Measures and functions in locally convex spaces

by

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Declaration

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiæ Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Contents

Introduction **1**

1 Preliminaries **4**

1.1 Locally Convex Spaces 4

1.1.1 Quotient and Normed Spaces 7

1.2 Vector Measures 8

1.2.1 Spaces of Measures 8

1.2.2 Stone Representation 9

1.2.3 p-semivariation 9

1.2.4 Bartle-Dunford-Schwartz-type Theorems 10

1.3 Polish Spaces 11

1.4 Vector-valued Measurable Functions 12

1.4.1 Integrability and Integrals 14

1.5 Nuclear maps and Nuclear spaces 15

2 Liapounoff Convexity-type Theorems **17**

2.1 Counterexample 18

2.2 Non-negative Scalar Measures 20

2.3 Vector measures 22

2.4 Liapounoff Convexity-type Theorems 27

3 Barrelled spaces **31**

3.1 Existence of the Dunford Integral 31

4 Nuclear spaces and Nuclear maps **35**

4.1 Measures 36

4.2 Measurability and Integrability 40

5 Factorization of Measurable Functions **45**

5.1 Core Results 46

5.2 Applications 51

5.2.1 Set-valued Operators 51

5.2.2 Operator-valued Measurable Functions 54

5.2.3 Conditional Expectation 56

5.2.4 Operators on $L_1(\mu)$ and $L_1(\mu, X)$ 57

Summary

In this dissertation we establish results concerning the following:

- Liapounoff convexity-type theorems for certain locally convex space-valued measures defined on fields of sets.
- The improved properties of the composition of nuclear mappings with measures and measurable functions as compared to the measures and measurable functions considered on their own.
- The factorization of various classes of measurable functions and mappings.

These results are explained in three parts.

In Chapter 2, we establish Liapounoff convexity-type results for locally convex space-valued measures defined on fields (of sets) or equivalently on Boolean Algebras.

A. Liapounoff [Lia40] showed that the range of a non-atomic vector measure, taking values in a finite dimensional space, is compact and convex. Liapounoff convexity-type theorems concern the compactness and convexity of the closure of the range of a vector measure. We specifically investigate such results for certain classes of locally convex space-valued measures defined on fields and fields of sets with the interpolation property.

We find that vector measures defined on fields with the interpolation property have properties very similar to the status quo. However, for vector measures defined on (general) fields, similar results may not hold.

Properties stronger than non-atomicity, specifically, the strong continuity property, yield results comparable to the status quo. Such properties are thoroughly considered. Then we investigate certain locally convex spaces for which some of the summability conditions can be relaxed.

In the second part of this dissertation, in Chapter 3, we consider the existence of weak integrals in locally convex space. In general, the existence of the Dunford (and Gel'fand) integral depends on whether the closed graph theorem for the dual of space under consideration holds. Since barrelled spaces can be characterized in terms of the validity of the closed graph theorem, we consider locally convex spaces whose duals are barrelled spaces.

J. Diestel [Die72] discovered that the composition of an absolutely summing map (between two Banach spaces) with a Pettis integrable function has "improved" integrability properties, compared to that of the integrable function considered on its own. J. Rodriguez [Rod06] has recently, generalized Diestel's result to the case of a composition with Dunford integrable function.

In Chapter 4, we investigate the "improved" properties of the composition of a nuclear map with a locally convex space-valued measure and we investigate the properties of nuclear space-valued vector measures.

A natural consequence is the investigation of the composition of nuclear maps with measurable functions. We also find that the measurability and integrability properties of locally convex space-valued measurable functions are improved with such a composition.

The third part of this dissertation concerns the factorization of measurable functions and certain mappings. In Chapter 5, we first consider the factorization of Polish space-valued measurable functions along the lines of the "Doob-Dynkin's lemma" (cf. [Rao84, Proposition 3, p.7]), a result found in (scalar-valued) stochastic processes. This allows us to determine when, for two measurable functions, f and g it is possible to find a measurable function h, such that $g = h \circ f$.

Similar results are established for various classes of measurable functions. We discover similar factorization results for certain multifunctions (set-valued functions) and operator-valued measurable functions. Another consequence is a factorization scheme for operators on $L_1(\mu)$.