CHAPTER 2

DERIVATIVE SECURITIES – THE THEORY

The valuation of derivative securities has drawn the attention of mathematicians across the world and has become a field of research for many. In order to obtain the fair value of any derivative security, it is important to understand the concepts of a stochastic process, arbitrage, martingales and partial differential equations. In this chapter, the fundamentals of option pricing theory are briefly set out.

Since 1973, the Black-Scholes model (Black & Scholes, 1973) has been the most popular option pricing model. This model can be adjusted in order to price options on various underlying instruments. The theory of option pricing can be applied to derive the two Black-Scholes option pricing formulas for call and put options, given by equations (19) and (20) (in this chapter).

2.1 Basic theory

2.1.1 *Introduction*

One could consider a probability space $(\Psi,\mathscr{F},P)$, where Ψ is a sample space; $\mathscr{F}$ is a σ-field on Ψ, and $\mathscr{F}$ consists of a collection of subsets of Ψ, called events; and P denotes a probability measure on $(\Psi,\mathscr{F})$. The measure P is a countable additive set function assigning a non-negative number $P(A)$ to each set $A \in \Psi$.

A *random variable,* called u, is a measurable mapping of Ψ into $\mathbb{R}$. A sequence (u_n) of random variables is called a discrete time stochastic process. Let $\mathscr{F}_n$ be the set of events known at time t_n. A *filtration* of the probability space is an increasing sequence of σ-algebras $\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots \subseteq \mathscr{F}_n \subseteq \ldots \mathscr{F}$.

If each u_n is measurable with regard to the corresponding member of $\mathscr{F}_n$ of the filtration, then the stochastic process is said to be *adapted*. This means that u_n is measurable with regard to $\mathscr{F}_n$, but not necessarily in respect of $\mathscr{F}_{n-1}$. If an event $\mathscr{F}_n$ is not known, then one can find a $\mathscr{F}_{n-1}$ measurable approximation to u_n. This approximation is denoted by $E(u_n|\mathscr{F}_{n-1})$ and is called the conditional expectation of u_n in respect of $\mathscr{F}_{n-1}$.

An adapted sequence (u_n) of random variables is *predictable* if u_n is $\mathscr{F}_{n-1}$ measurable for all $n \geq 1$.

The integrable random variables are the subset of random variables for which the integral with regard to P exists, and

$$\int u \, dP < \infty$$

The integral of u is the unconditional expected value, denoted by $E(u)$. For any event Q in $\mathcal{F}$, and B_Q (the set of points in Ψ for which Q occurs), $B_Q \in \mathcal{F}$. The expected value may be defined as follows:

$$E(Q) = P(B_Q)$$

Assume that the price S_t of an asset is a stochastic variable and follows an Ito process described by the following stochastic differential equation:

$$dS_t = \mu(S,t)\,dt + \sigma(S,t)\,dW \tag{1}$$

where W is a Wiener process with a drift rate of 0 and a variance rate of 1.0. A variance rate of 1.0 means that the variance of the change in W in an interval of length T equals T. The variable S_t has an expected drift rate (average drift per unit time) of μ and a variance rate (variance per unit of time) of σ^2 and satisfies the equations

$$P\left[\int_0^T |\mu|\,dt < \infty\right] = 1$$

$$P\left[\int_0^T \sigma^2\,dt < \infty\right] = 1$$

If N assets are traded in a market and the i-th asset is defined as a risky asset and priced at S_t^i at time t, the riskless asset can be defined as an investment at the risk-free rate, r, which gives a price of $S_t^0 = e^{rt}$, at time t, where $S_0^0 = 1$. This is the 'zero-th' share. The market price of all assets is given by

$$S_t = (S_t^0, S_t^1, \ldots, S_t^N)^T$$

A *trading strategy* is a predictable N-dimensional stochastic process,

$$\Theta_t = (q_t^0, q_t^1, \ldots, q_t^N)^T$$

denoting the holding or position in each asset at time t. The value of a portfolio Π at time t is given by the following equation:

$$\Pi_t = \Theta_t \cdot S_t = q_t^0 S_t^0 + \sum_{n=1}^{N} q_t^n S_t^n \qquad (2)$$

For two different periods in time, the strategic position of a portfolio is given by the equation

$$\Theta_t = \begin{cases} \Theta_0, & 0 \le t \le t_1 \\ \Theta_{t_1}, & t_1 < t \le t_2 \end{cases}$$

The change in value of the portfolio at time t_1 is therefore

$$(\Theta_{t_1} - \Theta_0) \cdot S_{t_1}$$

If this product is zero, the portfolio is defined to be self-financing or is called a *self-financing trading strategy*. The strategy is represented by the following equation

$$\Theta_{t_1} \cdot (S_{t_1} - S_0) = \Theta_{t_1} \cdot S_{t_1} - \Theta_0 \cdot S_0$$

or, generally

$$\Pi_{t_{i+1}} - \Pi_{t_i} = \Theta_{t_i} \cdot (S_{t_{i+1}} - S_{t_i})$$

A self-financing trading strategy in continuous time is therefore a strategy where

$$\Pi_t - \Pi_0 = \int_0^t \Theta_t dS_t \tag{3}$$

A strategy Θ is *admissible* if it is self-financing and if $\Pi_t(\Theta) \geq 0$ for any t. An admissible strategy with zero initial value and non-zero final value is called an *arbitrage strategy.* In such a strategy a riskless profit can be made, without initially investing anything .

A *derivative security* is defined as an $\mathscr{F}_T$-measurable random variable $u(T)$. The derivative is *attainable* if there is a self-financing trading strategy Θ_u such that $\Pi_T(\Theta_u) = u(T)$ with a probability of one. This self-financing trading strategy is then called a *replicating strategy.* If all derivative securities in an economy are attainable, the economy is called *complete.* If there are no arbitrage opportunities in an economy, the value of an attainable derivative $u(T)$, is given by the value of the unique replicating strategy.

Any tradable asset which has a strictly positive price (and pays no dividends) for all $t \in [0,T]$ is called a *numeraire.* Generally, the riskless money-market account is the numeraire, although the choice of numeraire is arbitrary. The price of any tradable asset (S^i) can be denominated in terms of a numeraire, for example S^0. The relative price is denoted by $(S^i)' = S^i/S^0$.

2.1.2 *Markov chains*

If S_t is an Ito process satisfying equation (1) and $f(\cdot)$ is any bounded function, and if the information set $\mathscr{F}_t$ contains all information about S_t until time t, then S_t satisfies the Markov property, provided that

$$E[f(S_{t+h}) \mid \mathscr{F}_t] = E[f(S_{t+h}) \mid S_t], \qquad \text{where } h > 0$$

A Markov chain is a stochastic process where the only information useful for predicting future values is the current value. The stochastic process S_t is a Markov chain if it satisfies the Markov property.

2.1.3 *Brownian motion*

A Brownian motion is a real-valued continuous stochastic process, (S_t), $t \in [0,T]$ (also called a Wiener process) with independent increments, such that the increments

$$x = S_{t_2} - S_{t_1}$$

have a normal distribution with mean zero and variance $|t_2 - t_1|$:

$$S_{t_2} - S_{t_1} \sim N(0, t_2 - t_1)$$

with $S_0 = 0$.

A Brownian motion is *standard* if

$$S_0 = 0 \qquad E(S_t) = 0 \qquad E(S_t^2) = t$$

In this case, the density function of a variable x is given by

$$f(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t} \right)$$

2.1.4 *Martingales*

Consider a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, where any information is generated by all observed events up to time t. Assume that S is a stochastic process where S is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and for all t we have

$$E(|S(t)|) < \infty$$

A martingale is defined as a zero-drift stochastic process. Therefore, $S(t)$ is a martingale if $\mathrm{E}(S(t)|\mathscr{F}_t)$ is defined and for each t and $h > 0$ the following relation holds

$$E(S(t+h)|\mathscr{F}_t) = S(t)$$

If

$$E(S(t+h)|\mathscr{F}_t) \leq S(t)$$

$S(t)$ is called a super-martingale and if

$$E(S(t+h)|\mathscr{F}_t) \geq S(t)$$

$S(t)$ is called a sub-martingale.

Consider the set $\wp$ which contains all probability measures P^* such that

- P^* and P have the same null-sets and are therefore equivalent; and
- the relative price processes $(S^i)'$ are martingales under P^* for all i, therefore

$$E^*[(S^i)'(T)|\mathscr{F}_t] = (S^i)'(t) \qquad \text{for } t \leq T \tag{4}$$

The measures $P^* \in \wp$ are called equivalent martingale measures.

Derivative securities are defined as those securities for which the expectation of the payoff is well-defined. A derivative security is therefore an $\mathscr{F}_T$-measurable random variable, $u(T)$, such that

$$E^{P^*}[|u(T)|] < \infty$$

A continuous trading economy is free of arbitrage opportunities and every derivative security is attainable if $\wp$ contains only one equivalent martingale measure. This was proved by Harrison and Pliska (1981).

For a given numeraire M with a unique equivalent martingale measure P_M, the value of a self-financing trading strategy

$$\Pi_t'(\Theta_u) = \frac{\Pi_t(\Theta_u)}{M(t)}$$

is a P_M-martingale. For a replicating strategy Θ_u that replicates the derivative security $u(T)$, it holds that

$$E^{P_M}\left(\frac{u(T)}{M(T)}\middle|\mathscr{F}_t\right) = E^{P_M}\left(\frac{\Pi_T(\Theta_u)}{M(T)}\middle|\mathscr{F}_t\right)$$

$$= \frac{\Pi_t(\Theta_u)}{M(t)}$$

Therefore,

$$\Pi_t(\Theta_u) = M(t)\, E^{P_M}\left(\frac{u(T)}{M(T)}\middle|\mathscr{F}_t\right) \tag{5}$$

2.2 Principles

2.2.1 *Girsanov's theorem*

Girsanov's theorem can be used to determine equivalent martingale measures by changing the probability measure and therefore the drift of a Brownian motion.

Theorem: For any stochastic process $\omega(t)$ such that with a probability of 1,

$$\int_0^t \omega(s)^2 ds < \infty$$

one can state that under the measure $dP^* = \rho dP$ the process

$$W^*(t) = W(t) - \int_0^t \omega(s)ds$$

is also a Brownian motion, where the Radon-Nikodym derivative is given by

$$\rho(t) = \exp\left\{\int_0^t \omega(s)du(s) - \frac{1}{2}\int_0^t \omega(s)^2 ds\right\}$$

It therefore follows that

$$dW = dW^* + \omega(t)dt \tag{6}$$

■

2.2.2 *Ito's lemma*

Theorem: X_t is an $\mathbb{R}$-valued Ito process if the following relation holds for all $t \geq 0$,

$$X_t = X_0 + \int_0^t \mu ds + \int_0^t \sigma dW_s \tag{7}$$

where μ and σ are functions of X and t. This stochastic integral is usually interpreted as the stochastic differential equation

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \tag{8}$$

Then, for a sufficiently differentiable function, $(t,X(t)) \to f(t,X(t))$ of the process X, for which the partial derivatives are continuous with respect to $(t,X(t))$, the function f has a stochastic differential given by the following equation (Björk, 1999)

$$df(t,X(t)) = \sigma\frac{\partial f}{\partial x}dW(t) + \left(\mu\frac{\partial f}{\partial x} + \tfrac{1}{2}\sigma^2\frac{d^2 f}{dx^2} + \frac{\partial f}{\partial t}\right)dt \tag{9}$$

The Ito-formula can also be written as follows:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)du + \frac{1}{2}\int_0^t f''(X_u)d\langle X,X\rangle_u \qquad (10)$$

where, by definition

$$\langle X,X\rangle_t = \int_0^t \sigma^2 X_u^2 du$$

■

2.2.3 *The Feynman-Kač proposition*

If one assumes that f is a solution to the boundary value problem

$$\frac{\partial f}{\partial t}(t,x) + \mu(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2} = rf\ , \qquad f(T,x) = \Phi(X)$$

and one furthermore assumes that the process $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is in $\mathcal{L}^2$, then f can be represented as

$$f(t,x) = e^{r(T-t)} E_{(t,x)}[\Phi(X_T)] \qquad (11)$$

where X satisfies the stochastic differential equation

$$dX_s = \mu(s,X_s)dt + \sigma(s,X_s)dW_s$$

$$X_t = x$$

The process is fully described by Björk (1999). ■

2.2.4 *The Ornstein-Uhlenbeck process*

If X solves the stochastic differential equation

$$dX_t = -\frac{1}{2}X_t dt + dW_t$$

then X is an Ornstein-Uhlenbeck process. Such a process has the normal distribution as its invariant measure. ■

2.3 The Black-Scholes model

2.3.1 *An exact solution for European options*

In the Black-Scholes economy it is assumed that there are two tradable instruments: the riskless money market instrument $M(t)$ (where $M(0) = 1$) and a stock $S(t)$. The value of the money market instrument is strictly positive and can therefore serve as a numeraire. Since the money market instrument is assumed to be riskless if it has a constant risk-free interest rate and no stochastic term, its price is described by

$$dM = rMdt \tag{12}$$

One can assume that the stock price follows a geometric Brownian motion

$$dS = \mu Sdt + \sigma SdW \tag{13}$$

with constant drift μ and volatility σ.

Since the stock price can be expressed in terms of the numeraire,

$$S' = \frac{S}{M}$$

it follows from Ito's Lemma that:

$$dS' = \left[\frac{M\frac{dS}{dt} - \frac{dM}{dt}S}{M^2}\right]dt + \frac{\mu S}{M}dt + \frac{\sigma S}{M}dW$$

$$= -\frac{rS}{M}dt + \frac{\mu S}{M}dt + \frac{\sigma S}{M}dW$$

$$= (\mu - r)S'dt + \sigma S'dW$$

For $\sigma \neq 0$ Girsanov's theorem can be used to turn the relative stock price into a martingale. Therefore a unique measure P^* is used, where $\partial P^* = \rho dP$ with $\omega(t) = -(\mu - r)/\sigma$, to obtain

$$dW = dW^* - \frac{\mu - r}{\sigma}dt$$

Therefore,

$$dS' = \sigma S'dW^* \tag{14}$$

The stochastic process S' is therefore a martingale, and, consequently, this economy is arbitrage-free and complete for $\sigma \neq 0$. The original price process S follows, under the measure P^*, the process

$$dS = \mu S dt + \sigma S\left(dW^* - \frac{\mu - r}{\sigma}dt\right) \tag{15}$$

or

$$dS = rSdt + \sigma SdW^* \tag{16}$$

Equation (16) shows that under the equivalent martingale measure, the drift μ is replaced by the risk-free rate r. The equivalent form is

$$S(t) = S(0) + \int_0^t rS(u)du + \int_0^t \sigma S(u)dW^* \tag{17}$$

If

$$f(S_t) = \ell n(S_t)$$

where S_t is an Ito process and a solution of equation (17), and the Ito formula is applied to this equation, the following equation results:

$$\ell n(S_t) = \ell n(S_0) + \int_0^t \frac{dS_u}{S_u} + \frac{1}{2}\int_0^t \left(\frac{-1}{S_u^2}\right)\sigma^2 S_u^2 du$$

Using equation (16), it follows that

$$\begin{aligned} \ell n(S_t) &= \ell n(S_0) + \int_0^t (r - \sigma^2/2)du + \int_0^t \sigma dW^* \\ &= \ell n(S_0) + (r - \frac{1}{2}\sigma^2)t + \sigma W^* \end{aligned}$$

Consequently,

$$S(t) = S(0) \exp\left[(r - \frac{1}{2}\sigma^2)t + \sigma W^*\right] \tag{18}$$

is a solution of equation (17), and therefore a solution of equation (16). The random variable $W^*(t)$ has a normal distribution with mean 0 and variance t.

If one defines a contingent claim of maturity T by giving its payoff $u \geq 0$, which is $\mathscr{F}_t$ -

measurable, then a European call option on the underlying price of the stock, S, with strike K, at the exercise time T has a payoff of

$$u(T) = \max\{S(T) - K, 0\}$$

In this case, u is a function of the underlying price at time T only. Some options depend on the whole path of the underlying asset, for instance Asian options. From equation (5) it follows that the price of a call option c, at time 0 is given by

$$c = E\left[\max\{S(T) - K, 0\}/M(T)\right]$$

If one uses the explicit solution of $S(T)$ given in equation (18), one gets

$$c = \int_{-\infty}^{\infty} e^{-rT} \max\{S(0)e^{(r - \sigma^2/2)T + \sigma x} - K, 0\} \frac{e^{-\frac{1}{2}\frac{x^2}{T}}}{\sqrt{2\pi T}} dx$$

The payout is non-zero if

$$S(0)\, e^{(r - \sigma^2/2)T + \sigma x} - K > 0$$

$$\therefore \quad \ln \frac{S(0)}{K} > -(r - \tfrac{1}{2}\sigma^2)T - \sigma x$$

$$\therefore \quad x > -\left[\frac{\ln\left(\frac{S(0)}{K}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma}\right]$$

Therefore, it follows that

$$c = \int_{-\frac{\ell n(S(0)/K) + (r-\sigma^2/2)T}{\sigma}}^{\infty} e^{-rT}\{S(0)e^{(r - \sigma^2/2)T+\sigma x} - K\}\frac{e^{-\frac{1}{2}\frac{x^2}{T}}}{\sqrt{2\pi T}}\,dx$$

$$= S(0)\int_{-\frac{\ell n(S(0)/K) + (r-\sigma^2/2)T}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi T}}\, e^{-\frac{1}{2}\sigma^2 T + \sigma x - \frac{1}{2}\frac{x^2}{T}}\,dx$$

$$- e^{-rT}K\int_{-\frac{\ell n(S(0)/K) + (r-\sigma^2/2)T}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi T}}\, e^{-\frac{1}{2}\frac{x^2}{T}}\,dx$$

$$= I_1 - I_2$$

If one changes variables in I_1 and I_2, this results in

$$I_1 = 1 + S(0)\int_{-\infty}^{\frac{\ell n(S(0)/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\rho^2}\,d\rho$$

and

$$I_2 = 1 + e^{-rT}K\int_{-\infty}^{\frac{\ell n(S(0)/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\omega^2}\,d\omega$$

Therefore,

$$c = S(0)\,N(d) - e^{-rT}K\,N(d - \sigma\sqrt{T}) \qquad (19)$$

where

$$d = \frac{\ell n\left(\frac{S(0)}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

which is the well-known Black-Scholes call option pricing formula. The price of a put option is given by

$$p = e^{-rT}K\,N(-d + \sigma\sqrt{T}) - S(0)\,N(-d) \qquad (20)$$

2.3.2 *The Black-Scholes partial differential equation*

In the case of path-dependent options, one cannot use the exact solution, and therefore it is necessary to use a numerical solution of the Black-Scholes partial differential equation. If one assumes that a stock price, S, follows a Wiener process, where the drift and volatility are dependent on the level of the stock price,

$$dS = \mu S dt + \sigma S dW \tag{21}$$

Then the variable S has a lognormal distribution, where ℓn S follows a generalized Wiener process.

If f is the value of a derivative security dependent on S, it follows from Ito's Lemma that

$$df = \left(\mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW \tag{22}$$

The discrete versions of equations (21) and (22) for a small interval Δt are

$$\Delta S = \mu S \Delta t + \sigma S \Delta W \tag{23}$$

and

$$\Delta f = \left(\mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} \right) \Delta t + \sigma S \frac{\partial f}{\partial S} \Delta W \tag{24}$$

where ΔS and Δf are the changes for a small time interval Δt. If one chooses a portfolio of the stock and the derivative as follows:

- short 1 derivative, and
- long K shares,

then the value of the portfolio, Π, is

$$\Pi = -f + KS \tag{25}$$

and the change in the value of the portfolio in time Δt is

$$\Delta\Pi = -\Delta f + K\Delta S \tag{26}$$

Substituting equations (23) and (24) gives

$$\Delta\Pi = -\left(\mu S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{d^2 f}{dS^2}\right)\Delta t - \sigma S\frac{\partial f}{\partial S}\Delta W + K\mu S\Delta t + K\sigma S\Delta W \tag{27}$$

Choosing K = $\frac{\partial f}{\partial S}$ eliminates the Wiener process and results in

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \tag{28}$$

The portfolio is therefore riskless for the short period of time Δt. In order to agree with the principle of no-arbitrage, it follows that the portfolio should earn the risk-free rate, r, in this period:

$$\Delta\Pi = r\Pi\Delta t$$

Substituting for $\Delta\Pi$ and Π, gives

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t$$

or

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{29}$$

Equation (29) is known as the Black-Scholes partial differential equation (Black & Scholes, 1973). When Black and Scholes published this equation in 1973, they made a major breakthrough in the pricing of any derivative dependent on non-dividend paying stock. Equation (29) can be solved using the Feynman-Kač proposition, to give the exact solution in Section 2.3.1.

2.3.2.1 Black's model

In 1976, Black published a paper describing an adjustment to the Black-Scholes model in order to price options on futures. Options on commodities, say beef, can be difficult to deliver at expiry of the option, therefore it is easier to have an option on the future, and have cash settlement at expiry. Since options on futures tend to be more attractive to investors than options on spot prices, Black's model became widely used in the option market.

If one assumes that the underlying instrument of the option is the future price, F, of the stock on the expiry date of the option, and one assumes that the futures price, F, follows a geometric Brownian motion, then

$$dF = \mu F dt + \sigma F dW \tag{30}$$

Since f is a function of F and t, it follows from Ito's lemma that

$$df = \left(\mu F \frac{\partial f}{\partial F} + \frac{\partial f}{\partial t} + \tfrac{1}{2}\sigma^2 F^2 \frac{d^2 f}{dF^2} \right) dt + \sigma F \frac{\partial f}{\partial F} dW \tag{31}$$

Consider a portfolio consisting of

- short one option, and
- long K futures contracts.

Since it costs nothing to enter into a futures contract, the cash value of the portfolio at $t = 0$ is given by the price of the option contract

$$\Pi = -f \tag{32}$$

The *wealth* of the portfolio can change in time Δt by the amount

$$\Delta\Pi = K\Delta F - \Delta f$$

Using the discrete versions of equations (30) and (31), it follows that by choosing $K = \frac{\partial f}{\partial F}$,

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 \right) \Delta t \tag{33}$$

This change is riskless, therefore to ensure that the arbitrage-free assumption holds, the return should be equal to the risk-free rate of interest

$$\Delta\Pi = r\Pi\Delta t \tag{34}$$

If one substitutes equations (32) and (33), this gives a partial differential equation for the price of an option on a futures price:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 = rf \tag{35}$$

In the case of exchange traded options where a margin is paid, $\Delta\Pi$ in equation (33) equals 0, and

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 = 0 \tag{36}$$

This equation can be solved analytically for European options and numerically for American options.

2.4 Numerical methods

The exact solution of the Black-Scholes model gives the price of a European option, which can only be exercised on the expiry date of the option. American options can be exercised at any time before or on the expiry date of the option. This implies that whenever the intrinsic value of the option is more than the value of the option, it would be profitable to exercise the option early. The problem with the exact solution of the Black-Scholes model, as set out in section 2.3 above, is that it does not provide for American options with an early-exercise value. Two numerical methods that solve the partial differential equations in Section 2.3 and which support American options are the binomial method and the finite difference method.

2.4.1 *The binomial method*

If S is the price of a non-dividend paying stock, and f is the value of an option on the stock, and the life of the option is divided into intervals of length Δt, then in each time-interval the stock price moves from its initial value of S to either Su or Sd with a probability of p and 1-p respectively. This process is shown in Figure 2.1.

In a risk-neutral world, the expected rate of return from an investment should be the risk-free rate, r. Therefore

$$Se^{r\Delta t} = pSu + (1-p)Sd$$

which gives

$$e^{r\Delta t} = pu + (1-p)d \tag{37}$$

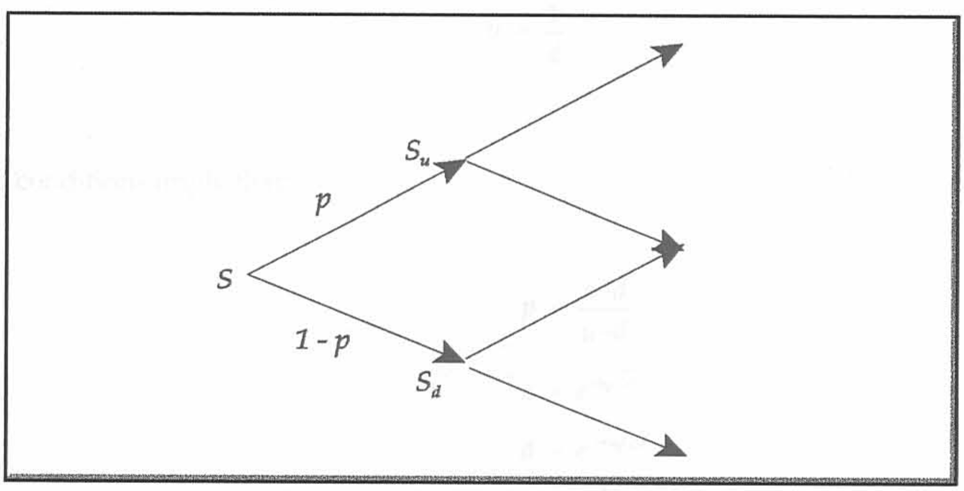

Figure 2.1: The binomial tree for stock price movement

The variance of a parameter S is given by

$$Var(S) = E(S^2) - [E(S)]^2 \tag{38}$$

where

$$Var(S) = S^2 e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1)$$
$$E(S) = Se^{r\Delta t}$$
$$E(S^2) = pS^2u^2 + (1-p)S^2d^2$$

or

$$e^{2r\Delta t + \sigma^2 \Delta t} = pu^2 + (1 - p)d^2 \tag{39}$$

Equations (37) and (39) give two conditions for u, d, and p. Cox, Ross and Rubinstein (1979) proposed a third condition:

$$u = \frac{1}{d}$$

These conditions imply that:

$$p = \frac{a-d}{u-d}$$
$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = e^{-\sigma\sqrt{\Delta t}}$$
$$\text{where} \quad a = e^{r\Delta t}$$

A tree of stock prices can be constructed, starting at time zero, and calculating the possible stock prices at time Δt, $2\Delta t$ and so on. In general, at time $i\Delta t$, the i+1 stock prices are:

$$Su^j d^{i-j} \qquad j = 0,1,\ldots,i$$

The value of the call option at time T is given by

$$\max(S_T - X, 0)$$

and for a put option by

$$\max(X - S_T, 0)$$

The value of the option is then calculated by working back through the tree. In a risk-neutral world, the value of the option at time $T - \Delta t$ can be calculated by discounting the value at time T at the short term rate r. The same is done for the following time steps. For American options one must check at each node that the early-exercise value is not bigger than the value of the option.

2.4.2 *The finite difference method*

A finite difference method solves a partial differential equation by converting it into a set of difference equations, which are then solved through an iterative process. Consider the differential equation for the value of an option:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \qquad (40)$$

Since the time t and the stock price S are the two variables in equation (40), N equally spaced time intervals can be chosen between zero and T, the expiry date of the option, and M price intervals can be chosen between zero and S_{max}. This results in a finite difference grid of (M+1) x (N+1) points. The (i,j) point corresponds to time $i\Delta t$ and stock price $j\Delta S$ and $f_{i,j}$ is the value of the option at the (i,j) point.

2.4.2.1 The Implicit finite difference method

The value of $\frac{\partial f}{\partial S}$ at (i,j) is given by an average of the forward and backward differences:

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \tag{41}$$

The value of $\frac{\partial f}{\partial t}$ at (i,j) is given by the forward difference approximation:

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \tag{42}$$

The finite difference approximation for $\frac{\partial^2 f}{\partial S^2}$ at the (i,j) point is

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j-1} - f_{i,j}}{\Delta S} \right) \Big/ \Delta S$$

or

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2} \tag{43}$$

Substituting equations (41), (42) and (43) into equation (40) gives, after some manipulation

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \tag{44}$$

where

$$a_j = \frac{1}{2}rj\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t$$

$$b_j = 1 + \sigma^2 j^2 \Delta t + r\Delta t$$

$$c_j = -\frac{1}{2}rj\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t$$

The value of a put option at time T is $\max[X - S_T, 0]$ or $\max[S_T - X, 0]$ for a call option , therefore

$$f_{N,j} = \max[k(X - j\Delta S), 0] \qquad j = 0, 1, \ldots, M \tag{45}$$

where $k = 1$ for a put and $k = -1$ for a call option.

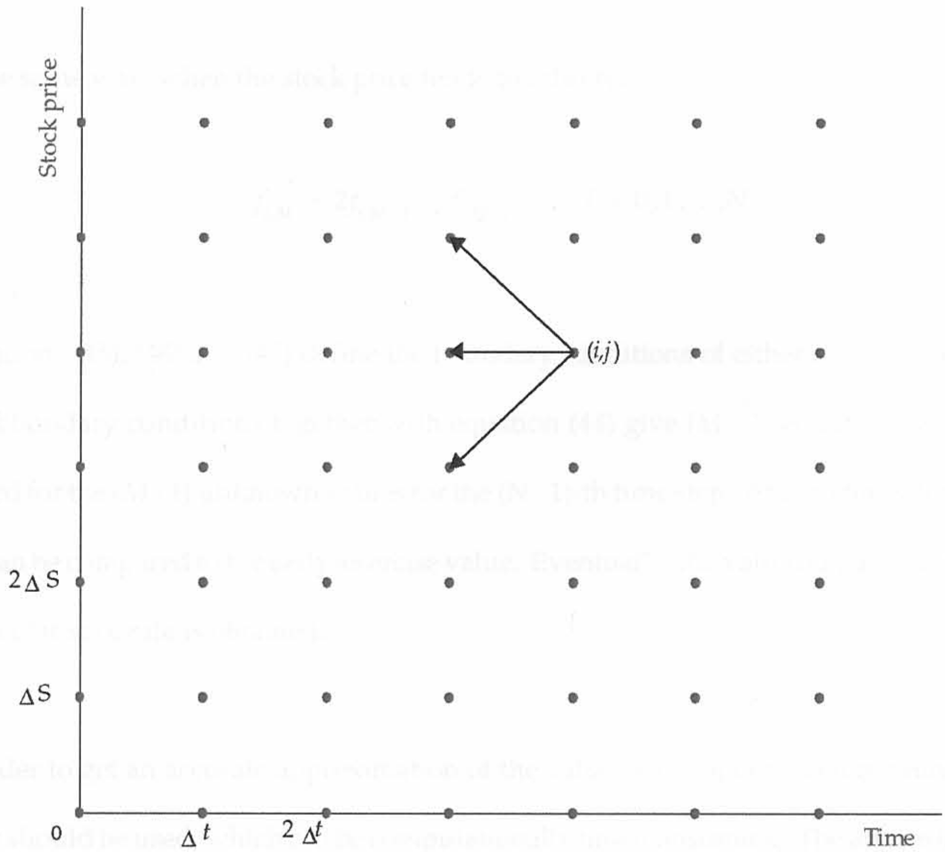

Figure 2.2: Finite difference grid

When the stock price is zero or tends to infinity, the delta of the option tends to zero or 1 respectively. In order to find the value of the option at zero and infinity, one assumes therefore that the second derivative at these points is approximately zero. Hence, when the stock price is zero,

$$\frac{\partial^2 f}{\partial S^2} \approx 0 = \frac{f_{i,2} + f_{i,0} - 2f_{i,1}}{\Delta S^2}$$

or

$$f_{i,0} = 2f_{i,1} - f_{i,2} \qquad i = 0,1,\ldots,N \tag{46}$$

In the same way, when the stock price tends to infinity:

$$f_{i,M} = 2f_{i,M-1} - f_{i,M-2} \qquad i = 0,1,\ldots,N \tag{47}$$

Equations (45), (46) and (47) define the boundary conditions of either a put or a call option. The boundary conditions together with equation (44) give (M - 1) equations which can be solved for the (M - 1) unknown values for the (N - 1)-th time step. At each time step, the value of f can be compared to the early-exercise value. Eventually, the value for f at time $t = 0$ for the particular spot rate is obtained.

In order to get an accurate approximation of the value of the option, a large number of time steps should be used, which can be computationally time-consuming. The approximate value for a very small time step can be obtained by solving the problem for two different time steps, say $\Delta t = 0.1$ and $\Delta t = 0.01$. These values are then linearly extrapolated to give an approximate value for $\Delta t \sim 0$.

The control variate technique can be used when there is an analytic solution to a similar problem, as with a European option. The approximation error is therefore calculated and can be used as a correction term to adjust the numerical value obtained for problems where there is no analytical solution available, for instance, for American options.

An explicit finite difference method can also be used if the implicit scheme is found to be time-consuming. The explicit method is similar to a trinomial tree approach. Unfortunately, often one or more of the two probabilities are negative, which can result in instability and inconsistencies in the solution. For the purposes of this study, an implicit finite difference method which is unconditionally stable is used.

This chapter provides the basic theory for pricing derivative securities. It forms the basis for pricing bond options in Chapters 5, 6 and 7. Before one can value options on interest rates accurately, it is, however, necessary to understand the underlying instrument. Therefore, Chapters 3 and 4 discuss the theory of the term structure of interest rates.