CHAPTER 4

THE ITERATIVE BOOTSTRAP METHOD[1]

The determination of a smooth zero-coupon yield curve in a market where only coupon bonds are traded can be a difficult and time-consuming process. When only a few data points are available, it is especially difficult to obtain a smooth forward curve.

The standard bootstrap technique was evaluated empirically, using South African yield curve data, which motivated the formulation of a more efficient technique. In this chapter, the formulation of the iterative bootstrap method is discussed, and the convergence of the iterative sequence is proved. Empirical results illustrate the use of the method.

[1]The results of the research discussed in this chapter were published in *RISK* (Smit & Van Niekerk, 1997).

4.1 Disadvantages using the standard bootstrap method

The problem with the standard bootstrap procedure is that it is assumed that sufficient data are available to determine the present value of all coupons. As such data are not available in the South African market, interpolation techniques (such as the Newton Raphson technique) must be used to find intermediate data points (for bootstrap purposes) before fitting the final curve. There can be any number of intermediate data points, even twenty or more, depending on the number of coupons between two data points. If the data points do not form a smooth curve, it is possible that the curve from which coupons are discounted will differ from the final fitted curve, causing a discrepancy. Another disadvantage is that the interpolation of data points in the standard bootstrap technique is time-consuming.

Once the zero-coupon rates have been determined, the question arises as to which approximation technique to use. Polynomial approximation and spline fitting are the most commonly used techniques, but they are not always suitable for the South African yield curve, due to structural inefficiencies in the fixed income market and the resultant dispersion of data points.

A solution to these problems was developed in this study. This solution involves constructing a zero-coupon curve using an iterative bootstrap method (IBS-method), where the entire curve is *simultaneously* bootstrapped, starting with a first guess. Each iteration results in a sequence of implied zero-coupon rates which are then fitted using least squares approximation, and used again in the next iteration. This approach is described in the sections below.

4.2 Iterative bootstrapping – introduction

A standard bootstrap procedure follows a process where the coupons of each individual bond in the data set is bootstrapped to obtain a fixed zero-coupon rate for a specific term. This rate is again used in the bootstrap process for the next bond. The method therefore progresses along the time-axis to find the discrete zero-coupon rates, which are then approximated by a curve. Interpolation methods are used to discount coupons at intermediate maturity dates.

In order to overcome problems with the standard bootstrap method, a method is suggested that follows an iteration process. The entire data set is bootstrapped simultaneously, using implied zero-coupon rates obtained in the previous iteration, by starting with a first guess for the zero-coupon yield curve. For each iteration, this again results in a set of implied zero-coupon rates (one data point for each coupon bond). A least squares approximation technique is used to obtain a smooth curve which is employed to discount cashflows for the next iteration. These iterations converge and ultimately yield a unique zero-coupon curve for the particular approximation technique[2]. The iterative bootstrap method is a dynamic method compared to the more static standard bootstrap method.

The advantage of bootstrapping the bonds simultaneously in the iteration process is that, for each iteration, different cashflows are discounted from the same smooth curve to find the implied zero-coupon rates for the next iteration. The final fitted zero-coupon curve is therefore obtained by bootstrapping from the same curve. Therefore, there is no discrepancy between the curve that has been used for bootstrapping, and the final fitted zero-coupon yield curve. The replacement of the interpolation of data points with a method where a fitted curve

[2]Different approximation techniques result in slight differences in the resultant term structure.

determines the points speeds up the whole process. The use of numerical methods, such as the Newton Raphson method, also becomes unnecessary.

4.3 The iterative bootstrap method

The following assumptions are made:

- It is possible for the yield curve $z(t)$ to have any shape (positive, negative).
- All interest rates are positive.
- A bond pays a nominal value of 1 unit at the end of its term.
- The term-to-maturity, t, is given in years.
- Continuously compounded interest rates are used.
- Market participants take advantage of arbitrage opportunities as they occur.

If all fixed income securities meet the no-arbitrage principle, the price P_k of an arbitrary coupon-bearing bond, k, should equal the sum of the n cashflows, discounted at the particular zero-coupon rate, $z(t)$:

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)} z(t_i^{(k)})} + (1 + \gamma_k) e^{-t_n^{(k)} z(t_n^{(k)})} \tag{1}$$

where γ_k is the coupon payment. The price of the bond can also be determined using the market yield-to-maturity, η_k, for the bond k (as traded in the market), therefore

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)} \eta_k} + (1 + \gamma_k) e^{-t_n^{(k)} \eta_k} \tag{2}$$

Since the final zero-coupon yield curve, $z(t)$ is not yet known and the IBS-method does not use the interpolation of data points, a curve from which to bootstrap is needed. If it is assumed that $y_j(t)$ is the j-th approximate fit for the zero-coupon yield curve (for the j-th iteration), starting at $y_1(t)$ as a first guess, equation (1) can be reformulated as follows:

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)} y_j(t_i^{(k)})} + (1 + \gamma_k) e^{-t_n^{(k)} y_j^*(t_n^{(k)})} \tag{3}$$

where P_k is known from equation (2) (the market price) and $y_j^*(t_n^{(k)})$ is the implied zero-coupon yield for the term $t_n^{(k)}$ years and the only unknown parameter. Equation (3) holds at any time throughout the iteration process. If the curve $y_j(t)$ is different from the correct zero-coupon curve, the point $y_j^*(t_n^{(k)})$ deviates from this curve, in order to give the correct price, P_k.

Coupons are bootstrapped simultaneously using $y_j(t)$ for each iteration j and all bonds k, $k=1,\dots,m$, where m is the number of bonds. From equation (3), it is possible to solve $y_j^*(t_n^{(k)})$ for all bonds in the data set, to get an implied array of zero-coupon yields for each term $t_n^{(k)}$ in each iteration:

$$y_j^*(t_n^{(k)}) = -\frac{1}{t_n^{(k)}} \ln\left[\frac{P_k - \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)} y_j(t_i^{(k)})}}{1 + \gamma_k}\right] \tag{4}$$

where it is assumed that $t_n^{(1)} \neq t_n^{(2)}$.

To serve as input for the next iteration, the zero-coupon rates $y_j^*(t_n^{(k)})$ are approximated by a fit $y_{j+1}(t_n^{(k)})$. By repeating the process for $j = 1,2,\dots$ a sequence of implied zero-coupon rates is found for each term, $t_n^{(k)}$, in the data set. The theorem discussed below states that these implied zero-coupon curves converge to the zero-coupon curve implied by the coupon bond market.

4.4 Theorem

Given an arbitrary guess for the function, $y_1(t)$, the approximate linear interpolated fit $y_j(t)$ will converge to the zero-coupon yield curve, $z(t)$, on condition that

$$0 \leq \gamma < (e^{\tau z(\tau)} - 1)^{-1}$$

for any bond maturing at time $\tau < t_m$, where t_m is the maximum range of the term structure, and $y_j(t)$ interpolates the implied zero-coupon rates $y_{j-1}^*(t_i)$, $j > 1$, $0 < t_i < t_m$.

4.4.1 *Proof*

If P is the price of a bond maturing at time t_3, paying coupons, γ, at time t_1, t_2 and t_3, and at time t_1 the zero rate $z(t_1)$ is known, it is possible to prove that the theorem holds for this bond. If the theorem holds for time t_3, it is possible to demonstrate that it will hold for any time $t_i \leq t_m$, where t_m is the maximum term of the term structure.

Using the first guess $y_1(t)$ and assuming that $y_1(t_2) > z(t_2)$, it follows from equations (1) and (3) that

$$z(t_3) > y_1^*(t_3) \tag{5}$$

If one assumes that $y_1(t_2) < z(t_2)$, it implies that

$$z(t_3) < y_1^*(t_3) \tag{6}$$

According to equations (1) and (3)

$$\gamma e^{-t_2 y_j(t_2)} + (\gamma + 1)e^{-t_3 y_j^*(t_3)} = \gamma e^{-t_2 z(t_2)} + (\gamma + 1)e^{-t_3 z(t_3)} \tag{7}$$

Therefore, for $j = 1$,

$$\frac{e^{-t_2 z(t_2)} - e^{-t_2 y_1(t_2)}}{e^{-t_3 y_1^*(t_3)} - e^{-t_3 z(t_3)}} = \frac{1+\gamma}{\gamma}$$

or

$$\frac{1 - e^{-t_2[z(t_2) - y_1(t_2)]}}{1 - e^{-t_3[y_1^*(t_3) - z(t_3)]}} = \frac{1+\gamma}{\gamma}\left[e^{-t_3 z(t_3) + t_2 y_1(t_2)}\right] \tag{8}$$

Next, if one assumes that convergence does not occur, and

$$z(t_2) - y_1(t_2) \leq y_1^*(t_3) - z(t_3) \tag{9}$$

then, for $t_2 < t_3$,

$$[z(t_2) - y_1(t_2)]\, t_2 < [y_1^*(t_3) - z(t_3)]\, t_3$$

Since e^{-x} is a decreasing function, it follows that

$$e^{-t_3[y_1^*(t_3) - z(t_3)]} < e^{-t_2[z(t_2) - y_1(t_2)]}$$

Therefore,

$$\frac{1 - e^{-t_2[z(t_2) - y_1(t_2)]}}{1 - e^{-t_3[y_1^*(t_3) - z(t_3)]}} < 1 \tag{10}$$

If equation (10) holds, it follows from equation (8) that

$$\frac{1+\gamma}{\gamma}\, e^{-t_3 z(t_3) + t_2 y_1(t_2)} < 1$$

Therefore,

$$-t_3 z(t_3) + t_2 y_1(t_2) < \ln\left(\frac{\gamma}{1+\gamma}\right)$$

$$\Rightarrow \quad y_1(t_2) < \frac{1}{t_2}\left[t_3 z(t_3) + \ln\left(\frac{\gamma}{1+\gamma}\right)\right]$$

Since $y_1(t) > 0, \quad \forall\, t > 0$, it follows that

$$z(t_3) > \frac{1}{t_3} \ln\left(\frac{1+\gamma}{\gamma}\right)$$

$$\therefore \gamma > (e^{t_3 z(t_3)} - 1)^{-1}$$

However, this violates the assumption that

$$\gamma < (e^{t z(t)} - 1)^{-1}$$

This implies that equation (9) does not hold, therefore

$$y_1^*(t_3) - z(t_3) < z(t_2) - y_1(t_2) \tag{11}$$

On the other hand, since $y_1(t_2) < z(t_2)$, and $y_1^*(t_3) > z(t_3)$, it is possible to say

$$-(z(t_2) - y_1(t_2)) < y_1^*(t_3) - z(t_3) \tag{12}$$

From equations (11) and (12), it follows that

$$|y_1^*(t_3) - z(t_3)| < z(t_2) - y_1(t_2) \tag{13}$$

Since the function $y_{j+1}(t)$ interpolates $y_j^*(t_i)$, $\forall j$, one can substitute $y_1^*(t_i)$ with $y_2(t_i)$, i = 1,2,3.... The iteration process therefore results in the following:

$$\begin{aligned} |z(t_2) - y_1(t_2)| &> |y_1^*(t_3) - z(t_3)| \\ &= |y_2(t_3) - z(t_3)| \end{aligned} \tag{14}$$

Although many sophisticated interpolation techniques can be used to interpolate the implied zero-coupon yields $y_j^*(t)$, it is possible to assume for the purposes of the proof that one interpolates linearly between the points $(t_1, z(t_1))$ and $(t_3, y_1(t_3))$. Then

$$y_1(t_2) = \frac{t_2 - t_1}{t_3 - t_1}\left(y_1(t_3) - z(t_1)\right) + z(t_1) \tag{15}$$

and

$$z(t_2) = \frac{t_2-t_1}{t_3-t_1}\left(z(t_3) - z(t_1)\right) + z(t_1) \tag{16}$$

Therefore,

$$|z(t_2) - y_1(t_2)| = \frac{(t_2-t_1)}{(t_3-t_1)}|z(t_3) - y_1(t_3)|$$

Since $t_2 < t_3$, it follows from equation (14) that

$$|y_2(t_3) - z(t_3)| < |z(t_3) - y_1(t_3)| \tag{17}$$

Finally, it is important to show that

$$|y_3(t_3) - z(t_3)| < |y_2(t_3) - z(t_3)|$$

Since $y_1^*(t_2) = y_2(t_2) > z(t_2)$, it follows as in equation (5) that $y_2^*(t_3) < z(t_3)$. If one supposes that

$$y_2(t_2) - z(t_2) \leq z(t_3) - y_2^*(t_3)$$

then, as in equation (10), it follows that

$$\frac{1 - e^{-t_2[y_2(t_2) - z(t_2)]}}{1 - e^{-t_3[z(t_3) - y_2^*(t_3)]}} < 1 \tag{18}$$

Using equation (7) one can write:

$$\frac{1 - e^{-t_2[y_2(t_2)-z(t_2)]}}{1 - e^{-t_3[z(t_3)-y_2^*(t_3)]}} = \frac{1+\gamma}{\gamma}\left[e^{-t_3 y_2^*(t_3) + t_2 z(t_2)}\right]$$

Therefore,

$$\frac{1+\gamma}{\gamma}\left[e^{-t_3 y_2^*(t_3) + t_2 z(t_2)}\right] < 1$$

$$=> \quad y_2^*(t_3) > \frac{1}{t_3}\left[t_2 z(t_2) - \ln\left(\frac{\gamma}{\gamma+1}\right)\right]$$

Since $z(t_3) > y_2^*(t_3)$, it follows that

$$z(t_3) > \frac{1}{t_3}\left[t_2 z(t_2) - \ln\left(\frac{\gamma}{\gamma+1}\right)\right]$$

$$\therefore \quad \frac{\gamma+1}{\gamma} < \frac{e^{t_3 z(t_3)}}{e^{t_2 z(t_2)}} \leq e^{t_3 z(t_3)}$$

Therefore,

$$\gamma > (e^{t_3 z(t_3)} - 1)^{-1} \tag{19}$$

Equation (19) again violates the assumption. Hence,

$$z(t_3) - y_2^*(t_3) < y_2(t_2) - z(t_2)$$

Since $y_1(t_2) > z(t_2)$, $y_2^*(t_3) < z(t_3)$ and $y_2^*(t_3) = y_3(t_3)$, it follows that

$$|y_3(t_3) - z(t_3)| < |y_2(t_2) - z(t_2)| \tag{20}$$

Linear interpolation between $(t_1, z(t_1))$ and $(t_3, y_2(t_3))$ gives equations (15) and (16) with y_2 instead of y_1. Therefore,

$$|y_3(t_3) - z(t_3)| < |y_2(t_3) - z(t_3)| \tag{21}$$

In general:

$$|y_j(\tau) - z(\tau)| < |z(\tau) - y_{j-1}(\tau)| \quad \forall\, \tau < t_n,\ j \geq 1$$

which proves that the sequence $\{|\, y_j(\tau) - z(\tau)|\}$ converges to zero, which proves that

$$\lim_{j \to \infty} y_j(\tau) = z(\tau)$$

□

4.5 An illustration of the method

The following example illustrates the use of the iterative process to determine a zero-coupon yield curve. One could suppose that the interest rates for three risk-free securities in the money market are known (maturing in 1, 6 and 12 months):

$$y_1(0.08) = 13.95\%; \; y_1(0.5) = 14.48\%; \; y_1(1.0) = 14.88\%$$

Since the money market instruments are zero-coupon rates, it follows that

$$y_j(t) = z(t) \qquad \forall\, j, \quad t \leq 1 \tag{22}$$

If four different coupon-bearing bonds are traded in the market, maturing in 3, 5, 8 and 10 years respectively, and the bonds, with a nominal value of 1 unit, pay semi-annual coupons of γ_k units and are priced at present at P_k:, then

$P_1 = 0.9751097, \qquad \gamma_1 = 0.075$

$P_2 = 0.9845960, \qquad \gamma_2 = 0.08$

$P_3 = 0.8766290, \qquad \gamma_3 = 0.07$

$P_4 = 0.8080316, \qquad \gamma_4 = 0.065$

To start the iteration process, a continuous extrapolation is guessed for $y_1(t)$, $1 < t \leq 10$, where $y_1(3) = 15.3\%$ and $y_1(5) = 15.6\%$ and $y_1(8) = 15.9\%$ and $y_1(10) = 16.1\%$.

Each of the four bonds in the example implies a zero-coupon yield $y_j^*(t_n)$, where $t_n = 3, 5, 8$ and 10 respectively for each iteration j. For example, the first bond (maturing in three years), gives, for $j = 1$,

$$y_1^*(3) = -\frac{1}{3}\,\ell n\left[\frac{0.9751097 - 0.075\left(e^{-0.5z(0.5)} + e^{-z(1)} + e^{-1.5y_1(1.5)} + e^{-2y_1(2)} + e^{-2.5y_1(2.5)}\right)}{1.075}\right]$$

In the same way $y_1^*(5)$, $y_1^*(8)$ and $y_1^*(10)$ can be found. Using these results as well as the data points $y_1(t_i)$, t_i = 0.08, 0.5, 1.0, a second approximate fit, $y_2(t)$, is found. Repeating this process, results in a sequence $y_j(t)$ as shown graphically in Figure 4.1 (overleaf). Table 4.1 shows the numerical results for each iteration. The results show clearly that the sequence $y_j^*(t)$ converges.

Table 4.1: Implied zero-coupon yields for five iterations starting with a first guess y_1

Term, t_i	y_1 (%)	y_1^* (%)	y_2^* (%)	y_3^* (%)	y_4^* (%)	y_5^* (%)
0.08	13.95	13.95	13.95	13.95	13.95	13.95
0.5	14.48	14.48	14.48	14.48	14.48	14.48
1	14.88	14.88	14.88	14.88	14.88	14.88
3	15.30	15.55	15.53	15.53	15.53	15.53
5	15.60	16.07	15.97	15.98	15.98	15.98
8	15.90	16.79	16.42	16.51	16.50	16.50
10	16.10	17.30	16.62	16.85	16.79	16.80

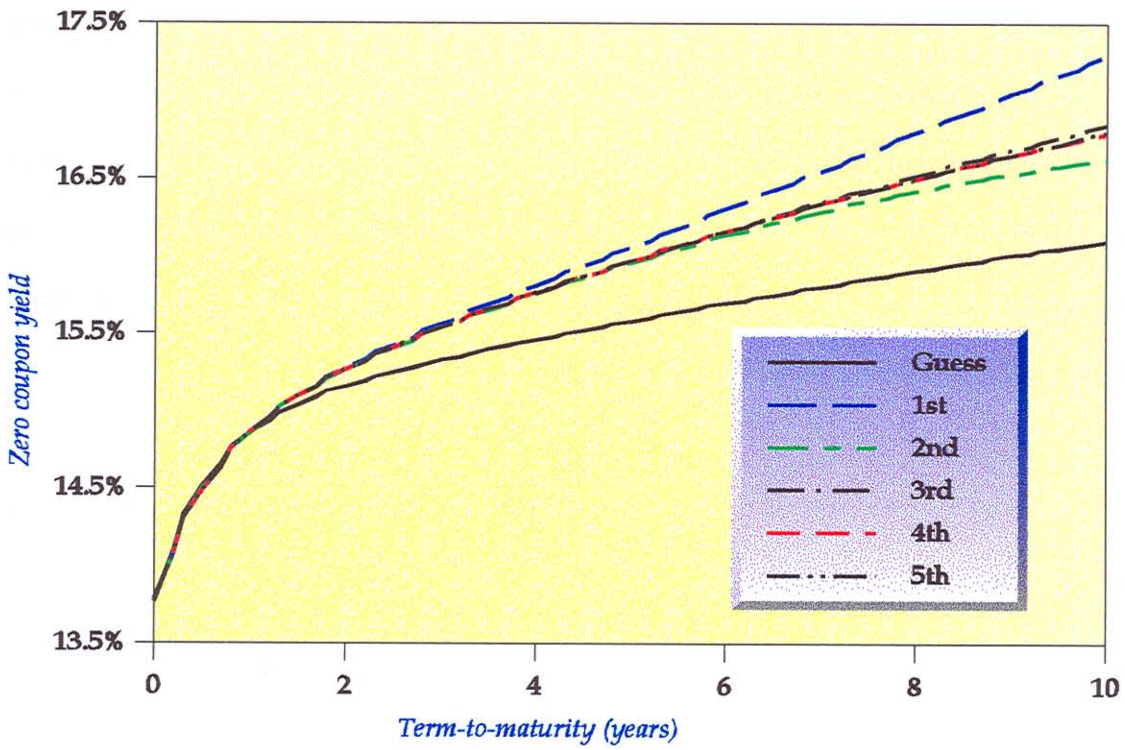

Figure 4.1: Results of the iterative process

4.6 Empirical results

The IBS-method developed in the previous sections was used to derive a zero-coupon yield curve for the South African fixed income market empirically. Daily closing rates over a three-year period were used to evaluate the method. Money market instruments were used to obtain data points between $t = 0$ and $t = 1$, while actively traded bonds were used to obtain information for the remainder of the term structure.

To compensate for market data that do not form a smooth curve, a least squares approximation technique was used in order to obtain a reasonably accurate fit of the data points which then served as input in order to interpolate for the next iteration. Appendix A sets out a discussion of the least squares approximation technique. It is important to realise that the success of the iterative method depends on a reasonably accurate interpolation of data points for bootstrapping purposes in each iteration. It is possible, for instance, to obtain an implied negative interest rate if the curve fitting technique oscillates or diverges from the data points.[3]

The empirical results of the study show that the technique yields a smooth spot rate curve and that the curve approximates the data points sufficiently well. The iterative method was compared to the standard bootstrapping technique, using a least squares fit. Market data from 1996 were used, which resulted in similar results for both methods, as is shown in Figure 4.2. The iterative method, however, provides a more accurate result in the region where data points do not form a smooth curve, due to interpolation discrepancies when the standard bootstrap technique is used. The difference between the two curves in Figure 4.2 increases when the data points are less smooth. The iterative method also proved to be computationally more efficient.

Figure 4.3 shows the results of a par-bond curve in November 1999, as derived from the zero-coupon curve. The forward curves implied by the zero-coupon curve in the above examples are sufficiently smooth. The implied forward swap curves for a 10-year and a 5-year swap are shown in Figure 4.4. When the standard bootstrap method is used, these curves are usually irregular with sudden changes in the slope of the curve.

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[3]This is why the theorem assumes a linear interpolation.

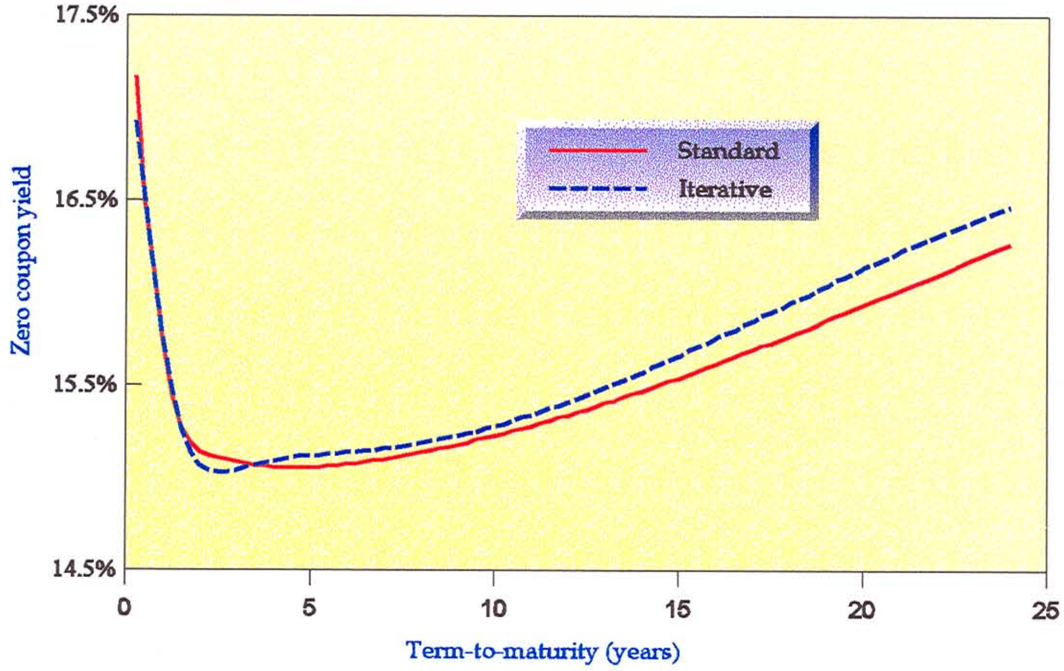

Figure 4.2: Comparison between standard bootstrap method and iterative bootstrap method

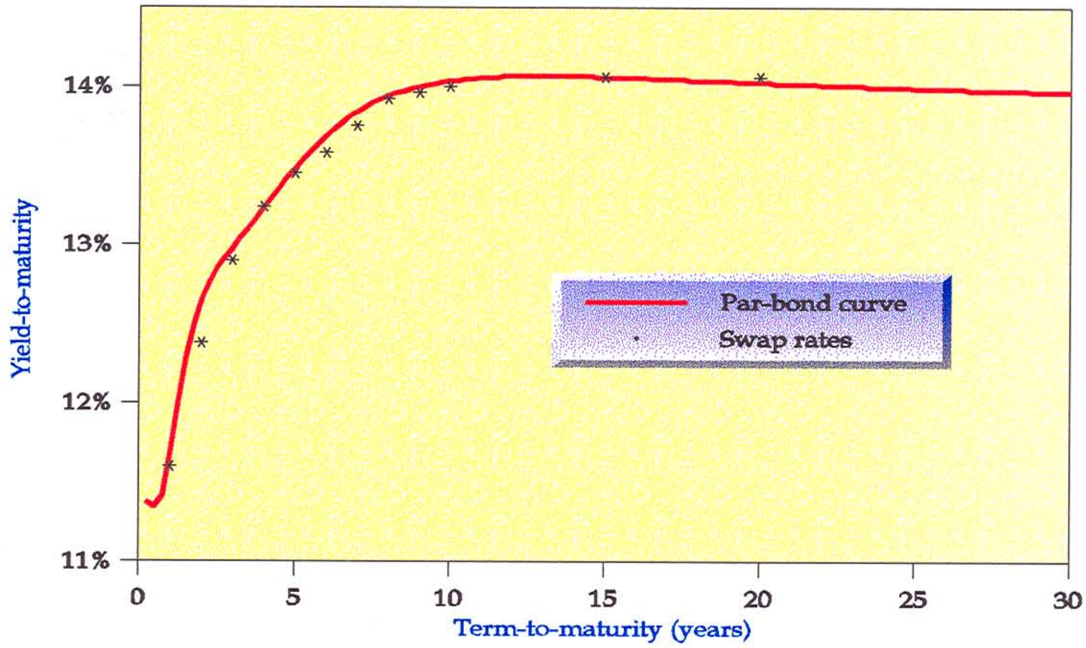

Figure 4.3: Par-bond yield curve in November 1999

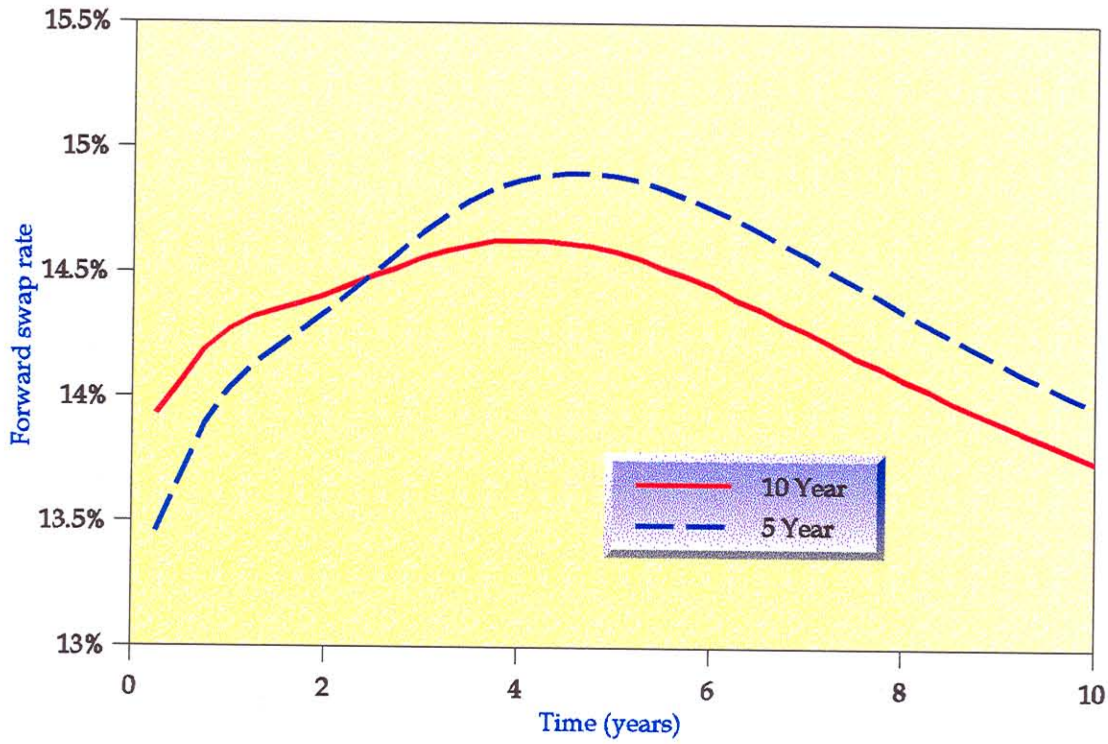

Figure 4.4: Implied forward swap curves for the yield curve in Figure 4.3

The empirical results from the South African market show that the method performs sufficiently well and yields better results than alternative techniques. Some of the advantages of the method are that it produces a smooth term structure, a smooth forward curve and that it is flexible and computationally efficient. It can therefore be applied in volatile and illiquid emerging fixed income markets to identify mispricings and arbitrage opportunities.

In order to evaluate the accuracy of the approximation of the zero-coupon yield curve, the zero-coupon yield curve obtained is used to calculate the implied yield-to-maturity, based on this curve. The sum of the squares of the errors in these rates was in the order of 1.4×10^{-4}. The individual deviations from the actual market rates varied between zero basis points for the more liquid bonds and 15 basis points for less liquid bonds containing a liquidity premium.

4.7 Credit premium

For the valuation of most derivatives, it is usually assumed that there is no risk of counter-party default. The no-default assumption does not, however, apply for bonds, and therefore the risk of a default on the coupons and/or nominal must be accounted for. This is done by adding basis points to the yield of the bond, in order to compensate for the credit risk. Bonds that are less tradable, on the other hand, also trade at a liquidity spread to the more liquid bonds.

The South African government bonds have the highest credit rating in the country. The government bonds can therefore be used to give a homogeneous zero-coupon yield curve with the same credit rating. All other bonds are priced from this curve to determine their yield (plus the credit and/or liquidity premium). Non-government organisations, for example Transnet, Eskom and Telkom, have a fairly big credit spread to the government curve, although some have government guarantees (Brown, 1999).

One advantage of the zero-coupon yield curve is that any bond can be priced from the zero-coupon yield curve, as determined from government bonds. An implied yield-to-maturity can therefore be found for any other bond. The difference between this *implied* yield and the market yield equals the credit spread added to compensate for the credit risk. (It is assumed that bonds with similar liquidity are compared in this example, in order to be able to ignore the liquidity premium.)

The evolution of the credit spread was investigated for some non-government bonds over time. Table 4.2 shows the credit spread over a 3-year period for three Transnet bonds, maturing in 2002, 2008 and 2010 respectively. The liquidity of these bonds is comparable to the smaller RSA government issues, and one can therefore ignore the liquidity spread. It is evident that the credit spread increased over the last three years, and must be taken into account when pricing these bonds. The 1998 emerging market crisis emphasised the importance of appropriate credit spreads for non-government bonds.

Table 4.2: Credit spreads for Transnet bonds

Date	Credit spread above government yield curve (basis points)		
	T001	T004	T011
November 1996	11	10	4
October 1997	12	22	15
February 1998	10	20	20
November 1998	32	37	41
May 1999	32	32	43
November 1999	43	29	37
May 2000	21	23	36

4.8 Concluding Remarks

The standard bootstrap method displays some inefficiencies when it is applied to a yield curve where there are only coupon bonds and irregular data points. An iterative method was therefore developed, which starts with a first guess, and then converges to the actual zero-coupon yield curve. This method is more efficient than the standard method.

The IBS-method developed in this chapter can be used to price all vanilla fixed income instruments. It can also serve as input in pricing many derivative securities, for instance options on fixed income vanilla products.

The next few chapters concentrate on the valuation of options on fixed income products, where the zero-coupon yield curve is an important input for some models.