, \varphi, \tau)$ is ergodic if and only if $c_1 \neq 1$ and $c_2 \neq 1$.

A.2.1 Remark. It is easily seen that τ is not a homomorphism, namely

$$\tau(AB) = \begin{pmatrix} a_{21}b_{12} + a_{22}b_{22} & c_1(a_{11}b_{12} + a_{12}b_{22}) \\ c_2(a_{21}b_{11} + a_{22}b_{21}) & a_{11}b_{11} + a_{12}b_{21} \end{pmatrix}$$

while

$$\tau(A)\tau(B) = \begin{pmatrix} a_{22}b_{22} + c_1c_2a_{12}b_{21} & c_1(a_{22}b_{12} + a_{12}b_{11}) \\ c_2(a_{21}b_{22} + a_{11}b_{21}) & c_1c_2a_{21}b_{12} + a_{11}b_{11} \end{pmatrix}.$$

In fact, unless $c_1c_2 = 1$, it follows that we don't even have $\tau(A^2) = \tau(A)^2$ for all A. Nor, for that matter, do we have $\tau(A^*) = \tau(A)^*$ for all A, unless $c_2 = \overline{c_1}$. This is opposed to the situation for a measure theoretic dynamical system as defined in Section 2.1, where τ in equation (1.1) of that section is always a $*$-homomorphism. It therefore makes sense not to assume that τ is a $*$-homomorphism in Definition 2.3.1, since we now have an example where it isn't.■

A.2.2 Remark. We note that $\varphi(\tau(A)) = \varphi(A)$, i.e. φ is τ-invariant, but this fact in itself does not imply that $\varphi(\tau(A)^*\tau(A)) \leq \varphi(A^*A)$, since τ is not a $*$-homomorphism, by Remark A.2.1.

Furthermore, $\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathfrak{A}$, so φ is commutative (so to speak) even though $\mathfrak{A}$ is not. Also, while $\tau(AB) \neq \tau(BA)$ for some $A, B \in \mathfrak{A}$, we still have $\varphi(\tau(AB)) = \varphi(AB) = \varphi(BA) = \varphi(\tau(BA))$, so τ is noncommutative (so to speak), but with respect to φ it is again commutative. We conclude that while $\mathfrak{A}$ is noncommutative, $(\mathfrak{A}, \varphi, \tau)$ is still in many respects commutative simply because $\varphi(AB) = \varphi(BA)$ for all A and B.■

A.2.3 Question. Is there an example of an ergodic $*$-dynamical system $(\mathfrak{A}, \varphi, \tau)$ in which $\varphi(AB) \neq \varphi(BA)$ for some $A, B \in \mathfrak{A}$?■

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The remark at the end of each reference indicates where in this thesis (apart from the Introduction) the reference appears.