Modelling default-risky bonds

By

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0.1 Declaration

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Signature of Candidate

Date: 2002-10-31

0.2 Acknowledgements

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0.3 Abstract

In this dissertation, we examine current models used to value default-risky bonds. These models include both the *structural* and the *reduced-form* approaches. We begin by examining various issues involved in modelling credit risk and pricing credit derivatives. We then explore the various dimensions of structural models and reduced-form models and we provide an overview of four models presented in the literature on credit risk modelling. Both the theoretical and empirical research on default-risky bond valuation is summarized. Finally, we make suggestions for improving on the credit risk models discussed.

0.4 Preface

Building and implementing a model of credit risk requires choices along a variety of dimensions. To clarify these dimensions, this dissertation will examine, in detail, several existing credit risk models.

This dissertation is divided into six chapters. The first presents an overview of credit risk and credit derivatives. The second chapter studies the fundamentals of credit modelling. In essence, this describes the various dimensions of a credit risk model and categorizes credit risk models into two groups: *traditional credit models* and *market based models.* Market based models are then further divided into two groups: *structural models* and *reduced-form models.* The third chapter presents the fundamentals of interest rate modelling. The fourth chapter studies two structural models in the area of default-risky bond pricing: Merton (1974) and Longstaff and Schwartz (1995). A special section in the fourth chapter provides a comparison of these two models. The fifth chapter studies two reduced-form models in the area of default-risky bond pricing: Jarrow, Lando and Turnbull (1997) and Duffie and Singleton (1999). A comparison of structural and reduced-form models is provided in Chapter 6. Finally, Chapter 7 gives conclusions and suggests a few directions for further research.

Contents

Chapter 1

Introduction

1.1 Credit Risk

Credit risk or *default risk* refers to the possibility that the borrower will fail to service or repay debt on time. Default occurs when a borrower cannot fulfill key financial obligations, such as making interest payments to bondholders or repaying bank loans. The most fundamental example of a credit-linked security is a corporate bond.

The risk of default affects virtually every financial contract. Therefore, the pricing of credit risk has received increasing attention recently from practitioners, academics, and regulators. Practitioners need to quantify credit risk accurately in the market in order to be properly compensated for bearing it. Academics need to develop a strong modelling framework for credit risk, and regulators have a great interest in whether and how credit-sensitive transactions should be regulated. One can often read in academic literature comments such as "The modelling of credit risk, credit derivatives and non-hedgeable securities in general, is currently in a poor state" (Wilmott, (1998)). "Unlike market risk, the modelling of credit risk is a very difficult task because credit risk is not the simple manifestation of one single source or driver of the risky event" (Ong, (1999)). "The current situation with credit derivatives is still awaiting its Black-Scholes, there is no consensus on the best way of pricing them [credit derivatives]" (James, (1998)). The focus has mainly been on defaultable debt, derivatives on financial securities subject to credit risk and, more recently credit derivatives. The pricing of credit risk is essential to the valuation and hedging of each of these types of securities, therefore a model of credit risk must be developed.

Over the past decade, credit markets have seen tremendous growth in both geographical reach and range of new products. As a result, it has become imperative for market participants to understand credit risk and how to monitor and properly manage it. The taking of credit risk is a fundamental function of banks. For example, when a bank extends a loan to a customer, it is exposed to the risk that the customer will default on the loan. Traditionally, banks have dealt with this risk by requiring borrowers to meet certain underwriting standards. Another traditional approach used by banks to manage credit risk is to diversify the risk across different borrowers in different geographical regions and different industries. The development of markets for securitised assets and for loan sales has provided banks with another method for managing credit risk. This method allows banks to sell some of their loans into securitised pools or directly to outside investors. Although these methods reduce banks' credit risk exposure, they do not provide a formal method for valuing credit risk.

It is not only banks that are exposed to credit risk but investors in corporate, municipal and sovereign bonds as well. There is always a chance that a bond issuer will not meet its obligations to pay principal and interest. Although this risk is small for the typical high-grade borrower, investors are still exposed to credit spread risk, the risk of a decline in a bond's credit rating. A bond's credit rating is a general measure of the credit risk of the issuing firm. Rating agencies, like Standard & Poor's (S&P), Moody's, Fitch Ibca and Duffs & Phelps categorize bonds from the best rating AAA/Aaa to the worst rating C. A downgrade in a credit rating usually results in an immediate drop in the value of the bond. A bond issuer's cost of borrowing crucially depends on his credit rating. Therefore, not only does a downgrade in a credit rating affect bond investors, it also increases the bond issuer's cost of borrowing. Participants in the various bond markets have developed an informal model of credit risk that determines the *credit risk premium* associated with bonds of differing credit quality. The credit risk premium is the spread over default-free securities, such as government bonds, that bonds of differing credit ratings trade at. The credit spread is commonly expressed as a yield differential and it indicates the compensation to the lender attributable to credit risk.

The massive growth of the OTC derivatives market has led to a heightened awareness among banks of another type of credit risk: counterparty risk. Counterparty risk represents the risk that a party with a negative mark-to-market value on an OTC contract will default on their obligations to the other party of the contract. To deal with this counterparty risk, some banks insist on doing business only with highly rated counterparties. In this case, the OTC derivatives are traded at market rates that do not reflect the credit rating differences between the counterparties. A major development in the OTC derivatives market has been the setting up of AAA-rated special-purpose credit subsidiaries by the major investment banks. For example, in December 1991, Merrill Lynch launched Merrill Lynch Derivative Products and in early 1993, Salomon Brothers set up Swapco. The AAA-rated subsidiary essentially guarantees derivative transactions between the parent investment bank and credit-sensitive counterparties. However, the setting up of these AAA-rated subsidiaries does not address the banks' evaluation of risky counterparties that it trades with.

1.2 Credit Derivatives

Recent developments in the derivatives market have revolved around instruments that are used to trade credit risk, which in the process, is separated from other features of a financial instrument. These instruments, known as credit derivatives, are derivatives on the credit risk of a given bond, loan or issuer. The advent of credit derivatives has made most banks realize that more formal models of credit risk must be developed to price and trade these instruments. It has become clear to most banks that the traditional methods of managing credit risk, such as underwriting, diversification, loan sales and asset securitisation offer only a partial solution to controlling credit risk exposure.

Broadly defined, a *credit derivative* is a financial contract outlining either a certain or a potential exchange of payments in which at least one leg is linked to the performance of a specified underlying credit-sensitive asset or liability. The underlying market instruments include bank loans, corporate, emerging market, and municipal debt, convertible securities as well as the credit exposures generated from other derivatives-linked activities.

Credit derivatives provide users with an efficient means of hedging or acquiring credit risk. They permit investors to manage credit exposures by separating their views on credit from other market variables without jeopardizing relationships with borrowers; they also provide access to those investors who may be precluded from the underlying debt markets. Credit derivatives could also allow a company whose business depends substantially on another company to gain some protection from the other company's failure. Companies planning to issue debt can use credit derivatives to lock in a maximum financing rate at some issuance time in the future. Clearly, credit derivatives have many potential uses.

The market for credit derivatives has grown rapidly over the past few years. It was virtually nonexistent in 1994, had reached an estimated $20 billion by 1995, jumped to $350 billion by 1998 and was estimated at $740 billion by end of 2000.[1] This growth has been driven by the ability of credit derivatives to provide valuable new methods for managing credit risk.

The potential users of credit derivatives include commercial and investment banks, insurance companies, corporations, money managers, mutual funds, hedge funds and pension funds.

1.3 Evolution of Credit Derivatives

Credit derivatives have been in existence for a long period of time in the form of letters of credit (LCs), loan guarantees, bond insurance and option-embedded corporate debt securities. Under an LC, an issuer pays a bank an annual fee in exchange for the bank's promise to make debt payments in case of default. Under a bond insurance contract, a debt issuer pays an insurer to guarantee performance on a bond. Under an option embedded contract, a debt issuer or a debt holder has the right to redeem the debt prior to maturity at a pre-specified price in response to a credit rating change. The following table[2] shows the evolution of credit derivatives.

[1] These numbers were taken from a 1999 survey by the British Bankers' Association.

[2] This table was adapted from Beder-Iacono (1997).

TIME	EVENTS
1750's	Standby letters of credit and performance bonds were widely used in trade finance
1840's	London Guarantee and Accident Co & New York Guaranty Co were formed to issue credit insurance
1840's	Governments become large guarantors of bonds for railway construction
1900's	Export credit insurance began to be widely used
1900's	Letters of credit emerge
1960's	Increase in government guarantees
1960's	Foreign Credit Insurance Association begins to offer foreign political and commercial guarantees
1970's	State insurance guaranty fund system begins
1970's	Government National Mortgage Association (GNMA) begins issuing guaranteed pass-through mortgages
1970's	AMBAC Financial Group begins to insure municipal bonds
1970's	Farmer Home Administration (FmHA) begins guaranteeing loans made by commercial lenders
1980's	Credit supported commercial paper begins to be issued
1980's	Callable and puttable floating rate notes issued
1990's	First credit default swaps and credit-linked notes

Table 1.1: Evolution of credit derivatives

1.4 Credit Derivative Structures

In this section, we describe a variety of common credit derivatives and their uses in financial markets.

1.4.1 Credit Swap

A credit swap enables two parties to swap the credit risk associated with a reference security (or a portfolio of securities) without transferring the security itself. The credit risk buyer receives a fee, or periodic fixed payments from the credit seller. In exchange, the credit buyer promises to make a payment if the reference security experiences a credit event. A credit event may be a rating downgrade, in which case the credit swap is called a *rating option*, or it may be default, in which case the credit swap is called a *default swap* or *default option*. The contingent default payment can be linked to the price movement of the underlying asset, or it can be a fixed predetermined level based on the expectation of the loss rate.

Default swaps can be used to free up credit lines by reducing exposure to a single borrower or group of borrowers without the borrower's knowledge or consent (which may be required when loans are sold outright). Similarly, investors who need to protect themselves against default but cannot or do not want to sell the particular security, for accounting, regulatory, liquidity or tax reasons, can buy a credit default swap. Companies that have available credit lines but are unable to lend or invest because of balance sheet constraints can sell default swaps without breaching balance sheet limits.

1.4.2 Spread Swap

Buying a credit swap can virtually eliminate credit exposure, but this transaction also reduces return. Credit portfolio managers can achieve revenue-neutral diversification and increase risk-adjusted return by exchanging default obligations that are not closely correlated. A *spread swap* or *exchange option* is an instrument in which two parties exchange default obligations. Suppose A is exposed to B's default and X is exposed to Y's default. A and X exchange default obligations, possibly for a fee, so that A recompenses X for any adverse consequence of Y's default and X similarly recompenses A for losses resulting from B's default. Assume the default probabilities of B and Y are not jointly related; a spread swap is equivalent to an exchange of default swaps on B and Y. If Y is chosen to be a risk-free issuer, then a spread swap becomes a default swap.

1.4.3 Total Return Swap

Two parties enter into a *total return swap* in order to swap all the economic risks associated with a reference security, that is both market and credit risk, without transferring the security itself. The receiver in the swap will be long of the total economic risk of the security, and will receive the positive cash flows from that asset (coupons or dividends, plus any appreciation in capital value). The payer in the swap will be paid some spread over a reference rate such as Libor, as well as any depreciation in the capital value.

For investors seeking exposures to a specific asset, total return swaps are the synthetic equivalent of buying the asset and locking in term financing. For investors seeking to eliminate exposure to a specific asset, total return swaps are the synthetic equivalent of selling the asset and locking in a return.

1.4.4 Credit Spread Option

A *credit spread call (put) option* gives the purchaser the right but not the obligation to buy (sell) an underlying credit-risk-sensitive asset or credit spread at a predetermined price for a predetermined period of time. For example, a corporate note issuer might purchase a credit spread put to hedge the risk of widening spreads.

1.4.5 Credit Linked Note

Credit linked notes are debt instruments issued by highly rated issuers in which the coupon or the redemption value of the note is linked to the performance of a reference asset or index. They can be used by the seller to hedge against credit risk and by the buyer to achieve higher yields.

Credit linked notes on multiple assets can be used to enhance return as well as reduce risk. A *first-of-default credit note* is an instrument whose default is linked to a basket of high credit rated assets. In the event that any one of the reference credits defaults, the redemption value of the structured note is reduced by either a predetermined amount or the fall in value of the defaulted security. In effect, it may have a default risk equivalent to a credit rating below the minimal credit rating imposed upon the investor by regulations, even though each of the component assets has a credit rating exceeding this threshold. In the present regulatory environment, the investors would be allowed to hold the first-of-default note, thereby circumventing the minimal rating regulations. On the other hand, a *last-of-default note* is an instrument where default is triggered only when all reference credits default. Consequently, this may have a default risk equivalent to a higher credit level than any of the assets comprising the note.

1.5 Modelling Issues

As mentioned previously, the most fundamental example of a credit linked security is a corporate bond. The price of a corporate bond is subject to three types of risk:

1. Interest rate risk - this is the most essential part for the valuation of any security in fixed income markets. There are several models dealing with interest rates, most notably the Heath-Jarrow-Morton (1992), model, which provides a universal framework for interest rate risk modelling and risk management.

2. Credit risk or the likelihood of default - this risk pertains to the pure possibility of the bond entering default, irrespective of the magnitude of the loss from default. Default risk may be reflected in the borrower's credit rating or other macroeconomic variables. In general, credit ratings provide a good proxy for the default risk. Likelihood of default increases as credit rating decreases.

3. Recovery risk - different seniority debt for a particular firm can have dif-

ferent recovery rates in the event of default. This is difficult to model because it also depends on the market value of residual assets of a firm in default.

These three factors are the features that must go into a stochastic default-risky bond model. The first requirement is often over-looked, as it does not directly pertain to default.[3] The requirement is crucial because the value of default-risky bonds is as much a function of risk-free rates as it is a function of credit spreads. In practice, most default-risky bond models do not model the risk-free rate directly, but specify a class of stochastic interest rate models that could be used. A default-risky bond model should also have the ability to price any contingent claim whose cash flow is subject to default.

[3] Although risk-free rates and default probabilities may be correlated.

Chapter 2

Fundamentals of Credit Modelling

2.1 Introduction

Most default-risky bonds will deliver some cash flow other than the promised cash flow when default occurs. This will necessarily have a present value that is less than that of the promised cash flows. If this happens, then we say that there is a *partial recovery*.[1] The present value of the default cash flows at the time of the default is often referred to as the *recovery value of the bond*.

If there are no-arbitrage opportunities in the market, default-risky bonds must trade at values that are less than their risk-free counterparties.[2] This implies that their yields will be higher than the corresponding risk-free yield. The difference in yields is referred to as a *credit spread* (or sometimes just spread).[3]

Default risk and recovery risk together determine credit spreads on a bond. It

[1] For corporate debt, partial recovery is usually awarded in a bankruptcy settlement well after the promised cash flows are due. For sovereign debt, partial recovery is usually in the form of a "restructuring," which means that the country pays part of its obligations with new debt of a lesser value.

[2] That is risk-free bonds with the same promised cash flows.

[3] In most cases spreads can be thought of as the market's "view" of the likelihood of default.

is important to segregate them since credit derivatives may be written on either or both risks - hence from a modelling viewpoint, it is essential to not treat them as one composite entity. Moreover, the sources of empirical information in the modelling process will be quite disparate. Ratings and other industry-level information are quite effective in providing market participants with a good idea of default likelihood, whereas a more accurate assessment of the individual firm's recovery risk is necessary in understanding why spreads tend to be different for firms with the same rating category and industry.

An important variable in credit modelling is the time of default. This can be difficult to define in practice. There are many scenarios in which one could interpret the time of default in multiple ways. For example, let us say that an issuer declares its intention to default on a certain obligation before any payment is due. Does default occur at the time of the announcement or at the next payment date (assuming that the full cash flow is not delivered then)? This sort of issue needs to be dealt by on a case by case basis. Credit models must answer these questions when they define default. Actual credit derivatives must specify what they mean by default in their defining contracts.[4]

It is desirable that credit risk models possess the following attributes. First, they should be arbitrage-free and they should reflect current market information. That is, we should be able to fit a credit risk model to the current term structure of credit spreads. This is akin to fitting an interest rate model to the current term structure of interest rates.

Second, the models should produce default rates (sometimes called hazard rates) that are plausible. Third, models should be computationally tractable where the inputs to the model are readily estimable.

Now that we have the fundamentals, we are ready to present methods of credit risk modelling. The current methods of modelling can be divided into two distinct approaches, namely "traditional" and "market based" models.

[4]This question is relevant for many credit derivatives. An example is a credit default swap.

2.2 Traditional Credit Models

Traditional models use historical data to determine both default probabilities and recovery rates specific to certain debt classes. Traditional credit analysis for a debt class is a combination of both industrial and financial analyses. Industrial analysis focuses on economic cyclicality, growth prospects, R&D expenses, competition, source of supply, degree of regulation, labour, and company accounting factors, whereas financial analysis looks closely at various financial ratios, equity returns, foreign exposure, management quality and other factors. Credit analysts compare these factors to their historical values as well as for competing companies in the same industry when drawing conclusions about the creditworthiness of a company

Rating agencies like Standard and Poor's (S&P), Fitch Ibca, Duff and Phelps and CA Ratings[5] use traditional credit analysis in assigning ratings to borrowers. The ratings are ordinal in nature and do not quantify the default probability. The rating agencies publish observed historical defaults that can be used to infer the default probability for a specific rating. The higher credit ratings exhibit extremely low observed default frequencies, and therefore the historical experience is only really statistically significant for lower quality credits. For example, for the period 1981 to 1995, Standard & Poor's only had one default within one year of an A-rated or better company. The rating designations are shown in Table 2.1 below.

2.3 Market Based Models

The market based models use information from the market (equity values and credit spreads) to derive values for the default probabilities and recovery rates. Market based models attempt to describe the dynamics of default within the rigorous framework of financial mathematics. The key ingredients of this approach are credit events (e.g. defaults or downgrades) and payments on contracts made at such events. The mathematical modelling of credit risk involves making assumptions about the stochastic process driving default, the process generating the payoff upon default, and the evolution of risk-free interest rates.

[5]South Africa only.

Moody's	S&P	Meaning
Aaa	AAA	Highest Quality. Smaller degree of risk. Interest payments protected by a large or stable margin.
Aa	AA	High Quality. Margin of protection slightly lower than Aaa (AAA).
A	A	Upper Medium Grade. Adequate security of principal and interest. May be susceptible to impairment in future.
Baa	BBB	Medium Grade. Neither highly protected nor poorly secured. Adequate security for the present. Speculative features.
Ba	BB	Lower Medium grade. Speculative elements. Future not well secured.
B	B	Speculative. Lack characteristics of desirable investment.
Caa	CCC	Poor standing. May be in default or danger with respect to principal or interest.
Ca	CC	High degree of speculation. Often in default.
C	C	Lowest-rated class. Extremely poor chance of ever attaining any real investment standing.
-	D	In default.

Table 2.1: Ratings assignments and their meanings

In an arbitrage-free complete financial market, the price at time t, X_t, of a promised payoff X paid at a terminal time T is

$$X_t = \widetilde{E}\left[X \exp\left(-\int_t^T r_u du\right) \mid F_t\right] \tag{2.1}$$

where $(r_s, s \geq 0)$ is the spot interest rate and $\widetilde{E}[\cdot]$ is the expectation taken under the *equivalent martingale measure* (Harrison and Kreps (1979), and Harrison and Pliska(1981)), and F_t is the information available to agents at time t.

In the default risk framework, a default appears at some random time τ. We denote by $I(T < \tau)$ the indicator function of the set $\{T < \tau\}$ equal to 1 if the default occurs after T and equal 0 to otherwise. A default free contingent claim consists of a nonnegative random variable which represent the amount of cash paid at a pre-specified time to the owner of the claim. For a defaultable contingent claim, the promised payment is actually done only if the default did not occur before maturity. If the default occurs before maturity, some payment other than the promised payment is done. In general, the payment of a defaultable claim consists of two parts:

1. Given a maturity date $T > 0$, a random variable X, which does not depend on τ represents the promised payoff - that is, the amount of cash the owner of the claim will receive at time T, provided that the default has not occurred before the maturity date T.
2. A predictable process φ, prespecified in the default-free world, models the payoff which is received if default occurs before maturity. This process is called the *recovery process*.

The value of the defaultable claim is, provided that the default has not occurred before time t,

$$X_t = \widetilde{E}\left[XI(T < \tau)\exp\left(-\int_t^T r_u du\right) + \varphi_\tau I(\tau \leq T)\exp\left(-\int_t^T r_u du\right) \mid F_t\right] \tag{2.2}$$

where F_t is the information at time t. It is assumed that the owner of the contingent claim knows when the default appears. At time t, the owner of the claim knows if the default has occurred before; if the default has not yet occurred, he has no information on the time when it will happen.

As mentioned before, the time of default τ is an important variable in default risk modelling and market based models differ in their modelling of the default time. Once an assumption has been made on the evolution of the default time, the valuing of the defaultable claim (equation (2.2)) reduces to the problem of computing the expectation of $XI(T < t)$ under the risk-neutral probability.

2.3.1 Types of Market Based Models

The problem of modelling default risk is well represented in the literature. There are two distinct approaches. The first, pioneered by Merton (1974), attempts to model the default process by specifying two processes: one for the market value of the firm's assets and one for a benchmark of default. This benchmark of default is related to the firm's liabilities, and default is said to occur when the value of the firm's assets falls below this benchmark. Models of this type are often called *structural models*. The difficulty of modelling both the conditions under which default occurs, and in the event of default, the division of the value of the firm among claimants has led to the development of an alternative modelling approach.

Under the alternative approach, no direct reference is made to a firm's asset value; instead, default is modelled as an unpredictable event governed by a hazard rate process. The hazard rate process and the recovery rate are exogenously specified. Models of this type are often called *reduced-form models*. Reduced form models are especially practical when it is difficult to gather the asset and liability information needed by a structural model.

The distinction between structural and reduced-form models is only one of the many distinctions one must take into account when developing a model of credit risk. One must also model the type of payoff upon default. Different approaches have provided for fractional recoveries of par, a default-free version of the bond, or the market value at time of default. Recovery can also be modelled as a function of the debt's priority of claim in the capital structure (e.g. senior or subordinated) or the credit rating that is given to the debt by one of the major credit rating agencies. Consideration may also be given to the type of default. Both business cycles and firm-specific events influence defaults. However, firm-specific defaults can be unrelated to the business cycles. These may arise from events related to a firm's business activities or product liability lawsuits. Therefore, default may be triggered by some unexpected information that cannot be observed from economic variables only. Clearly, the modelling of default is very complex and should take into account as many of these issues as possible.

Another consideration in modelling credit risk is whether to use an *equilibrium* or *arbitrage-free* model. Equilibrium models focus on investor preferences, and assume that the economy tends to gravitate towards a state where all investors have allocated their resources optimally. In any other circumstance, investors with suboptimal allocations will attempt to improve their positions, thus creating instability. This instability will only disappear once the economy enters into a steady-state where no market participants are motivated to cause disturbances (i.e. when all investors have achieved optimal wealth creation). Equilibrium models require that the parameters to a given model be estimated empirically. The equilibrium model is then used to price securities. Equilibrium models require significant data and econometric techniques to estimate, and their output will not equate to market prices in all (or any) cases.

In contrast to the equilibrium models, arbitrage-free models begin by assuming that the prices of a small number of securities are given, and then deduce prices of other instruments by attempting to match their behaviour with these "basic" securities. The main assumption employed is that markets are free from opportunities to earn riskless profits (i.e. arbitrage). This leads to the result that any two portfolios producing identical payoffs under all scenarios have the same price (otherwise, riskless profits are possible by purchasing the cheaper portfolio and selling the more expensive one). Arbitrage-free models use market prices of securities to infer a model's parameters. Therefore, the arbitrage-free model will be calibrated in such a fashion as to produce the given market prices. Arbitrage-free models are easy to get data for and are useful in hedging derivatives. However, they can be misled by market imperfections such as illiquidity.

There are two main advantages possessed by the arbitrage-free pricing methodology over its equilibrium counterpart. The first is that arbitrage pricing does not require any assumptions regarding investor preferences, aside from the basic axiom that market participants will always prefer more wealth to less (referred to in economics as *insatiability*). While equilibrium models are suitable for economic theory, in applications to derivatives pricing it is not often justifiable to assume a specific form of preference function for a given investor. The second advantage of arbitrage pricing is that it provides an explicit algorithm whereby the violation of the arbitrage-free results will lead (at least in theory) to a risk-free profit. The key to choosing between these two types of models is whether one is more concerned about estimating intrinsic value (equilibrium models) or value relative to current market prices (arbitrage-free models).

When valuing defaultable debt, it is important not only to model the credit risk, but also the interest rate risk. For that reason, most models of credit risk are integrated with an interest rate model. There are two distinct approaches in the literature on the models of the interest rate curve. The first is the Vasicek (1977), and its variants, which focus on the dynamics of the short-term interest rate from which the whole yield curve is reconstructed. Models of this form are commonly referred to as *short rate models*. The second approach initiated by Heath, Jarrow and Morton (1992), takes the full forward rate curve as a dynamic variable, driven by one or several continuous-time Brownian motions. Models of this form are commonly referred to as *Heath, Jarrow and Morton models* (HJM models). Most interest rate models are based more on their mathematical tractability rather than on their ability to describe the data. Since it is difficult to build a model of credit risk, most people choose relatively simple models of interest rate risk to accompany the model of credit risk. The simplest approach assumes constant interest rates.

Another important question when implementing a model of credit risk is what technique will be used to calculate prices? The most popular approaches are analytic (or closed form) solutions, a lattice (or tree) framework, finite difference methods and Monte Carlo methods. Analytic solutions are convenient to use and provide quick intuition on important variables, but usually are too simple or too inflexible in practical situations where complex payout or exercise contingencies are present. The lattice framework provides more flexibility and is computationally feasible if the problem can be solved with a recombining binomial or trinomial tree. Finite difference methods are suitable for problems with two or three random factors. They are similar to lattice framework in that the computations work back from end of the life of the security to the beginning. However, they are more flexible than the lattice framework because there are many ways to improve finite difference methods making them faster and more accurate. Finally, Monte Carlo simulation provides the most flexibility and is useful for solving complex path-dependant problems or high-dimensional problems. However, Monte Carlo analysis can be slow and computationally intensive. The method that is chosen in practice is likely to depend on the characteristics of the problem being evaluated and the accuracy required.

Chapter 3

Introduction to Interest Rate Modelling

3.1 Fundamentals

This section presents the fundamentals of interest rate modelling as they pertain to work in this dissertation.

A *zero coupon bond* is an obligation to pay the holder one dollar at a fixed maturity date T. We write the value of the zero coupon bond at time t as $P(t,T)$.

We assume (for this chapter) that the payment will be made with absolute certainty. At any time $t_0 < t$, we let $P(t,T)$ denote the price of a zero coupon bond at time t maturing at time T. The no arbitrage condition gives

$$P(t,T) = P(t,\tau)P(\tau,T) \tag{3.1}$$

for all $\tau \in (t,T)$

A general bond may have *coupons*. These are payments of the same amount c_i which are paid at times t_i, where the t_i's are less than or equal to the final maturity date T. The bond will also pay some *principal amount* p at maturity.[1] All bonds of the above form can be written as a linear combination of zero

[1] The principal amount is generally referred to as the *par value* of the bond.

coupon bonds. If β_t is the value at time t of a bond with principal p, maturity T and coupons c_i to be paid at times t_i, then we have

$$\beta(t) = \sum_{t \leq t_i \leq T} c_i P(t, t_i) + pP(t, T) \tag{3.2}$$

This implies that it is sufficient to restrict our attention to zero coupon bonds because all coupon bonds are just linear combinations of zero coupon bonds.

The *continuously compounded zero coupon yield*, $y(t, T)$ is given by

$$y(t, T) = -\frac{1}{T-t} \ln(P(t, T)) \tag{3.3}$$

For a fixed t, the function $T \longmapsto y(t, T)$ is called the *(zero coupon) yield curve*.

An *instantaneous forward rate* $f(t, T)$ is defined as

$$f(t, T) = -\frac{\partial \ln(P(t, T))}{\partial T} \tag{3.4}$$

which implies

$$P(t, T) = \exp\left\{-\int_t^T f(t, s) ds\right\} \tag{3.5}$$

The *spot rate*[2] $r(t)$ is defined to be

$$r(t) = \lim_{T \downarrow t} f(t, T) \tag{3.6}$$

The spot rate can also be thought of as the rate of return of a bond with an infinitesimal time to maturity. That is

$$r(t) = -\lim_{T \downarrow t} \frac{1}{T-t} \ln(P(t, T)) \tag{3.7}$$

3.1.1 The Wiener Process

The definition below and related concepts are taken from (and covered in much more detail in) Brzeźniak and Zastawniak (1999). The *Wiener process (or Brownian motion)* is a stochastic process $W(t)$ with values in $\Re$ defined for $t \in [0, \infty)$ such that

[2]The spot rate is sometimes referred to as the *short rate*.

1. $W(0) = 0$ a.s.;

2. the sample paths $t \longmapsto W(t)$ are continuous a.s.;

3. $W(t)$ has stationary independent, normally distributed increments: If

$$0 = t_0 < t_1 < t_2 \ldots < t_n$$

and

$$Y_1 = W(t_1) - W(t_0), Y_2 = W(t_2) - W(t_1), \ldots, Y_n = W(t_n) - W(t_{n-1})$$

then

- $Y_1, Y_2, \ldots, Y_n$ are independent,
- $E[Y_j] = 0 \; \forall j$,
- $Var[Y_j] = t_j - t_{j-1} \; \forall j$.

3.1.2 Itô's Lemma

Let $W(t)$ be a Wiener Process. Let $x(t)$ be an Itô Process with $dx = a(x,t)dt + b(x,t)dW$. Let $V = V(x,t)$, then,

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt \\ &= \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt + \frac{\partial V}{\partial x}b(x,t)dW \end{aligned}$$

Proof: Using a Taylor expansion

$$dV = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial^2 V}{\partial t \partial x}dxdt + h.o.t \quad (3.8)$$

Any term of order $(dt)^{\frac{3}{2}}$ or higher is denoted by h.o.t. and is small relative to terms of order dt . Note that $(dW)^2 = dt$.

So,

$$\begin{aligned} (dt)^2 &= h.o.t \\ dxdt &= a(x,t)(dt)^2 + b(x,t)dWdt = h.o.t. \\ (dx)^2 &= b(x,t)^2(dW)^2 + h.o.t = b(x,t)^2 dt + h.o.t. \end{aligned}$$

Model	$v(r_t, t)$	$\sigma(r_t, t)$
Merton [4]	θ	σ
GBM[5]	θr_t	σr_t
Ho and Lee	θ_t	σ
Vasicek	$\theta + \alpha r_t$	σ
Brennan & Schwartz	$\theta + \alpha r_t$	σr_t
Cox-Ingersoll-Ross	$\theta + \alpha r_t$	$\sigma\sqrt{r_t}$

Table 3.1: Short rate models

Therefore,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt \tag{3.9}$$

3.2 Short Rate Models

A common approach to stochastic modelling of interest rates is to take the short rate $r(t)$[3] to be a stochastic process. Models of this form are commonly referred to as *short rate models*. The process r_t is generally taken to be a diffusion process defined by the stochastic differential equation

$$dr_t = v(r_t, t)dt + \sigma(r_t, t)dW_t \tag{3.10}$$

driven by a Wiener process W. We can interpret v as an instantaneous rate of return. Some examples with their specification of v and σ are shown in Table 3.1 above.

It has been shown (in Vasicek (1977), for example) that, in this formulation, the value of a zero coupon bond at time t maturing at time T must be the solution to the partial differential equation

$$\begin{aligned} P_t + \frac{1}{2}\sigma^2(r_t,t)P_{rr} + (v(r_t,t) + \lambda(r_t,t)\sigma(r_t,t))P_r - rP &= 0 \\ P(t,T) &= 1 \end{aligned} \tag{3.11}$$

[3]Sometimes written as r_t.

where λ is the market price of risk.[6] The Feynman-Kač equation gives

$$P(r_t, t, T) = E_{(r_t,t)}\left[\exp\left\{-\int_t^T r_s ds\right\}\right] \tag{3.12}$$

where r_t is now the solution to the SDE

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t \tag{3.13}$$

where $\mu = v + \lambda\sigma$ is a risk-neutral drift.

3.3 HJM Models

Another approach is to model the instantaneous forward rates $f(t, T)$ as the underlying stochastic variables. Models which apply this approach are generally referred to as "Heath, Jarrow and Morton," models (or HJM models) after the authors of Heath, Jarrow and Morton (1992).

Mathematically, an HJM model can be described as follows. Forward rates are modelled as a stochastic process given by

$$df(t, T) = \mu(f, t, T)dt + \sum_{i=1}^{n} \sigma_i(f, t, T)dW_t^i \tag{3.14}$$

where $W_t^1, \ldots, W_t^n$ are independent Brownian motions, the $\sigma_i(f, t, T)$'s are specified by the modeller, and the $\mu(f, t, T)$'s are determined by the no arbitrage condition. This is the requirement that

$$\mu(f, t, T) = \sum_{i=1}^{n} \sigma_i(f, t, T) \int_t^T \sigma_i(f, t, s)ds \tag{3.15}$$

This condition is often referred to as the "HJM drift condition."

[6]There is a theorem which states that the no-arbitrage condition implies that the difference between the instantaneous rate of return of any asset and the spot rate divided by the asset's volatility must be a function of the state variables and calendar time. That function is called the *market price of risk*.

3.4 The Yield Curve

Interest rates vary for different maturities of debt. A graph of the spot yield for different maturities is called a *yield curve*. In general, there is a distinct difference between short and long term interest rates. There are a number of economic theories that are cited to explain the shape of the yield curve. However, the *expectations theory* and the *market segmentation theory* have evolved as the major theories that explain the shape of the yield curve.

The *expectations theory* is based on the premise that current interest rates are somehow related to the market's expectations of future rates. These future interest rates are affected by economic factors such as money supply figures, inflation and trade deficit figures. Market participants have different views on the expected future behaviour of these economic factors and this determines their anticipations of the future interest rates. These expectations are evident from the shape of the yield: a downward sloping yield curve implies that the short term interest rate is expected to fall, whereas the opposite is expected from an upward sloping yield curve. In general, the short term interest rate is more sensitive to the economic factors than the long term interest rate.

The *market segmentation theory* relies on the idea that some investors have restrictions (either legal or practical) on their maturity structure. Examples include money market funds (short-term maturities) or life insurance companies (long-term maturities). The shape of the yield curve is therefore determined by the supply and demand for securities within a given maturity segment.

In any nation the lowest interest rates on local currency denominated debt apply to those loans assumed by the sovereign government. These loans take place through the sale of government bonds. Provided that the debt is issued in the sovereign currency, the government has the option of printing money to meet any payments that are due. It is for this reason that sovereign debt is assumed to have no risk of default. This means that the probability that the loan will not be paid is effectively zero and consequently, the interest rate offered on a sovereign loan is regarded as the risk-free rate. A yield curve constructed using government bonds is therefore called a *risk-free yield curve* or a *zero-coupon yield curve*.

The risk-free yield curve is a concept central to economic and financial theory and the pricing of interest rate contingent claims. Together with the no-arbitrage theory, it provides a mechanism for comparing cash flows occurring at different times. Any risk-free financial asset comprised of specified tranches can be assigned a present value that is arbitrage-free. This is because it is possible to lend (borrow) the appropriate amount now that will match each tranche as it occurs. We will illustrate this concept with the arbitrage-free pricing of a South African government bond

3.5 Pricing A South African Government Bond

South African government bonds have fixed rate coupons which are paid semi-annually up to and including the maturity date at which time the principal or face value is also repaid. The coupons are quoted in percentages and indicate the percentage of the principal to be repaid annually. Therefore a 13% semi-annual coupon means that 6.5% of the principal is repaid every six months with the final coupon payment and the repayment of the principal at maturity. South African bonds are priced by *yield-to-maturity* - the price of a bond is quoted as a semi-annual interest rate and the cash price is obtained by discounting the cash flows of the bond to the present using this yield-to-maturity as the interest rate for the discounting.[7]

We now provide the framework for pricing a South African bond, with the assumption that the principal on the bond is 100.

- $P(t, y, n)$ = trading price of a bond
- t = current time
- n = coupons still to be received
- Δn = fractional number of half-years before the first coupon will be received
- c = coupon rate of the bond (13 for 13%)

[7] The yield-to-maturity can thus be regarded as the 'internal rate of return' of the bond's cash flows.

- y = quoted yield of the bond expressed as an interest rate compounded semi-annually

The quoted yield is consistent with the traded price if the following equality holds,

$$P(t,y,n) = \frac{1}{(1+y/2)^{\Delta n}}\left(\sum_{i=0}^{n-1}\frac{c/2}{(1+y/2)^i} + \frac{100}{(1+y/2)^{n-1}}\right) \tag{3.16}$$

A bond has a specified time structure of payments. Assume a function $r(t,T)$ which represents the risk-free, continuously compounded interest rate at time t applicable to loan maturing at time T. Using no-arbitrage theory, the present value of each cash flow of the bond can then be determined. A coupon received at time t_i will have a present value of $(c/2)\exp(-r(t,t_i)(t_i-t))$. The sum of the present values of all of the cash flows comprising the bond will be an arbitrage-free value for the bond. Let $P(t,n)$ be the arbitrage-free price for the bond. Then,

$$P(t,n) = \left(\sum_{i=1}^{n}\frac{c}{2}e^{-r(t,t_i)(t_i-t)}\right) + 100e^{-r(t,t_n)(t_n-t)} \tag{3.17}$$

3.6 Determinants of the Risk-Free Yield Curve

Fundamental to the pricing of interest rate derivative instruments and the management of their risk is the construction of a risk-free yield curve. In liquid fixed-income markets, zero-coupon bonds and money market rates are typically used to construct the risk-free yield curve; in markets where a limited number of zero-coupon bonds are traded, there are usually enough coupon bearing bonds traded to use in constructing this curve. In the South African fixed-income-market, however, only a limited number of liquid financial instruments are available to construct the risk-free yield curve. Under the efficient market hypothesis the most liquid of these instruments will be trading at arbitrage-free prices. A risk-free yield curve must be consistent with these prices. The present value of the cash flows of these instruments should sum to their trade price as in equation (3.16).

The primary financial instruments of South Africa's money market[8] that may be used to reliably fix interest rates at the short end of the risk-free yield curve are the Johannesburg Interbank Acceptance Rate (JIBAR), Negotiable Certificates of Deposits (NCDs) and Treasury Bills (T-bills). The JIBAR J_t is the rate of interest that banks will offer to each other for a t -month loan that begins on that particular day. The most popular period is 3 months but 1, 6, and 12-month JIBAR rates are also available. NCDs, the most liquid of the instruments, are issued by all major banks through private placements. T-bills are issued by the government using an auction and usually have a maturity of 91 days. The secondary market for T-bills is relatively illiquid because local banks use them to meet reserve requirements. This lack of liquidity in the T-bills has led market participants to use Forward Rate Agreements (FRAs) in constructing the short end of the risk-free yield curve instead of T-bills. A Forward Rate Agreement is a forward contract where two parties agree that a certain interest rate will apply to a certain notional loan or deposit during a specified future period of time. A 3×6 FRA is an agreement to fix the rate for the period between three and six months time (i.e., for the 3 month period starting in 3 months time). Other FRAs frequently traded in the South African Market are 6×9's and 9×12's. Settlement is against the relevant JIBAR rate. FRAs are settled at the start of the future period, when the FRA yield rate (i.e., the rate agreed upon in advance under the FRA) and the JIBAR rate are compared. If there is a difference between these rates a discounted cash settlement based on the difference is made.

The JIBAR rate is quoted as a yield rate. This means that a discount bond maturing in three months time would be traded as

$$P(t, t+3) = \frac{100}{\left(1 + J_3 \frac{91}{365}\right)} \tag{3.18}$$

where $P(t, T)$ is a zero coupon bond of maturity based on a notional principal of 100.

FRAs are also quoted as yield rates implying

[8]The money market is the universe of instruments for the relatively short term (<2years) borrowing and lending of cash.

$$P(t,t+3i)=\frac{100}{\left(1+J_3\frac{91}{365}\right)\left(1+FRA_{3\times 6}\frac{91}{365}\right)\ldots\left(1+FRA_{3(i-1)\times 3i}\frac{91}{365}\right)} \quad (3.19)$$

NCDs are quoted as yield rates i.e.,

$$P(t,t+s)=\frac{100}{\left(1+NCD_s\frac{s}{365}\right)} \quad (3.20)$$

where s is the term of the NCD in days.

T-Bills are quoted as discount rates i.e.,

$$P(t,t+s)=\left(1-TB_s\frac{s}{365}\right)100 \quad (3.21)$$

where s is the term of the T-Bill in days.

The arbitrage-free price of $P(t,T)$ is $100e^{-r(t,T)(T-t)}$, where the (continuously compounded) risk-free rate $r(t,T)$ of the bond is related to its price by

$$r(t,T)=\frac{1}{T-t}\ln\left(\frac{100}{P(t,T)}\right) \quad (3.22)$$

Assume that there are k of these money market instruments. Using equation (3.22), the zero coupon bond values for these instruments can be converted into continuously compounded zero coupon interest rates. This implies that the risk-free rate $r(t)$ is known at distinct times $\{t_j^*: j=1,\ldots,k\}$ where at each t_j^* there is the following restriction on $r(t)$

$$r(t_j^*)=r_j^*\ \forall j=1,\ldots,k \quad (3.23)$$

In South Africa we can use the government bond market and the interest rate swaps markets to obtain information about interest rates for longer periods. The South African bond market is a relatively developed fixed income market with bond maturities of up to 30 years but it suffers from a lack of a complete set of well-traded bonds with well-spaced maturities. For sectors of the yield curve that don't offer good tradable liquidity to reliably fix interest rates, market participants use interest rate swaps - which can be seen as par yield bonds. An interest rate swap is an exchange of cash flows based upon different interest rate indices denominated in the same currency on a pre-set notional amount with a pre-determined schedule of payments and calculations. Usually, one party will receive fixed flows (the swap rate) in exchange for making floating payments (according to JIBAR). In South Africa, interest rate swaps are quoted form 1 to 25 years, with the most liquid swaps being in the 1 to 10 years area. The most popular interest rate reset period is for 3 months, but reset periods can also be 1, 6 or 12 months. Settlement is against the relevant JIBAR rate. On every reset date the agreed swap rate and the JIBAR rate are compared, and if there is a difference between these rates, the settlement is made at the end of the reset period.

3.7 Estimating the Risk-Free Yield Curve

The Fundamental Theorem of Asset Pricing [see Dybvig and Ross (1989), for example] implies that in a world of certain cash flows, $c(t)$, and frictionless markets, absence of arbitrage is equivalent to the existence of a linear pricing rule, $\delta(t) > 0 \ \forall t$, such that

$$P = \sum_{t=1}^{T} c(t)\delta(t) \tag{3.24}$$

If markets are incomplete, there exists multiple sets of $\delta(t)$ which satisfy this equation. In the term structure of interest rates literature, $\delta(t)$, called the "discount function" is usually transformed into a zero coupon curve by $r(t) = -\ln\delta(t)/t$ The discount function $\delta(t)$ is the current price of a risk-free zero coupon bond paying one unit of currency at time t. Clearly if we exclude the possibility of negative interest rates we must have the following for a discount function $\delta : [0,\infty) \longmapsto [1,0)$:

$$\begin{aligned} \delta(0) &= 1, \\ \delta(t_1) < \delta(t_2) &\iff t_1 > t_2 \end{aligned}$$

Estimating the risk-free yield curve requires three decisions:

1. A pricing function relating instrument market prices, P_i, to the discount rate function, $r(t_j)$, via promised cash flows, c_i at time t_j, for $1 \leq j \leq K$.

2. A functional form to be used to approximate the yield curve function, $r(t)$, or the discount function, $\delta(t)$.

3. An econometric method for estimating the parameters of the yield curve function.

3.7.1 Pricing function

The simplest pricing function, appropriate to a world with complete markets and no taxes or transaction costs, is just the present value of the promised cash flows:

$$\begin{aligned} P_j &= \sum_{j=1}^{K} c_j \delta(t_j) \\ &= \sum_{j=1}^{K} c_j \exp(-t_j r(t_j)) \end{aligned} \tag{3.25}$$

Let $\{B_i\}_{1 \leq i \leq N}$ be a set of observed market instruments, let $\tau_1 < \tau_2 < \ldots < \tau_K$ be the set of dates at which cash flows occur, let $c_{i,j}$ be the cash flow of the i^{th} instrument on date τ_j, and let P_i be the market price of the i^{th} instrument. The pricing function becomes

$$P_i = \hat{P}_i + \varepsilon_i \tag{3.26}$$

where $\hat{P}_i$ is defined by

$$\begin{aligned} \hat{P}_i &= \sum_{j=1}^{K} c_{i,j} \delta(\tau_j) \\ &= \sum_{j=1}^{K} \exp(-\tau_j r(\tau_j)) \end{aligned} \tag{3.27}$$

Since equation (3.25) omits such obvious factors as taxes and liquidity, the error term, ε_i, will contain both systematic and random factors.

3.7.2 Approximating function

After deciding on the appropriate pricing function, the next step is to decide on the functional form to be used to approximate the yield curve function $r(t)$ or the discount function, $\delta(t)$. It is not possible to estimate the value of the yield curve at each possible horizon as the number of cash flows points will usually exceed the number of available instruments. The usual practice is to select an approximating function and then estimate the parameters of this function. Examples of approximating functions include polynomials, cubic splines, step functions, piecewise linear and exponential forms.

Given a proposed yield curve function $\hat{r}_\psi(t)$ such that

$$\hat{r}_\psi(\tau_j^*) = \hat{r}_j^* \ \forall j = 1, \ldots, K$$

the resultant theoretical price for the i^{th} instrument is, from equation (3.25)

$$\hat{P}_i(\psi) = \sum_{j=1}^{K} c_{i,j} \exp(-\tau_j r_\psi(\tau_j)) \tag{3.28}$$

The yield curve function, ψ, could be chosen such that it minimizes the objective function

$$E_\psi = \sum_{i=1}^{n} (P_i - \hat{P}_i(\psi))^2 \tag{3.29}$$

The problem of solving for the optimal representation of the "true" yield curve becomes an exercise in finding the most efficient technique for choosing $\hat{r}_{\psi+1}(\tau)$ such that

$$E_{\psi+1} < E_\psi$$

and that convergence occurs "rapidly enough".

3.7.3 Estimation method

Lastly, the method of approximating the parameters of the approximating function must be selected. Methods used in the past include weighted least squares, maximum likelihood and linear programming. Related discussions include error weighting functions and how to handle the bid-ask spread (usually by collapsing the bid and ask quotes into a single price by taking their mean).

3.8 Summary

According to Vasicek and Fong (1982), the objective when estimating the term structure empirically is to fit a zero-coupon yield curve that both fits the data sufficiently well and is a smooth function. In this chapter, we introduced some techniques for determination of the zero-coupon yield curve that have these requirements as their objective. We have shown that the modelling is difficult, and in general not computationally straightforward or unique. In South Africa, *bootstrapping* is a popular technique for determining the zero coupon yield curve. The fundamental idea behind bootstrapping is to discount the coupons prior to maturity from a bond using the zero coupon rates already determined from money market instruments. The zero-coupon rate for a specific term obtained this way is then used in the bootstrap process for the next bond. In this way rates for longer and longer periods are obtained and these rates are then approximated by a curve.

The problem with the bootstrap procedure described above is the assumption of the existence of a complete series of regularly spaced coupon bearing bonds - this is not the case in South Africa. Also, according to Smit & van Niekerk (1997), the commonly used approximating functions such as polynomials and cubic splines are not always suitable for the South Africa yield curve due to structural inefficiencies in the fixed-income market and the resultant dispersion of data points. The problem of yield curve determination, especially in a sparse and illiquid market such as South Africa, is not trivial and represents significant opportunities for research for students of financial economics.

Chapter 4

Overview of Structural Models

4.1 Introduction

The key characteristic shared by structural models is their reliance on economic arguments for why firms default (e.g. the firm's value does not cover its obligations). These economic models provide the framework to derive a relationship between defaultable debt prices (or credit spreads) and market variables. Our investigation focuses on two structural models that are designed to price default-risky bonds: Merton (1974), and Longstaff and Schwartz (1995). We chose to investigate these two models because of their analytical tractability and the fact that the Longstaff and Schwartz model combines many distinctive features of other models. Like Merton, they assume that the firm values follows a diffusion process, as in Black and Cox (1979), they allow for early default before maturity of default-risky debt and as in Shimko, Tejima and van Deventer (1993), the riskless rate is assumed to follow the Vasicek (1977), model. Before reviewing these models, we list below the main issues that need characterization in a structural model.

1. Asset value process.

2. Issuer's capital structure.

3. Recovery process.

4. Terms and conditions of the debt issue.

5. Default-risk-free interest rate process.

6. Correlation between the default-risk-free interest rate and the asset price.

7. Correlation between interest rate risk and default risk.

4.2 Merton (1974)

Beginning with the groundbreaking Black-Scholes (1973), insight that the debt of a firm can be viewed as a contingent claim on the assets of the firm, Merton provided one of the first in-depth valuation models for default-risky bonds. The contingent claims approach requires the specification of three processes. First, a process for the total asset value process of the firm has to be explicitly modelled. Second, the bankruptcy process has to be modelled completely. That is, the "when" and "how" of bankruptcy have to be made explicit. Third, the payoffs to creditors in the event of default have to be specified in detail.

The following assumptions were made in Merton's valuation framework:

1. Riskless interest rate is constant, i.e. $r(t) = r \ \forall t \geq 0$;

2. Firm value V dynamics: $dV_t = \mu V_t dt + \sigma V_t dW_t, V_0 > 0$; μ is the instantaneous expected rate of return on the firm per unit time; σ^2 is the instantaneous variance of the return on the firm per unit time; W is a standard Gauss-Wiener process;

	Assets	Bonds	Liabilities
No default	$V_T \geq B$	B	$V_T - B$
Default	$V_T < B$	V_T	0

Table 4.1: Payoffs to the firm's liabilities at maturity

3. Firm has a single outstanding issue of debt promising B at T. Default occurs when $V_T < B$. Debt covenants grant bondholders *absolute priority*: in the event of default, bondholders get the entire firm and the shareholders get nothing.

Merton also assumes that the firm is neither allowed to repurchase shares nor to issue any new senior or equivalent claims on the firm. This assumption implies that at the debt's maturity T we have the payoffs in Table 4.1 above to the firm's liabilities.

If at time T the asset value V_T exceeds or equals the face value B of the bonds, the bondholders will receive their promised payment B and the shareholders will get the remaining $V_T - B$. However, if the value of assets V_T is less than B, the ownership of the firm will be transferred to the bondholders. Equity is then worthless (because of limited liability of equity, the shareholders cannot be forced to pay the shortfall $B - V_T$). Summarizing, the value of the default-risky bond issue $f(V_T, T)$ at time T is given by

$$f(V_T, T) = \min(B, V_T) = B - \max(0, B - V_T) \tag{4.1}$$

which is equivalent to that of a portfolio composed of a default-free loan with face value B maturing at time T and a short European put position on the assets of the firm V with strike B and maturity T. The value of the equity E_T at time T is given by

$$E_T = \max(0, V_T - B) \tag{4.2}$$

which is equivalent to the payoff of a European call option on the assets of the firm V with strike B and maturity T. With the payoff specifications just described, we are able to value corporate liabilities as contingent claims on the firm's assets.

At this point, Merton (1974), considers the formation of a zero net investment portfolio consisting of a claim whose price is the value of the assets of the firm, the debt of the firm, and the riskless debt. These are held in proportions such that the return on the portfolio is deterministic and the portfolio requires zero net investment. The expected rate on the portfolio must be zero to avoid arbitrage. This condition is sufficient to derive the PDE that the price of any contingent claim on V must satisfy. Since Merton's paper, Harrison and Kreps (1979), and Harrison and Pliska (1981), developed martingale methods for pricing derivatives. Instead of following Merton's PDE method to derive the closed form solution for the price at time t, $f(V_t, T)$ of the default-risky bond, we will follow the general martingale pricing techniques outlined in Musiela and Rutkowski (1998). The aim of introducing the martingale measure is twofold: firstly, it simplifies the explicit evaluation of arbitrage prices of derivative securities; secondly, it describes the arbitrage-free property of a given pricing model for primary securities in terms of the behaviour of relative prices.

Taking as given some risk-free short rate process r_t, we suppose that there is a security with value $\beta_t = \exp\left(\int_0^t r_s ds\right)$ at time t, which provides a riskless investment opportunity. Assuming that there are no arbitrage opportunities in the financial market, modelled by some probability space (Ω, F, P)[1], there exists a probability measure $\widetilde{Q}$, such that the processes of security prices, discounted with respect to β, are $\widetilde{Q}$-martingales (Harrison and Kreps (1979), and Harrison and Pliska (1981)). $\widetilde{Q}$ is called the equivalent martingale measure, and we let $\widetilde{E}[\cdot]$ denote the corresponding expectation operator. This gives us

$$
\begin{aligned}
\frac{f(V_t, T)}{e^{\int_0^t r ds}} &= \widetilde{E}\left[\frac{\min(V_T, B)}{e^{\int_0^T r ds}} \mid F_t\right] \\
&= \widetilde{E}\left[\frac{B - \max(B - V_T, 0)}{e^{\int_0^T r ds}} \mid F_t\right]
\end{aligned}
$$

Therefore,

[1] It is customary in financial models to regard F_t as a model for all the information available to agents at time t.

$$
\begin{aligned}
f(V_t, T) &= \widetilde{E}\left[\frac{B - \max(B - V_T, 0)}{e^{\int_t^T r\,ds}} \mid F_t\right] \\
&= \widetilde{E}\left[\frac{B}{e^{\int_t^T r\,ds}}\right] - \widetilde{E}\left[\frac{\max(B - V_T, 0)}{e^{\int_t^T r\,ds}} \mid F_t\right] \\
&= Be^{-r(T-t)} - \widetilde{E}\left[\frac{\max(B - V_T, 0)}{e^{\int_t^T r\,ds}} \mid F_t\right] \qquad (4.3)
\end{aligned}
$$

Merton (1974), using the Black-Scholes (1973), insight that the debt of a firm can be viewed as a contingent claim on the assets of the firm, observed that

$$
\widetilde{E}\left[\frac{\max(B - V_T, 0)}{e^{\int_t^T r\,ds}} \mid F_t\right] \qquad (4.4)
$$

is the value of a European put option on the assets of the firm with strike B and maturity T. Thus by the Black-Scholes formula, we have

$$
\widetilde{E}\left[\frac{\max(B - V_T, 0)}{e^{\int_t^T r\,ds}} \mid F_t\right] = Be^{-r(T-t)}\Phi(-d_2) - V_t\Phi(-d_1) \qquad (4.5)
$$

where $\Phi(x)$=standard normal cumulative distribution function,

$$
\begin{aligned}
d_1 &= \frac{\ln(V_t/B) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\ln(V_t/B) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
&= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

Substitution of equation (4.5) into equation (4.3) gives

$$
\begin{aligned}
f(V_t, T) &= Be^{-r(T-t)} - Be^{-r(T-t)}\Phi(-d_2) + V_t\Phi(-d_1) \qquad (4.6) \\
&= Be^{-r(T-t)}\left[(1 - \Phi(-d_2)) + \Phi(-d_2)\left(\frac{V_t\Phi(-d_1)}{Be^{-r(T-t)}\Phi(-d_2)}\right)\right] \\
&= Be^{-r(T-t)}[(1 - \Phi(-d_2)) + \Phi(-d_2)\delta] \qquad (4.7)
\end{aligned}
$$

where

$$\delta = \frac{V_t\Phi(-d_1)}{Be^{-r(T-t)}\Phi(-d_2)} \tag{4.8}$$

Proposition 1 *Firm Value Dynamics*
The SDE for firm value V dynamics:

$$\begin{aligned} dV_t &= \mu V_t dt + \sigma V_t dW_t \\ V_0 &> 0 \end{aligned} \tag{4.9}$$

has a unique solution given by

$$V_t = V_0 e^{mt+\sigma W_t} \tag{4.10}$$

where

$$m = \mu - \frac{1}{2}\sigma^2 \tag{4.11}$$

Proof.
We consider the process $X_t = \mu t + \sigma W_t$. Clearly, this is a solution to $dX_t = \mu dt + \sigma dW_t$. After making the transformation $Y = e^X$, an application of Itô's lemma gives the SDE for Y, $dY_t = Y_t(\mu + \frac{1}{2}\sigma^2)dt + Y_t\sigma dW_t$. Now we consider the process $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$ and make the same transformation. Itô's lemma confirms that $dY_t = Y_t\mu dt + Y_t\sigma dW_t$.

Using equation (4.10), we can explicitly write down the actual default probability . From the definition of default,

$$\begin{aligned} p &= P[V_T < B] \\ &= P[V_0 e^{mT+\sigma W_T} < B] \\ &= P\left[mT + \sigma W_T \leq \ln\frac{B}{V_0}\right] \\ &= P\left[W_T < \left(\frac{\ln(\frac{B}{V_0}) - mT}{\sigma}\right)\right] \\ &= \Phi\left(\frac{\ln(B/V_0) - mT}{\sigma\sqrt{T}}\right) \end{aligned} \tag{4.12}$$

The result in equation (4.11) uses the fact that W_T is normally distributed with mean zero and variance T.[2] Setting $\mu = r$ in equation (4.11), gives the risk-neutral default probability $\widetilde{p}$

$$\widetilde{p} = \widetilde{P}[V_T < B] = \Phi(-d_2) = 1 - \Phi(d_2) \tag{4.13}$$

We can now interpret the equation for defaultable debt (4.7) in an intuitive way. The value of defaultable debt is the value of otherwise similar, default-risk-free debt times the risk-neutral probability of no default plus the payoff in the case of default times the risk-neutral probability of default. δ is the implied recovery rate in the Merton model.

Defining $s(V_t, T)$ as the spread above the risk-free rate at which the debt trades at t, we can rewrite equation (4.7) for $t < T$ as

$$f(V_t, T) = B\exp(-(r + s(V, t))(T - t)) \tag{4.14}$$

where

$$\begin{aligned} s(V_t, T) &= -\frac{1}{T-t}\ln\left[\frac{1}{d}\Phi(-d_1) + \Phi(d_2)\right] \\ d &= \frac{B}{V_t}e^{-r(T-t)} \end{aligned} \tag{4.15}$$

d is the discounted debt-to-asset value ratio, which can be considered as a measure of the firm's leverage. Equation (4.15) defines a term structure of credit risk[3], which depends on the time to maturity of the debt, firm's asset volatility σ (the firm's business risk), and leverage d. In Merton's model, the credit spread increases as the leverage of the firm rises. This increase in the credit spread is natural because increased leverage heightens the probability that the firm may default. Higher default probability is reflected in an increase in the credit spread. Similarly, a rise in the volatility of the firm's value increases the probability that the firm may default, thus expanding the credit spread.

Furthermore,

[2] See Section 3.1.1 for properties of the standard Brownian motion.

[3] The term structure of credit risk is also called the risk structure of interest rates, the term structure of credit spreads or the risky term structure.

$$\lim_{h \downarrow 0} s(V_t, t+h) = 0 \tag{4.16}$$

This follows from the fact that

$$\begin{aligned} \lim_{(T-t)\to 0} d_1 &= \infty \\ \lim_{(T-t)\to 0} d_2 &= \infty \end{aligned}$$

and from standard properties of the normal distribution that state that

$$\begin{aligned} \Phi(+\infty) &= 1 \\ \Phi(-\infty) &= 0 \end{aligned}$$

From equation (4.16), we see that credit spreads for maturities going to zero are zero. Zero short spreads mean that default-risky bond investors do not demand a risk premium for assuming the default risk of an issuer, as long as the time to maturity is sufficiently short. This feature is not consistent with what is observed in the market. In the market, we see non-zero credit spreads for nearly all default-risky bonds regardless of maturity.

Despite its simplicity and intuitive appeal, Merton's model has many limitations. First, the credit spreads derived from the model are significantly lower than those implied by empirical evidence (Mason, Jones and Rosenfeld (1984)). That is, Merton's model underprices credit risk. Second, in the model the firm defaults only at maturity of the debt, a scenario that is at odds with reality. Also, most firms have complicated capital structures made up of a variety of security types, as opposed to a single debt issue. The Merton framework assumes that the absolute-priority rules are actually adhered to upon default in that debts are paid off in their order of seniority. However, empirical evidence (Franks and Torous (1989),(1994)) indicates that the priority rule is often violated.

Yet another problem with the Merton model is that the value of the firm, which is an input to the valuation model is difficult to ascertain since not all the firm's assets are either tradable or observable. These real life complications make the Merton framework less useful as a tool. However, it does not decrease the intuition behind modelling the default process. Merton's framework has spawned an enormous theoretical literature on defaultable debt pricing. Some examples are Black and Cox (1979), Kim, Ramaswamy and Sundaresan (1993), Shimko, Tejima and van Deventer (1993), Leland (1994), Longstaff and Schwartz (1995), Leland and Toft (1996), and Saá-Requejo and Santa-Clara (1999). The Merton model has also been loosely implemented in a commercial package which is marketed by KMV corporation. The KMV model draws its main strength from a judicious (but not-model consistent) use of a large database of historical defaults.

4.3 Longstaff and Schwartz (1995)

Longstaff and Schwartz (hereafter, LS) provide closed form expressions for the value of risky fixed and floating rate debt. LS address some of the weaknesses of the Merton model. In a way similar to Merton, LS assumed that the value of the firm follows a diffusion process

$$dV = \mu V\,dt + \sigma V\,dZ_1 \tag{4.17}$$

where σ is a constant, μ is the rate of return on the underlying asset value and Z_1 is a standard Wiener process. In contrast to Merton's assumption of constant interest rates, LS postulated that the short-term rate follow the mean-reverting Ornstein-Uhlenbeck process first used by Vasicek (1977).

$$dr = (\gamma - \beta r)dt + \eta dZ_2 \tag{4.18}$$

where γ, β and η are constants and Z_2 is another standard Wiener process. The authors chose the Vasicek model for the short-term rate because it incorporates mean reversion and facilitates the use of closed form solutions. A more general short-term rate model would require that defaultable debt prices be solved numerically. The instantaneous correlation between Z_1 and Z_2 is ρdt, i.e.,

$$dZ_1 dZ_2 = \rho dt \tag{4.19}$$

Priority of claim	Altman Study: 1985-1991 ω	Frank and Torous Study:1983-1990 ω
Bank Debt		0.136
Secured Debt	0.395	0.199
Senior Debt	0.477	0.530
Senior Subordinated Debt	0.693	
Cash-Pay Subordinated Junior Debt	0.72	0.711
Non-Cash-Pay Subordinated Debt	0.805	

Table 4.2: Historical values of ω from various bond classes

LS also assert that strict absolute priority to claims is rarely upheld in distressed organizations, which also differs from the Merton model. The model allows for a variety of liability classes with different coupon rates, priorities and maturity dates.

LS then assumed the existence of a (constant) threshold value of the firm, K, which serves as a financial distress boundary; if the value of the assets breaches this level, default is triggered (on all outstanding obligations), some form of restructuring occurs and the remaining assets of the firm are allocated among the firm's claimants. Thus contrary to Merton's model, default can occur prior to maturity. LS simplify their analysis by postulating that it is the ratio of V to K, rather that the absolute value of which governs financial distress and call this ratio X.

If a reorganization occurs during the life of a security, the security holder receives $1-\omega$ times the face value of the security at maturity, where ω represents the write-down on a particular security and is constant over all instruments issued by the firm. This type of payoff would be consistent with a reorganization which provided new securities in exchange for old claims. The model thus avoids the dependence of the payoff on the debt on underlying asset value. Values of ω can be obtained from historical information on various classes of bonds. The authors site two such studies:(see Table 4.2 above)

For completeness, we now use the assumption of perfect, frictionless markets in which trading takes place continuously to derive the fundamental PDE that the price of any derivative asset $H(V, r, t)$must follow. We will derive this PDE using Merton's derivation of the Black-Scholes model presented in Merton (1974).

Let $P_1(r_t, t)$ and $P_2(r_t, t)$ be the prices of two zero coupon bonds with different maturities. Then by applying Itô's lemma and using equation (4.18) we have

$$\begin{aligned} dP_i(r_t,t) &= \underbrace{\left((\gamma-\beta r)\frac{\partial P_i}{\partial r}+\frac{\partial P_i}{\partial t}+\frac{1}{2}\eta^2\frac{\partial_i^P}{\partial r^2}\right)}_{\mu_{P_i}P_i}dt+\underbrace{\left(\eta\frac{\partial P_i}{\partial r}\right)}_{\sigma_{P_i}P_i}dZ_2 \\ &= \mu_{P_i}P_i dt+\sigma_{P_i}P_i dZ_2 \end{aligned} \tag{4.20}$$

We now consider forming a riskless portfolio of these bonds. Let X_1 and X_2 be our holdings of P_1 and P_2 respectively. From equation (4.20) we have

$$d[X_1P_1+X_2P_2]=(X_1\mu_{P_1}P_1+X_2\mu_{P_2}P_2)dt+(X_1\sigma_{P_1}P_1+X_2\sigma_{P_2}P_2)dZ_2 \tag{4.21}$$

To eliminate interest rate risk, we now choose X_i such that

$$X_1\sigma_{P_1}P_1+X_2\sigma_{P_2}P_2=0 \Longrightarrow -X_2P_2=\frac{X_1\sigma_{P_1P_1}}{\sigma_{P_2}} \tag{4.22}$$

We have a riskless portfolio and thus

$$\begin{aligned} X_1\mu_{P_1}P_1+X_2\mu_{P_2}P_2 &= r(X_1P_1+X_2P_2) \\ \Longrightarrow X_1\mu_{P_1}P_1-\mu_{P_2}\left(\frac{X_1\sigma_{P_1}P_1}{\sigma_{P_2}}\right) &= rX_1P_1-r\left(\frac{X_1\sigma_{P_1}P_1}{\sigma_{P_2}}\right) \\ \Longrightarrow \mu_{P_1}-\mu_{P_2}\left(\frac{\sigma_{P_1}}{\sigma_{P_2}}\right) &= r-r\left(\frac{\sigma_{P_1}}{\sigma_{P_2}}\right) \\ \Longrightarrow \frac{\mu_{P_1}-r}{\sigma_{P_1}} &= \frac{\mu_{P_2-r}}{\sigma_{P_2}} \end{aligned} \tag{4.23}$$

Therefore

$$\frac{\mu_{P_1}(t)-r}{\sigma_{P_1}(t)}=\frac{\mu_{P_2(t)-r}}{\sigma_{P_2}(t)}=\lambda(r_t,t) \tag{4.24}$$

where $\lambda(r_t, t)$ must therefore be independent of the bond's maturity. Market participants commonly refer to as the *market price of interest rate risk*.

If $H(V,r,t)$ is the price of any derivative security contingent on V, r and t, then by applying Itô's lemma to H and using equations (4.17) and (4.18) we have

$$\begin{aligned}
dH(V,r,t) &= H_t dt + H_V dV + H_r dr \\
&\quad + \left[\frac{1}{2}H_{VV}(dV)^2 + H_{Vr}(drdV) + \frac{1}{2}H_{rr}(dr)^2\right] \\
&= H_t dt + H_V(\mu V dt + \sigma V dZ_1) + H_r((\gamma - \beta r)dt + \eta dZ_2) \\
&\quad + \left(\frac{\sigma^2}{2}V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2}H_{rr}\right) dt \\
&= \underbrace{\left(H_t + \mu V H_V + (\gamma - \beta r)H_r + \frac{\sigma^2}{2}V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2}H_{rr}\right) dt}_{\mu_H H} \\
&\quad + \sigma V H_V dZ_1 + \eta H_r dZ_2 \\
&= \mu_H H dt + \sigma V H_V dZ_1 + \eta H_r dZ_2
\end{aligned} \tag{4.25}$$

We now impose no-arbitrage conditions by selecting a portfolio such that the interest rate risk and asset risk are eliminated by taking positions in the underlying asset and the risk free zero coupon bond. Assume the riskless portfolio includes X_H, X_P and X_V units of the derivative security, zero coupon bond and the firm respectively. Once again by Itô we have

$$\begin{aligned}
d[X_H H + X_P P + X_V V] &= (X_H \mu_H H + X_P \mu_P P + X_V \mu_V V)dt \\
&+ (X_H H_V + X_V)\sigma V dZ_1 \\
&+ (X_H \eta H_r + X_P \sigma_P P)dZ_2
\end{aligned}$$

We now choose X_H, X_P and X_V such that we get a riskless portfolio

$$\begin{aligned}
X_H H_V + X_V = 0 &\implies X_V = -X_H H_V \\
X_H \eta H_r + X_P \sigma_P P = 0 &\implies X_P P = -\frac{\eta}{\sigma_P} X_H H_r
\end{aligned}$$

Standardizing to $X_H = 1$ gives

$$X_V = -H_V \tag{4.26}$$

and

$$X_P P = -\frac{\eta}{\sigma_P} H_r \tag{4.27}$$

Once the portfolio has been made riskless, the instantaneous return on the portfolio must equal the risk-free instantaneous interest rate. Therefore

$$\begin{aligned}
(X_H H + X_P P + X_V V)r &= X_H \mu_H H + X_P \mu_P P + X_V \mu_V V r H \\
&\quad - \frac{\eta}{\sigma_P} r H_r - r V H_V \\
&= H_t + \mu_V V H_V + (\gamma - \beta r) H_r + \frac{\sigma^2}{2} V^2 H_{VV} \\
&\quad + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2} H_{rr} - \frac{\eta}{\sigma_P} \mu_P H_r - \mu_V V H_V \\
0 &= H_t - rH + \left((\gamma - \beta r) - \eta \left(\frac{\mu_P - r}{\sigma_P} \right) \right) H_r \\
&\quad + r V H_V + \frac{\sigma^2}{2} V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2} H_{rr}
\end{aligned}$$

This leads to the following equation;

$$\frac{\sigma^2}{2} V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2} H_{rr} + r V H_V + (\alpha - \beta r) H_r - rH + H_t = 0 \tag{4.28}$$

where

$$\begin{aligned}
\alpha &= \gamma - \lambda \\
\lambda &= \eta \frac{\mu_P - r}{\sigma_P}
\end{aligned}$$

λ is the adjusted market price of interest rate risk.

Equation (4.28) is the fundamental PDE defining the price of any derivative security contingent on V, r and t. The value of the derivative security is obtained by solving equation (4.28) subject to the appropriate maturity condition.

Vasicek (1977), showed that the price of a riskless discount bond $D(r, T)$ when the dynamics of r are given by equation (4.18) is given by

$$D(r, T) = \exp(A(T) - B(T) r) \tag{4.29}$$

where

$$\begin{aligned} A(T) &= \left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta}\right)T + \left(\frac{\eta^2}{\beta^3} - \frac{\alpha}{\beta^2}\right)(\exp(-\beta T) - 1) \\ &\quad - \left(\frac{\eta^2}{4\beta^3}\right)\exp((-2\beta T) - 1) \\ B(T) &= \frac{1 - \exp(-\beta T)}{\beta} \end{aligned}$$

Given the LS framework, the value of a default-risky discount bond[4] is the solution to equation (4.28) with $H(V, r, T) = P(X, r, T)$ and $X = V/K$, subject to the following maturity payoff:

$$1 - \omega I(\tau \leq T) \tag{4.30}$$

where $I(\cdot)$ is an indicator function taking the value one if the first passage time τ of V to K is less than or equal to T, and zero otherwise. Based on the above assumptions, the value of a default-risky discount bond can be written as

$$P(X, r, T) = D(r, T)(1 - \omega Q(X, r, T)) \tag{4.31}$$

where $X = V/K$, $D(r, T)$ is the value of a riskless discount bond under the Vasicek model and $Q(X, r, T) = \widetilde{E}[I(\tau \leq T)]$ is the probability that the first passage time of $\ln X$ to zero is less than T, where the expectation is taken with respect to the risk-adjusted processes

$$d\ln X = \left(r - \frac{\sigma^2}{2} - \rho\sigma\eta B(T-t)\right)dt + \sigma dZ_1 \tag{4.32}$$

$$dr = (\alpha - \beta r - \eta^2 B(T-t))dt + \eta dZ_2 \tag{4.33}$$

Unfortunately, there is no known closed-form solution for $Q(X, r, T)$ when interest rates are stochastic, so LS proposed a numerical solution that is based on an implicit formula for the first passage density due to Buonocore, Nobile and Ricciardi (1987). The first passage density of $\ln X$ to zero at time τ starting from $\ln X$ at time zero, $q(0, \tau | \ln X, 0)$, is defined implicitly by the integral equation

[4] In this dissertation, we use the terms discount bond and zero coupon bond interchangeably.

$$\Phi\left(\frac{-\ln X - M(t,T)}{S(t)}\right) = \int_0^t q(0,\tau \mid \ln X, 0)\Phi\left(\frac{M(\tau,T) - M(t,T)}{S(t) - S(T)}\right) d\tau \tag{4.34}$$

where $\tau \leq t \leq T$. To obtain an explicit formula for the first passage density, the authors discretize time into n equal intervals, and define time $t_i = \frac{iT}{n}$ for $\{i = 1, 2, \ldots, n\}$. Discretizing equation (4.34) gives the recursive system for the terms below.

$$Q(X,r,T) = \lim_{n\to\infty} Q(X,r,T,n) \tag{4.35}$$

where

$$Q(X,r,T,n) = \sum_{i=1}^{n} q_i, \tag{4.36}$$

and the q_i are defined recursively by

$$\begin{aligned} q_1 &= \Phi(a_1), \\ q_i &= \Phi(a_i) - \sum_{j=1}^{i-1} q_j \Phi(b_{ij}), i = 2, 3, \ldots, n, \end{aligned}$$

The parameters a_i and b_{ij} are now given by

$$\begin{aligned} a_i &= \frac{-\ln X - M(iT/n, T)}{\sqrt{S(iT/n)}} \\ b_{ij} &= \frac{M(jT/n, T) - M(iT/n, T)}{\sqrt{S(iT/n) - S(jT/n)}} \end{aligned}$$

Here the authors use the functions M and S which are

$$\begin{aligned} M(t,T) &= \left(\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2}\right)t \\ &+ \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3}\right)\exp(-\beta T)\exp((\beta t) - 1) \\ &+ \left(\frac{r}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta}{\beta^3}\right)(1 - \exp(\beta t)) \\ &- \left(\frac{\eta^2}{2\beta^3}\right)\exp(\beta T)(1 - \exp(\beta t)) \end{aligned}$$

and

$$S(t) = \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2\right) - \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)(1-\exp(-\beta t)) + \left(\frac{\eta}{\beta^3}\right)(1-\exp(-2\beta t)) \tag{4.37}$$

LS propose using $n = 200$ as approximation to the infinite sum,

$$\lim_{n\to\infty} \sum_{i=1}^{n} q_i \tag{4.38}$$

The equation for defaultable debt, (4.31), has the intuitive structure that the value of risky debt can be viewed as the difference between the riskless bond and the discount for the default risk of the bond. The term, $\omega D(r, T)$, is the present value of the write-down on the bond in the event of a default. The term, $Q(X, r, T)$, is the risk-neutral probability of default.

According to Rogers (1999), LS's derivation for the price of the default-risky bond is *flawed* because they applied the results of Buonocore, Nobile and Ricciardi (1987), concerning the first-passage distributions of one dimensional diffusions to the log of the discounted firm value, but this process is not a diffusion. Also, Collin-Dufresne and Goldstein (1999), (henceforth, CG) assert that the numerical solution to $Q(X, r, T)$ proposed by LS is only valid for one-factor Markov processes, that is, when interest rates are non-stochastic. This means the LS formula is only an approximation to the true solution to their model. CG derived what they claim to be an efficient algorithm for computing the exact solution to the LS model. They report that the difference between the LS approximation and the exact solution to their model is economically significant for typical parameter values.

In the LS model, default risk is captured by the variable X, so bonds can be valued by conditioning on X directly rather than on the default status of other bonds. This implies that coupon bonds can be valued as the sum of a series of zero coupon bonds. From this model we can see that the price of a default-risky bond is an increasing function of X and a decreasing function of ω and T. Default-risky bonds have shorter durations[5] than their risk free equivalents

[5]The sensitivity of the bond price to changes in r provides a measure of the duration of the bond.

and this property also holds for the LS model. As r increases, $D(X, r, T)$ and $Q(X, r, T)$ become smaller. $Q(X, r, T)$ becomes smaller because the increase in r causes V to drift away from K at a faster rate. The model can also display that the duration of a default-risky discount bond need not be a monotone-increasing function of its maturity. In fact, it can display how the duration can decrease with time, given a moderate level of default risk. Therefore, it is clear that while default-risky bond prices are generally decreasing with increases of r , this can be reversed for extremely defaultable debt.

Findings cited by the authors show that the LS model allows for various term structures of credit spreads for different levels of default risk. The model displays a monotone increasing term structure of credit spreads for bonds with low default risk and a hump shaped structure for bonds with high default risk. Also, the model indicates that there should be a negative relationship between credit spreads and the level of interest rates.

Another important finding of the LS model is that the effect of a firm's correlation with interest rate changes can be very significant in determining the value of its debt. Exogenously specifying the write-down variable, ω, introduces another degree of freedom into the LS model so that it could, in principle, be made to fit any given level of the default spread observed in interest rates. More, problematic, however, is the assumption that ω is a constant. The violation of the absolute priority rule may imply stochastic values of ω, contrary to this assumption. The authors state that their model can easily be extended to allow for unsystematic stochastic values of ω which are uncorrelated with both business and interest rate risks. Setting the default trigger, K, to be a constant is not a satisfactory way of capturing the events that precipitate a firm into bankruptcy. Nevertheless, it precludes the undesirable property of simple versions of Merton's model that, before maturity, firm value can fall significantly below the face value of the bond without triggering default. Ideally, the critical level K should be a function of the liabilities outstanding at each point in time; ω should be stochastic. However, incorporation of such features may sacrifice the model's tractability without providing additional insight into the valuation of defaultable debt.

4.4 The Merton (1974) and LS(1995) Models: A Comparison

The relationship between Merton (1974), and LS (1995), is important since it not only provides a foundation for empirical comparisons, but also relates to some fundamental issues of pricing default-risky bonds. This section provides a comparison of these models. The three major findings are as follows. First, although the Merton model is less general that the LS model in terms of default probability, it is more general in terms of the recovery rate. Second, in both Merton (1974) and LS (1995), the condition triggering a default is not consistent with the no-arbitrage argument. Third, both models are restrictive due to predictable arrival times of default, which implies that the term structure of credit risk has to start from zero.

In LS, default happens when firm value $V(t)$, which follows a diffusion process with a continuous sample path, reaches a constant default threshold K from above. This results in two important features of the LS model. First, it permits default before the maturity date of default-risky debt. As a result, the LS model is more general than the Merton model which permits default only at the maturity date. Second, the probability of default is predictable, i.e., a currently solvent firm cannot default on its debt in the next instantaneous moment. The consequence of this feature is that when the time to maturity goes to zero, the LS model generates a term structure of credit risk that converges to zero too. This is also a restriction of the Merton model.

For comparison purposes, the pricing formulas for default-risky discount bonds under the Merton (1974) and LS (1995) models are given below. From equation (4.7), we have for the Merton model

$$P(V_t, T) = Be^{-r(T-t)}[1 - \Phi(-d_2) + \Phi(-d_2)\delta] \tag{4.39}$$

where

$$\delta = \frac{V_t\Phi(-d_1)}{Be^{-r(T-t)}\Phi(-d_2)} \tag{4.40}$$

is the implied recovery rate. B is the face value of a riskless discount bond and $\Phi(-d_2)$ is the probability of default in a risk neutral world. From equation (4.31), we have for the LS model

$$P(X, r, T) = D(r, T)((1 - Q(X, r, T)) + (1 - \omega)Q(X, r, T)) \quad (4.41)$$

where ω is the write-down proportion of the debt value in case of default, $X = V/K$, $D(r, T)$ is the value of a riskless discount bond under the Vasicek model and $Q(X, r, T)$ is the probability of default.

It is evident from equations (4.39) and (4.41) that the pricing formula for default-risky discount bonds has the same form in both the Merton and LS models. However, there are three key differences between equations (4.39) and (4.41). First, because of the assumption of constant interest rates in the Merton model, the price of a riskless discount bond is simply the present value of the face value of the bond, $Be^{-r(T-t)}$, whereas in the LS model the price of a riskless discount bond, $D(r, T)$, is given by the Vasicek model. Second, the Merton model, has a closed-form solution for the risk neutral probability of default, $\Phi(-d_2)$, whereas in the LS model, the probability of default, $Q(X, r, T)$, can be solved iteratively. Third, in the LS model, the recovery rate of default-risk debt, $1 - \omega$, is an exogenously specified constant whereas in the Merton model the recovery rate of default-risk debt, δ, is stochastic. In other words, the LS model assumes a zero covariance between recovery rate and probability of default. Therefore, the LS model is less general than the Merton model in terms of recovery rate.

Chapter 5

Overview of Reduced-Form Models

5.1 Introduction

Reduced-form models typically take as the basic ingredients the behaviour of default-free interest rates, the fractional recovery of defaultable bonds at default, as well as a stochastic intensity process λ for default. The intensity λ_t may be viewed as the conditional rate of arrival of default. For example, with constant λ, default is a Poisson arrival. In these models, the intensity process and recovery rates are modelled exogenously and hence the need to directly model the assets of the firm and understand the priority structure of the firm's funding is eliminated. The reduced-form models have been implemented in a commercial software package. The model is called *Credit Risk+* and it was developed by Credit Suisse Financial Products as a tool for the portfolio management of credit risk. In this model a default is triggered by the jump of a Poisson process whose intensity is randomly drawn for each debtor class.

A reduced-form model requires characterization of the following:

1. Issuer's default process (and/or corresponding intensity process).

2. Recovery process.

3. Default-risk-free interest rate process.

4. Correlation between the default-risk-free interest rate process and the default process.

Our investigation focuses on two reduced-form models that are designed to price default-risky bonds: Jarrow, Lando and Turnbull (1997), and Duffie and Singleton (1999). Before reviewing these two models, we will develop a pricing formula for a general contingent claim (that also includes the possibility of default), U. Following Duffie and Singleton's development, we define a defaultable claim to be a pair $((X,T),(X',T'))$ where the issuer is obligated to pay X (possibly a random variable) at time T. The second part of this pair says that T' is a (exogenously specified) stopping time at which the issuer defaults and claimholders receive X' (exogenously specified recovery). This means that a contingent claim (Z,τ) generated by a defaultable claim $((X,T),(X',T'))$ is defined by

$$\tau = \min(T,T'); \quad Z = XI(T'>T) + X'I(T' \leq T) \tag{5.1}$$

where τ is a stopping time at which Z is paid.

Under the assumption of arbitrage-free markets, there exists an equivalent martingale pricing measure $\widetilde{P}$ relative to the short-rate process r. We also assume that Z is F_τ measurable (which allows us to assume that Z can be determined given the information up to and including τ). This means, under the pricing measure $\widetilde{P}$, the price process for any contingent claim U described by (Z_τ,τ) is defined by $U_t = 0$ for $t \geq \tau$ and

$$\begin{aligned} \frac{U_t}{e^{\int_0^t r_u du}} &= \widetilde{E}\left[\frac{Z}{e^{\int_0^\tau r_u du}} \mid F_t\right] \\ U_t &= \widetilde{E}\left[e^{-\int_t^\tau r_u du}(XI(T'>T) + X'I(T' \leq T)) \mid F_t\right] \end{aligned}$$

$$
\begin{aligned}
&= \widetilde{E}\left[e^{-\int_t^T r_u du} X I(T' > T) + e^{-\int_t^{T'} r_u du} X' I(T' \leq T) \mid F_t\right] \qquad (5.2) \\
&= \widetilde{E}\left[e^{-\int_t^T r_u du} X I(T' > T) \mid F_t\right] + \widetilde{E}\left[e^{-\int_t^{T'} r_u du} X' I(T' \leq T) \mid F_t\right]
\end{aligned}
$$

When the interest rate process, r_u, the default process T' and the recovery process X' are specified, equation (5.2) fully characterizes the price of the contingent claim U. Also, the differences between reduced-form models are due to their assumptions for the processes followed by these parameters.

5.2 Jarrow, Lando and Turnbull (1997)

Jarrow, Lando and Turnbull (henceforth, JLT) present an arbitrage-free model of credit risk which characterizes the default process as a finite state Markov process in the firm's credit ratings. The authors begin the construction of their model by assuming that the markets for risk-free and risky debt are complete and arbitrage-free. The JLT model has three important characteristics:

- Different seniority debt for a particular firm can be modelled by assuming different recovery rates in the event of default.
- It can be combined with any default-free term structure model.
- Pseudo-probabilities (martingale, risk adjusted) for valuation are determined from historic transition probabilities for different credit rating classes.

For implementation of this model, the authors impose one major simplifying assumption. It is assumed that the process of the default-free term structure and the firm's bankruptcy (or more generally, financial distress) are statistically independent under the pseudo-probabilities. This means the Markov process for credit ratings is independent of the level of interest rates. The authors reference studies that show that while this assumption may hold for investment grade debt, it is not feasible for speculative grade debt.

The authors assume that default-risky discount bonds pay one dollar at maturity if there is no default, and pay $\delta < 1$ dollars at maturity in the event of default. δ represents the recovery rate on the bond and is taken to be an exogenously given constant. Under the JLT model, default-risky discount bonds are valued as follows:

$$v(t,T) = p(t,T)(\delta + (1-\delta)\widetilde{Q}_t(\tau^* > T)) \tag{5.3}$$

where $p(t,T)$ is the time t price of a default-free discount bond, $v(t,T)$ and is the time t price of a default-risky discount bond. τ^* represents the random time at which bankruptcy occurs and $\widetilde{Q}_t(\tau^* > T)$is the probability (under the martingale measure) that default occurs after date T . From equation (5.3) we see that the term structure of default-risky debt will be uniquely determined by specifying a distribution for the time of bankruptcy under the pseudo probabilities. JLT model the distribution of the time of bankruptcy as the first hitting time of a continuous time Markov chain with discrete states that consist of the different credit ratings and default (the absorbing state).

The authors use the following methodology to specify the bankruptcy process. They define a finite state space $S = \{1, \ldots, K\}$, which represents all of the possible classes of credit ratings, with state 1 being the highest, state $K-1$ being the lowest state and state K being the bankruptcy state. Examples of these different rating schemes can be seen in Table 2.1 on page 20 of this dissertation. They then specify a continuous time, time-homogenous Markov chain $\{\eta : 0 \leq t \leq \tau\}$ in terms of its $K \times K$ generator matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_{1,2} & \lambda_{1,3} & \cdots & \lambda_{1,K-1} & \lambda_{1,K} \\ \lambda_{2,1} & \lambda_2 & \lambda_{2,3} & \cdots & \lambda_{2,K-1} & \lambda_{2,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \lambda_{K-1,3} & \cdots & \lambda_{K-1} & \lambda_{K-1,K} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{5.4}$$

where $\lambda_{i,j \geq 0}$ $\forall i, j$ and

$$\lambda_i = -\sum_{i \neq j = 1}^{K} \lambda_{i,j}$$

The off-diagonal terms of the generator matrix, $\lambda_{i,j}$, represent the transition rates of jumping from credit class i to credit class j. To estimate the empirical generator matrix Λ, the authors suggest using historical results from Moody's or Standard & Poor's which are typically quoted in an annual fashion. The last row of zeros implies that bankruptcy (state K) is absorbing (i.e. once you enter it you can never leave). The $K \times K$ $t-$period probability transition matrix (under the real-world measure) for η is given by

$$Q(t) = \exp(t\Lambda) = \sum_{k=0}^{\infty} \frac{(t\Lambda)^k}{k!} \tag{5.5}$$

Although these real-world transition probabilities are Markovian, the transition probabilities under the martingale pricing measure could depend on the entire history of the process up to time t (i.e. non-Markovian). To facilitate empirical estimation and implementation, JLT assume that the transition probabilities are Markovian under the martingale pricing measure. In particular, they assume that the generator matrix under the martingale pricing measure is given by

$$\widetilde{\Lambda}(t) \equiv U(t)\Lambda \tag{5.6}$$

where $U(t) = diag(\mu_1(t), \ldots, \mu_{K-1}(t))$ is a $K \times K$ diagonal matrix whose first $K-1$ entries (corresponding to the $K-1$ credit ratings) are strictly deterministic functions of t that satisfy

$$\int_0^T \mu_i(t)dt < \infty, i = 1, \ldots, K-1 \tag{5.7}$$

Under the assumption in equation (5.6), the credit rating process is still Markov, but it is no longer time-homogenous. Although homogeneity is desirable, the authors make this trade-off so that the model can match any given initial term structure of credit risk spreads. The $\mu_i(t)$ are interpreted as risk premia, that is, the adjustments for risk that transform the actual probabilities into pseudo-probabilities suitable for valuation processes. To estimate the risk premium one could use the market price of the firm's default-risky discount bonds and back-out the implied risk premiums. This method calibrates the model to market prices in much the same way as arbitrage-free models of the risk-free term structure.

The $K \times K$ transition matrix from time t to time T for the Markov chain, η, under the equivalent martingale measure is given as the solution to the Kolgoromov differential equations

$$\frac{\partial \widetilde{Q}(t,T)}{\partial t} = -\widetilde{\Lambda}(t)\widetilde{Q}(t,T); \tag{5.8}$$

$$\frac{\partial \widetilde{Q}(t,T)}{\partial T} = \widetilde{\Lambda}(T)\widetilde{Q}(t,T); \tag{5.9}$$

$$\widetilde{Q}(t,t) = I \tag{5.10}$$

where I is the $K \times K$ identity matrix. We will denote the (i,j)th entry of $\widetilde{Q}(t,T)$ by $\widetilde{q}_{ij}(t,T)$. If we let the firm be in state i at time t, that is $\eta_t = i$, and define $\tau^* = \inf\{s \geq t; \eta_s = K\}$, then we have

$$\widetilde{Q}(\tau^* > T) = \widetilde{Q}[\tau^* > T|\eta_t = i] = \sum_{j \neq K} \widetilde{q}_{i,j}(t,T) = 1 - \widetilde{q}_{i,K}(t,T) \tag{5.11}$$

To facilitate their exposition, JLT assume a *recovery of treasury* (RT) recovery process that is given by

$$\varphi_{\tau^*} = \delta P(\tau^*, T) \tag{5.12}$$

where δ, the recovery rate is an exogenously specified constant and $P(\tau^*, T)$ is the price at time τ^* of an otherwise equivalent, riskless discount bond maturing at time T. Equation (5.12) says that claimholders receive \$1 at time T if default does not occur by T, and otherwise they receive δ dollars at time T. This is equivalent in saying that the claimholders invested the $\delta P(\tau^*, T)$ in a riskless discount bond that matures at time T.

Under these simplifying assumptions, equation (5.2) becomes

$$\begin{aligned}
U_t &= \widetilde{E}\left[e^{-\int_t^T r_u du}(XI(\tau^* > T) + \delta XP(\tau^*, T)I(\tau^* \leq T)) \mid F_t\right] \\
&= \widetilde{E}\left[e^{-\int_t^T r_u du}XI(\tau^* > T) + e^{-\int_t^T r_u du}\delta XP(\tau^*, T)I(\tau^* \leq T) \mid \eta_t = i\right] \\
&= \widetilde{E}\left[e^{-\int_t^T r_u du}XI(\tau^* > T) + e^{-\int_t^T r_u du}\delta XI(\tau^* \leq T) \mid \eta_t = i\right]
\end{aligned}$$

$$\begin{aligned}
&= \widetilde{E}\left[e^{-\int_t^T r_u du} X(I(\tau^* > T) + \delta I(\tau^* \leq T)) \mid \eta_t = i\right] \\
&= \widetilde{E}\left[e^{-\int_t^T r_u du} X(1 - I(\tau^* \leq T) + \delta I(\tau^* \leq T)) \mid \eta_t = i\right] \\
&= \widetilde{E}\left[e^{-\int_t^T r_u du} X(1 - (1-\delta) I(\tau^* \leq T)) \mid \eta_t = i\right] \\
&= \widetilde{E}\left[e^{-\int_t^T r_u du} \mid F_t\right] \widetilde{E}\left[X(1 - (1-\delta) I(\tau^* \leq T)) \mid \eta_t = i\right] \\
&= P(t,T)\widetilde{E}\left[X(1 - (1-\delta) I(\tau^* \leq T)) \mid \eta_t = i\right] \qquad (5.13)
\end{aligned}$$

where the second last equality holds because JLT assume that the process for default and the default-free term structure are independent under the martingale pricing measure. The third equality uses the fact that

$$P(\tau^*, T) = \widetilde{E}\left[e^{-\int_{\tau^*}^T r_u du} \mid F_t\right]$$

If the contingent claim U is a default-risky discount bond (i.e. $X = 1$), and $v^i(t,T)$ is the price of a default-risky discount bond that is now in credit class i , then equation (5.13) becomes

$$\begin{aligned}
v^i(t,T) &= P(t,T)\widetilde{E}\left[X(1 - (1-\delta) I(\tau^* \leq T)) \mid \eta_t = i\right] \\
&= P(t,T)(1 - (1-\delta)\widetilde{E}\left[I(\tau^* \leq T)) \mid \eta_t = i\right]) \\
&= P(t,T)(1 - (1-\delta)\widetilde{Q}\left[\tau^* \leq T \mid \eta_t = i\right]) \\
&= P(t,T)(1 - (1-\delta)(1 - \widetilde{Q}\left[\tau^* > T \mid \eta_t = i\right])) \\
&= P(t,T)(\delta + (1-\delta)\widetilde{Q}\left[\tau^* > T \mid \eta_t = i\right]) \\
&= P(t,T)(\delta + (1-\delta)\widetilde{Q}_t^i\left[\tau^* > T\right]) \qquad (5.14)
\end{aligned}$$

Equation (5.14) indicates that the higher the probability of default not occurring before maturity, the higher the value of the default-risky bond and therefore the lower the credit spread is.

Given that the forward rate for the default-risky discount bond in credit class i is defined by

$$f^i(t,T) \equiv -\frac{\partial}{\partial T} \ln v^i(t,T), \qquad (5.15)$$

equation (5.14) yields

$$
\begin{aligned}
f^i(t,T) &= -\frac{\partial}{\partial T}\ln\left(P(t,T)(\delta+(1-\delta)\widetilde{Q}_t^i[\tau^*>T])\right) \\
&= f(t,T)-\frac{\partial}{\partial T}\ln\left(\delta+(1-\delta)\widetilde{Q}_t^i[\tau^*>T]\right) \\
&= f(t,T)-I(\tau^*>t)\left(\frac{(1-\delta)\frac{\partial}{\partial T}\widetilde{Q}_t^i[\tau^*>T]}{\delta+(1-\delta)\widetilde{Q}_t^i[\tau^*>T]}\right) \\
&= f(t,T)+I(\tau^*>t)\left(\frac{(1-\delta)\lambda_{i,K}\mu_i(t)}{\delta+(1-\delta)\widetilde{Q}_t^i[\tau^*>T]}\right) \qquad (5.16)
\end{aligned}
$$

where

$$\frac{\partial}{\partial T}\widetilde{Q}_t^i(\tau^*>T)=\frac{\partial}{\partial T}\widetilde{Q}_t^i(\tau^*>T)=\lambda_{i,K}\mu_i(t) \qquad (5.17)$$

From the definition of $I(\tau^*>t)$, it follows that in bankruptcy,

$$f^i(t,T)=f(t,T) \qquad (5.18)$$

The credit risk spread on the short rate is given by

$$
\begin{aligned}
r^i(t)-r(t) &= \lim_{T\to t}(f^i(t,T)-f(t,T)) \\
&= \lim_{T\to t}\left(I(\tau^*>t)\left(\frac{(1-\delta)\lambda_{i,K}\mu_i(t)}{\delta+(1-\delta)\widetilde{Q}_t^i[\tau^*>T]}\right)\right) \qquad (5.19) \\
&= I(\tau^*>t)(1-\delta)\lambda_{i,K}\mu_i(t) \qquad (5.20)
\end{aligned}
$$

Equation (5.20) follows from equation (5.19) since

$$\lim_{T\to t}\widetilde{Q}_t^i(\tau^*>T)=1 \qquad (5.21)$$

$\lambda_{i,K}\mu_i(t)$ is the pseudo-probability of default. Contrary to market evidence, equation (5.20) implies that the credit risk premium is identical for all firms in a given credit class (rating category).

The main strengths of the JLT approach to credit risk modelling are its simplicity and computational tractability. The modelling of default based on credit rating transitions is intuitive, explicitly accounts for default risk and is not very computationally intensive. However, this simplicity is achieved through some assumptions whose validity is questionable. Most notable is the assumption of independence of the default process and the process for the default-free short rate under the martingale measure. This assumption certainly does not hold for lower rated bonds. Also, it is hard to believe that all bonds within a given credit rating class have identical transition probabilities. Clearly, some bonds will be more risky than others within a given class. Finally, there is the question of whether the transitions between credit classes are actually governed by a continuous-time Markov chain, since in practice there appears to be a tendency for a firm to continue to fall through changes in credit class. Also, modelling the transitions between credit classes as a continuous-time Markov chain means that the times in rating classes will be exponentially distributed, but more importantly, the probability of a downgrade given that the firm has just experienced one is higher than for a firm that has been in that class for some time. This is not supported by evidence. Therefore, it is clear that the JLT model will not be useful in making investment decisions among bonds of equal credit ratings, although it could be used to back out the relative credit risk imputed by the market. However, this model might be useful in discovering the term structure of credit risk for a given bond issuer and facilitate investment decisions and pricing and hedging of derivatives for that family of bonds.

5.3 Duffie and Singleton (1999)

JLT made some strong assumptions about the independence of the default process and the process for the riskless short rate under the pricing measure, and this led to a neat formula for the price of a default-risky discount bond. Under the JLT model, the default process is governed by a Markov process[1] under the pricing measure.

Unlike JLT, Duffie and Singleton (henceforth, DS) abstract from specifying the details of the default process. They treat default as an unpredictable event governed by an intensity-based or hazard-rate process and focus on the assumption made about the recovery process, which they assume obeys *recovery of*

[1] A Markov process is a stochastic process where only the present value of a variable is relevant for predicting its future.

market value (RMV). Under this assumption, for contingent claim U we have, for default at time τ

$$X' := (1 - L_\tau)U_{\tau-} \tag{5.22}$$

where X' is the payment claimholders receive given a default at time τ and $U_{\tau-} = \lim_{s \uparrow \tau} U_s$ is the price of the contingent claim "just before" default. The DS framework summarized below assumes the existence of the processes L_t, U_t, r_t, and h_t. The distribution of $X \equiv X_T$ under the pricing measure is also taken as given.

- h_t = risk-neutral hazard rate for default at time t
- $h_t \Delta t$ = conditional risk-neutral probability at time t of default over small time interval Δt, given no default before t
- L_t = loss in market value given a default
- $h_t L_t$ = risk-neutral conditional expected loss rate of market value
- r_t = risk-free short rate process
- $R_t = r_t + h_t L_t$ = default-adjusted short rate process

Let $A^t_{\Delta t}$ represent the event of a firm defaulting on its obligation for the first time in the interval $[t, t + \Delta t]$. Then a hazard rate of implies that

$$h_t = \lim_{\Delta t \to 0} \frac{\widetilde{E}[I(A^t_{\Delta t}) \mid F_t]}{\Delta t} \tag{5.23}$$

where $\widetilde{E}[\cdot]$ indicates the expectation under the equivalent martingale measure. One may also think of h_t as the jump arrival intensity at time t (under the equivalent martingale measure) of a Poisson process[2] whose first jump occurs at default.

The fundamental idea behind the hazard rate approach is that default comes by *surprise* (i.e., default involves a sudden loss in market value of an asset) and we only need to model the *intensity* or infinitesimal likelihood of a default. To incorporate this element of surprise, we define a default process that is

[2]See Appendix A

independent of the processes L_t, U_t, r_t, h_t, and of X. Thus, for the default process Λ_t which is 0 before the default and 1 afterward,

That is,

$$\Lambda_t = I(\tau \leq t) \tag{5.24}$$

the intensity process is given by

$$\{h_t(1 - \Lambda_{t-}) : t \geq 0\} \tag{5.25}$$

The assumption by DS that h_t and L_t do not depend on the value U_t of the contingent claim is typical of reduced-form models. The authors also assume that U does not jump at default τ. This means that, although there may be *surprise* jumps in the conditional distribution of the market value of the default-free claim (X, T), h, or L , these surprises occur precisely at the default time with probability zero. At this point, DS apply Itô's formula for jumping processes to the discounted gains process (which is a martingale under the pricing measure) to verify that

$$U(t,T) = \widetilde{E}\left[\exp\left(-\int_t^T (r_u + h_u L_u)du\right) X \mid F_t\right] \tag{5.26}$$

where the discounted gains process G is defined by

$$G_t = \exp\left(-\int_0^t r_u du\right) U_t(1 - \Lambda_t) + \int_0^t \exp\left(-\int_0^u r_s ds\right)(1 - L_u)U_{u-}d\Lambda_u \tag{5.27}$$

Equation (5.27) has the following intuitive meaning. The first term is the discounted price of the claim; the second term is the discounted payout of the claim upon default.

Instead of following DS's development of equation (5.26) using Itô's formula for jumping processes, we will follow Lando's (1998), more intuitive development of equation (5.26).

Lando (1998), showed that for the case of zero recovery, (i.e.,$L_\tau \equiv 1$) the expression of the contingent claim (equation (5.2)) is

$$
\begin{aligned}
U(t,T) &= \widetilde{E}\left[e^{-\int_t^T r_u du} X I(T' > T) + e^{-\int_t^{T'} r_u du} X' I(T' \leq T) \mid F_t\right] \\
&= \widetilde{E}\left[\exp\left(-\int_t^T r_u du\right) X I(\tau > T) \mid F_t\right], \tau = T', t < \tau \\
&= \widetilde{E}\left[\exp\left(-\int_t^T (r_u + h_u) du\right) X \mid F_t\right] \qquad (5.28)
\end{aligned}
$$

This establishes the result (5.26) in the special case $L \equiv 1$. We now use heuristic reasoning to establish the result (5.26) for all L. Suppose that the default time, happens exactly as before, with intensity h_t. Receiving a fraction $1-L_t$ of pre-default value in the event of default (at t) of a contract is equivalent, from a pricing perspective, to receiving the outcome of a lottery in which the full pre-default value is received with probability (under the martingale measure) $1-L_t$ and 0 is received with probability L_t, i.e. the event of default has been retained with probability L_t. This in turn may be viewed as a default process in which there is 0 recovery but where the default intensity has been thinned using the process L, producing a new default intensity of $h_t L_t$. Clearly, this way of thinking does not change the expectation in equation (5.2). However, we now can think of two types of default. Harmless default that occurs with intensity $h_t(1-L_t)$, and lethal default that occurs with intensity $h_t L_t$. As far as valuing the contingent claim prior to default is concerned, we are clearly only interested in lethal defaults, and we therefore price using the intensity of lethal defaults. Equation (5.28) becomes

$$
U(t,T) = \widetilde{E}\left[\exp\left(-\int_t^T (r_u + h_u L_u) du\right) X \mid F_t\right] \qquad (5.29)
$$

By discounting at the adjusted short rate R, both the timing and probability of default, as well as the effects of losses on default are all accounted for. Using this approach, defaultable contingent claims are treated as default-free when they are discounted at the default-adjusted short rate.

The key feature of the DS model is that h_t and L_t are exogenously specified. This allows the authors to derive a term structure model for default-risky debt which can be used in conjunction with common term structure models for risk-free debt such as BDT, Vasicek, CIR, and the HJM approach. The DS model does not allow for the effects of h and L separately since they enter the adjustment for default in the discount rate $R = r + hL$ in the product form hL. While it is clear that hL represents a credit spread between default-risky and risk-free debt, it is not clear what the individual contributions are to this spread. In order to learn more about the hazard and recovery rates in market prices, the loss percentage L could be modelled using historic default recovery rates, such as those in the Longstaff-Schwartz section of this dissertation and the default probability h could be estimated historically by studying the number of defaults within different classes of bonds. Another way to estimate these two parameters would be to back them out of the market prices of derivatives such as default-risky bond options whose payoffs depend nonlinearly on h and L. However, without a wide range of debt securities deriving value from the same issuer (e.g. liquidly traded bonds, credit derivatives), the components of the mean loss rate cannot be estimated separately. Given the paucity of credit data, efficient estimation of each individual parameter in the DS modelling framework can be a daunting task.

By modelling the default-adjusted rate $R_t = r_t + h_t L_t$ instead of the usual short rate r_t, more non-default factors which influence credit spreads may be incorporated in the model. Some of these factors could be due to liquidity, demand and supply, tax costs and embedded options. DS propose that all these non-default factors, or "liquidity" effects, be modelled with a stochastic process l, which represents the fractional carrying cost of the default-risky debt. The new adjusted short rate would then be adjusted for default and liquidity as follows:

$$R = r + hL + l \tag{5.30}$$

To gain insight into the term structure of $hL + l$, we could fit both a defaultable zero curve and a default free zero curve and compare the respective yields. However it will be difficult to infer anything about h, L and l individually. Responding to this, the authors suggest "extracting" information about the mean-loss-rate process hL from defaultable bond prices (before default) to infer the contribution of hL to the credit spread. The idea of relating credit spreads to firm-specific or macroeconomic variables such as stock prices, investor sentiment and capital investment is suggested as one possibility.

Although DS favour a reduced-form of credit risk model, they do mention that a general formula can be given for the hazard rate h_t in terms of the default boundary for assets, the volatility of the underlying asset process V at the default boundary and the risk-neutral conditional distribution of the level of the assets given the history of information available to investors. This brings us back in some fashion to the framework of the structural model where default is triggered by the firm value process.

DS's modelling approach is described in somewhat general terms, but they give various examples of how their framework can be applied to the valuation of default-risky bonds (callable and non-callable) and the pricing of credit derivatives such as credit-spread put options on default-risky bonds. The authors discuss several approaches to pricing default-risky bonds using equation (5.26). For example, one can either parametrize R directly, or parametrize the component processes r, h, and L. Pricing models that focus directly on R combine the effects of the changes in the default-free short rate r and the mean loss-rate process hL on bond prices. In contrast, pricing models that parametrize R and hL separately are able to "extract" information about mean loss rates from historical default-risky bond yields. Alternative specifications of the DS model focus on the different assumptions regarding the processes governing h_t, r_t, L_t and l_t.

The RMV assumption is central to the DS approach to modelling credit risk. We now define two other recovery assumptions before discussing the tractability of the RMV assumption. Let φ_τ denote the amount recovered (for every \$1 of face value owed) if default occurs at time τ. Under the *recovery of face value* (RFV) framework, the creditor receives a fraction $\varphi_\tau = (1 - L_\tau)$ immediately upon default. Under the *recovery of treasury* (RT) framework, the creditor receives a fraction $\varphi_\tau = (1 - L_\tau)P(\tau, T)$ immediately upon default. $P(\tau, T)$ is the time price of an otherwise equivalent, default-free bond.

The RMV assumption is accurate for products such as interest rate swaps, cross-currency swaps and discount bonds. These types of products are usually marked-to-market on a daily basis, and one could expect to receive a fraction of what the product was marked at just prior to default. Indeed, DS comment that "the RMV assumption is well matched to the legal structure of swap contracts in that standard agreements typically call for settlement upon default based on an obligation represented by an otherwise equivalent, non-defaulted, swap".While there may be cases where RT is more realistic than RMV, DS emphasize that under the RT assumption, the computational burden of computing equation (5.2) can be substantial. Largely for this reason, various simplifying assumptions regarding the relationships between h, r, and L have to be made. Finally, DS note that if one assumes liquidation at default and that absolute priority applies, then the RFV assumption may be more realistic since it implies equal recovery for bonds of equal seniority of the same issuer. The main attraction of the RMV model is that it is easier to use, because standard default-free term structure modelling techniques can be applied. The key thing to remember is what simplifications or assumptions one has made, and how this will affect the pricing of real world securities.

Chapter 6

Structural vs. Reduced-Form Models of Default

6.1 Introduction

In this chapter we review the strengths, drawbacks and inherent properties of structural and reduced-form models. In chapters 4 and 5, we showed that because of the intricate properties of default-risky debt, a model characterizing default-risky debt value requires a fair amount of complexity. The reason for this complexity lies in the number of factors driving default-risky debt value coupled with the interaction of these factors. The two main approaches to credit risk modelling (structural and reduced-form) differ in their treatment of the following factors that drive default-risky debt value, together with the interaction between the default process and the default- risk-free rate process.

- Default- risk-free rate process embodied in the short-rate, r_t
- Default process
- Recovery process

The assumptions made regarding these processes and their interactions affect the tractability, simplicity and practical applicability of the different models for default-risky debt. In general, models for pricing default-risky debt can be expressed simply using the following equation:

$$P(r, T, \cdot) = B(r, T) - \delta(\cdot)Q(\cdot)B(r, T) \tag{6.1}$$

where r is the riskless interest rate, T is maturity, $P(\cdot)$ is the price of a default-risky discount bond, $B(\cdot)$ is the price of riskless debt of the same maturity, $Q(\cdot)$ is the pseudo-probability of default and $\delta(\cdot) = 1 - \beta(\cdot)$, where $\beta(\cdot)$ is the recovery rate on default.

Structural models treat the value of the firm as the underlying stochastic process. $\delta(\cdot)$ and $Q(\cdot)$ are written as functions of firm value and the debt claims issued by the firm. While this approach is well-grounded in theory, it has the practical difficulty of being predicated on a difficult to observe stochastic process, the firm value. Reduced-form models treat $\delta(\cdot)$ and $Q(\cdot)$ as stochastic processes, utilizing the information about these functions that is embedded in observed credit spreads and recovery rates, such as in JLT.

Structural models of default posit some dynamics for the firm value process, and assume that there exists a lower threshold (constant or stochastic) which triggers default should firm value ever reach it. In contrast, reduced-form models of default abstract from the firm value process. They effectively assume that default is a jump process, and directly model the probability of such a jump occurring. Simply said, structural models rely on economic arguments of why firms default whereas reduced-form models eliminate the need for an economic explanation of default. The time at which default might occur in a reduced-form model is a random variable. Even in a structural model such as that of Longstaff and Schwartz (1995), the default time is not known in advance because the value of the firm is a random variable. Yet there are technical conditions that make a crucial distinction between the properties of the default time in most structural models and those in reduced-form models. In general, the default time in reduced-form models is more unpredictable than in structural models, where the time of financial distress can be foretold just before it occurs[1]

[1]More precisely, the time of default is predictable under structural models (based on a diffusion process), meaning that there is an increasing sequence of stopping times that converges

The strength of reduced-form models is also their weakness. Divorcing the firm from the intensity process enables modelling default without much information about why the firm defaults. Herein lies the strength. However, modelling default without theoretical guidance runs the risk of both ignoring market information and drawing erroneous conclusions without the tools to discover the appropriate explanation. Herein lies the weakness. The point to remember is that mathematical tractability - not economics - drives the choice of how to specify a reduced-form model.

Structural models for default-risky bonds are well suited if the relationship between prices of different securities issued by the firm is of importance, e.g. for convertible bonds or callable bonds that can be converted into shares when called by the issuer. Furthermore, the model allows the pricing of default-risky bonds directly from fundamentals, from the firm's value. Thus structural models can give a fair price of a default-risky bond as output. Further, questions from corporate finance like optimal capital structure design or the relative powers of shareholders and creditors can be addressed within a structural framework.

The main strength of the structural approach, the orientation towards fundamentals is also the model's weakness. Often it is hard to define a meaningful process for the firm's value, let alone observe it continuously. It can be very hard to calibrate such a firm's value process to market prices. Furthermore the model may very quickly become too complex to analyse in a real-world application. If one were to model the full set of claims on the value of the assets of a medium sized firm one may very well have to price twenty or more classes of claims: from banks, shareholders and private creditors down to workers' wages, taxes and supplier demands. This obviously quickly becomes unfeasible. Another drawback of the structural models is that they cannot incorporate credit-rating changes that occur quite frequently for default-risky bonds. Many default-risky bonds undergo credit downgrades by credit rating agencies before they actually default, and bond prices react to these rating changes either in anticipation or when they occur.

Both structural and reduced-form models cannot readily incorporate financial restructuring that often occurs upon default, such as renegotiating of the

to the default time, and therefore "foretells" the event of default.

terms of the debt contract by extending the maturity or lowering/postponing the promised payments, exchanging the debt for other forms of securities, or some combination of the above. Also, debt restructurings anticipated by the market will be priced into the value of a defaultable bond in ways that none of these models captures.

6.2 Credit Spreads and default Probability

Reduced-form models predict significantly different term structures of credit spreads than structural models. At the short end of the yield curve, structural models predict that the credit spread drops to zero as maturity goes to zero (i.e., upward sloping term structure of credit spreads), while reduced-form models predict that the spreads remain positive. The theoretical prediction that the term structure of credit spreads should be upward sloping at the short end is not an inherent property of the structural model framework, but rather is due to the assumption that the evolution of firm value follows a diffusion process.

Before considering the implications of the two frameworks at the long end of the yield curve, we describe below a simple reduced-form and a simple structural model that we will use in our analysis.

A simple reduced-form model assumes that default is triggered by a Poisson jump process with stochastic intensity h_t . Suppose we have a standard Poisson process N and define the counting process

$$N_t^* \equiv N(h_t) \tag{6.2}$$

where

$$H_t = \int_0^t h_s ds \tag{6.3}$$

The function h is the *intensity or hazard rate* function of the counting process N_t^* . Let $T_1, T_2, \ldots$ denote the arrival times of the jumps by N. We model the time τ of default as the first time that N^* jumps, so we have

$$H_\tau = T_1 \tag{6.4}$$

We now assume that the hazard rate process is defined in some way and then take an *independent* Poisson process N and define τ by way of equation (6.3) and equation (6.4). From this we have immediately, using *iterated expectations*, that the default probability is given by

$$\begin{aligned} P[\tau \leq T] &= P[T_1 \leq H(T)] \\ &= 1 - P[T_1 > H(T)] \\ &= 1 - E[P[T_1 > H(T)] \mid F_T] \\ &= 1 - E[\exp(-H(T))] \\ &= 1 - \underbrace{E\left[\exp\left(-\int_0^T h_s ds\right)\right]}_{\theta} \end{aligned} \tag{6.5}$$

where F is a sigma field with respect to which is h measurable but which is independent of N and θ is the survival probability.

From equation (6.5), we have that

$$P[\tau \leq \infty \mid F_t] = 1 \tag{6.6}$$

That is, reduced-form models predict that default will occur with certainty at some finite date. In other words, the probability of a firm never defaulting is zero in reduced-form models.

In contrast to reduced-form models, standard structural models assume that firm value V_t evolves dynamically as

$$\begin{aligned} \frac{dV_t}{V_t} &= \mu dt + \sigma dW_t \\ V_0 &> 0 \end{aligned} \tag{6.7}$$

where $\mu \in \Re$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W is a standard Brownian motion. For a given default threshold process $D = (D_t)_{t \leq 0}$ with $0 < D_0 < V_0$, the default time τ is given by

$$\tau = \inf\{t > 0 : V_t \leq D_t\} \tag{6.8}$$

so that τ is a random variable valued in $(0, \infty]$.

In order to calculate default probabilities in this model, we define the *running minimum log-asset process* $M = (M_t)_{t\geq 0}$ by

$$M_t = \min_{s\leq T}(ms + \sigma W_s), \tag{6.9}$$

that is, M keeps track of the historic low of the log-asset value. With equation (6.8) we then find for the default probability (using the unique solution for the SDE in (6.7) given by proposition 1 in Section 4.2.)

$$\begin{aligned} P[\tau \leq T] &= P\left[\min_{s\leq T} V_s \leq D_T\right] \\ &= P\left[\min_{s\leq T}(V_0 e^{ms+\sigma W_s}) \leq D_T\right] \\ &= P\left[M_T \leq \ln\left(D_T/V_0\right)\right] \end{aligned} \tag{6.10}$$

That is, the event of default by time T is equivalent to the running minimum log-asset value at time T being below the adjusted default threshold $\ln\left(D_T/V_0\right)$ at time T. Assuming that D is a deterministic function of time and using the fact that the distribution of M_t is inverse Gaussian[2], we have

$$\begin{aligned} P[\tau \leq T] &= 1 - \Phi\left(\frac{mT - \ln\left(D_T/V_0\right)}{\sigma\sqrt{T}}\right) \\ &+ e^{\frac{2m\ln\left(D_T/V_0\right)}{\sigma^2}}\Phi\left(\frac{\ln\left(D_T/V_0\right) + mT}{\sigma\sqrt{T}}\right) \end{aligned} \tag{6.11}$$

where $\Phi(\cdot)$ is the cumulative normal distribution function. If m is positive, which is a common occurrence in practice, then the structural model predicts that the probability of the firm never defaulting is

$$P[\tau > \infty] = 1 - e^{\frac{2m\ln\left(D_T/V_0\right)}{\sigma^2}} \tag{6.12}$$

The probability of the firm never defaulting is positive in structural models of default. The implication of this is that, if the firm does not default relatively 'soon', then structural models predict that the firm value will most likely continue to drift away from the default threshold forever.

[2]To find that distribution, one first calculates the joint distribution of the pair (W_t, M_t^W), where M^W is the running minimum of W, by the reflection principle. Girsanov's theorem is then used to extend to the case of Brownian motion with drift.

6.3 Previous Empirical Research

Despite the rich arrays of theories for pricing default-risky bonds, the empirical literature is rather thin. There is especially little to tell us how well different models perform and what is the nature of the errors they make in predicting credit spreads. Indeed, only a few empirical papers attempt to implement a structural or reduced-form model to test its ability to predict prices or credit spreads. Lack of good bond data, noisiness in even the best bond data, and the apparent inefficiency of the default-risky bond markets contribute to the dearth of good empirical evidence in this area.

6.3.1 Testing Structural Models

The empirical literature on structural bond pricing models is rather small, especially in comparison to the theoretical literature. Partly, this reflects the fact that reliable bond data have only become recently available to academics. The empirical studies fall into two categories: (1) tests of predictions that are generated by the structural models and (2) analyses of the empirical implementation of the models. The first group includes tests of the shape of the term structure of credit spreads and tests related to changes in bond prices. The latter group consists of papers by Jones, Mason and Rosenfeld (1984) and Wei and Guo (1997) and Eom, Hewege and Huang (2000).

Sarig and Warga (1989), estimated the term structure of credit spreads using a small number of zero coupon corporate bonds and zero coupon U.S. Treasuries. They demonstrated curve shapes (slightly upward sloping for investment-grade bonds, humped shaped for lower-grade bonds, and downward sloping for speculative-grade bonds) as predicted by Merton. Helwege and Turner (1999), show that Sarig and Warga's results largely reflect sample selection bias by maturity, and that the term structure of credit spreads facing low-grade issuers is actually mostly upward sloping, if one controls for firm specific credit risk. Helwege and Turner conclude that structural models place too much emphasis on the upside potential of speculative-grade bonds, perhaps through excessively high volatility parameter or from overstating the typical leverage of a speculative-grade bond.

Another aspect of the structural models that has been tested is the implied risk-neutral default rates. Delianedes and Geske (1998), find that rating migrations (using S&P credit ratings) and defaults are detected months before in the equity markets.

The second line of empirical research on structural models of default-risky bond prices involves implementation of the structural model with available data. These studies compare the actual prices in the market with those predicted by the model.

The most extensive attempt at implementation of a structural model is found in Jones, Mason and Rosenfeld (1984) (henceforth JMR). JMR's implementation of the model shows that model prices are too high, or alternatively, that credit spreads from the model are too low relative to those observed in the secondary market. The errors are largest for speculative-grade firms, but they conclude that the Merton model works better for low-grade bonds than high-grade bonds because the Merton model has greater incremental explanatory power for speculative-grade bonds than a naïve model (discounting cash flows at the risk-free rate). They also find that pricing errors are significantly related to maturity, estimated equity volatility, leverage and time period. Later Franks and Torous (1989), confirmed the finding that actual credit spreads were much greater than predicted credit spreads.

Wei and Guo (1997)], implement two structural models to determine their predictive abilities: Merton (1974), and Longstaff and Schwartz (1995). The authors find that neither model is able to predict credit spreads that are statistically equal to those found in the Eurodollar market[3], and the prediction errors are higher the longer the maturity. Wei and Guo draw two conclusions concerning the performance of the two models: (1) the two models have similar powers in predicting spreads and (2) the Longstaff and Schwartz model suffers from the assumption of a constant recovery rate, while benefiting from its more general treatment of the default event that the Merton model. The problem with this study concerns what the spread in the Eurodollar market actually represents. While some portion of that spread undoubtedly compensates for credit risk, other non-credit characteristics likely explain the bulk of this spread.

More recently, Eom, Hewege and Huang (2000), (henceforth EHH), test

[3] The Eurodollar market is largely a market of short-term debt issued by banks.

four structural models to determine their predictive abilities: a naïve one-factor model based on Merton (1974), Geske (1977), Longstaff and Schwartz (1995), and Leland and Toft (1996). EHH find that both the naïve one-factor model and the Geske model underpredict spreads (overprices bonds). In contrast, EHH find that under reasonable assumptions, the Longstaff and Schwartz model generates credit spreads that are too high on average and the Leland and Toft model, under all circumstances predicts excessively high credit spreads.

The conventional wisdom, while praising the theoretical insights into the default process gained from structural models, dismisses them as impractical for actual bond valuation. However, small sample sizes used in some of the empirical research, and doubts about the quality of bond pricing data leave us without conclusive evidence regarding the power of structural models. The resolution of these empirical issues awaits further research.

6.3.2 Testing Reduced-Form Models

Empirical implementation of reduced-form models is still in its infancy. Partly, this is due the fact that these models require that credit spread data accurately reflect market expectations about credit risk, recovery in the event of default, and liquidity. Accurate bond data is difficult to find.[4] The question remains whether reduced-form models can describe the behaviour of default-risky bonds successfully.

So far the papers that attempt to answer this question are by Duffee (1999), Frühwirth and Sögner (2001)(FS henceforth). Duffee estimates the parameters for the stochastic process of the credit spread for the Duffie and Singleton (1999), framework. FS estimate default intensities within the Jarrow and Turnbull (1995) framework. Duffee finds that reduced-form models based on the Duffie and Singleton (1997) framework have difficulty explaining the observed term structure of credit spreads across firms of different credit qualities. For example, the model produces both flat term structures of credit spreads for investment-grade bonds with less default risk and steeper term structures of credit spreads

[4]Duffee (1996) suggests that the firms' observed bond yields that make up his dataset are "flawed" because these yields are traders' indicative bid prices that appear to react very slowly to information in the firms' stock prices.

for investment-grade bonds with relatively more default risk. He also finds that the model's parameter estimates imply that regardless of how much the firm's financial health improves, the firm's credit spread remains positive. This suggests that the model successfully captures a non-default component in credit spreads. On average, the model appears to fit investment-grade, corporate bonds well; term structures of credit spreads for lower quality firms are more steeply sloped than are term structures of credit spreads for higher quality firms. Duffee makes significant strides in implementing this modelling framework; and concludes that "The results here can be used both as benchmarks for models of corporate bond pricing and as directions for future research." FS show that estimated default intensities strongly depend on the date of estimation and the bond. They also show that liquidity has no significant influence. FS also show that there is a statistically significant correlation between default intensity and default-free interest rates and they conclude that "... further research should engage in models where the default rate is a function of some relevant parameters and in the estimation of these models."

6.4 Summary

Currently, we can choose from many theoretical models to price default-risky bonds. The assembling and analyzing of quality pricing data is critical at this stage because without empirical results, choosing the best model remains a difficult task. A valuable extension to the structural approach would be to incorporate jumps in the value of the firm in a reasonable way. With jumps incorporated in the evolution of firm value, a firm can default instantaneously because of a sudden drop in its value. Within the reduced-form framework, we need to explore further parametric forms of the intensity and recovery processes.

Chapter 7

Conclusion

In this dissertation, we have tried to highlight the growing importance of valuing credit risk and give an overview of *structural models* and *reduced-form models*. A summary of the salient features of some of the models within these two categories is found in Appendix B. A summary of the models' strengths and drawbacks is found in Appendix C.

Despite the natural and elegant way of modelling default by the first time the firm value hits some barrier, structural models have been criticized for several reasons. The major points of criticism are that they require estimating the parameters of processes that cannot be directly observed, such as firm value and default boundary. Therefore, they are of limited use in arriving at precise valuations. They are, however, useful in gaining intuition about the effects of various variables on credit risk. The Merton model's strengths are its relative simplicity and intuition. However, its lack of interest rate dynamics, correlation modelling, and tractability limit its practical applications. The Longstaff and Schwartz model, though not closed form addresses some of the weaknesses of the Merton model.

Unlike structural models, reduced-form models are based upon more direct assumptions about the default process. These models can be parameterized to fit the current term structure of credit spreads. The Jarrow, Lando and Turnbull (JLT) model provided a simple framework that is easy to calculate. However, its simplicity results in inflexibility: bonds within a given class are considered homogenous and there is no correlation with interest rates modelled. The Duffie and Singleton (DS) model outlines an interesting framework for modelling credit risk, with an increasingly complex model of credit risk suggested. However very little specific guidance is given of model implementation, which leaves a great deal of unanswered questions. Of particular interest, in future research would be the possible combination of the JLT approach to modelling the default process (as a Markov chain in the credit ratings) in combination with the analytic tractability of the *recovery of market value* (RMV) assumption of DS on the recovery process.

While the structural approach is economically sound and often generates more conceptual insights on default behaviour, it implies less plausible credit spreads properties. The reduced-form approach is ad hoc though, but tractable and implies plausible credit spread properties. Can we have a model which not only has the flexibility of the reduced-form approach to fit data but also provides the theoretical insights on the economic mechanism behind default events of the traditional structural approach? Can we have a model which allows for both expected and unexpected defaults in a single framework? How can we reconcile the different implications of the traditional reduced-form and structural approaches? A valuable extension to the literature on credit risk modelling would be a model that answers these questions.

Madan and Unal (2000), (hereafter, MU) recently proposed a *structural reduced-form model* in closed form that attempts to answer these questions. The distinguishing feature of this model is that it incorporates the attractive features of structural models with the reduced-form approach. A key assumption of this model is that default is a consequence of a single jump loss event that drives the equity value to zero and requires cash outlays that cannot be externally financed. The authors provide examples such as the near collapse of Long Term Capital Management (LTCM) in 1998 and the collapse of Barings bank in 1995 to justify this questionable assumption. MU also state that such jump loss events can be the result of the outcome of lawsuits, sudden default of a creditor, supplier, or a customer and unexpected devaluations.

MU obtained parameter values for their model by calibrating the model to a small set of data on the term structure of credit spreads. They report that the resulting model for credit spreads is tractable and can be readily implemented.

When calibrated to data on credit spreads, the model yielded a variety of realistic credit shapes. An important addition to the literature on the empirical testing of credit risk models would be a full-scale empirical study of corporate bond yields using the MU model.

Appendix A

Poisson Processes

In reduced-form (hazard rate) models, the fundamental modelling tool is the Poisson process, and this appendix reviews some important aspects of Poisson processes. The definitions and concepts in this appendix are taken from (and covered in much more detail in) Rogers and Williams (1994).

A *homogeneous* Poisson counting process $\{N_t\}_{t\geq 0}$ is a non-decreasing process with right-continuous paths and values in Z^+ such that

1. $N_0 = 0$;
2. for any $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \ldots \leq s_n \leq t_n$, the random variables $X_i \equiv N(t_i) - N(s_i)$ are independent, and the distribution of each X_i depends only on the length $t_i - s_i$;
3. $\forall t \geq 0, N_t - N_{t-}$ is either 0 or 1.

    The definition of the Poisson process uniquely determines its distribution to within a single parameter λ called the rate of the process. When $\lambda = 1$, we speak of a standard Poisson process. Here are other key properties, in which the positive parameter λ appears explicitly.

4. the process $\widetilde{N}_t \equiv N_t - \lambda t$ is a martingale;
5. the inter-event times $T_n - T_{n-1}$ are independent with common exponential

(λ) distribution:

$$P[(T_n - T_{n-1}) > t] = e^{-\lambda t} \; \forall t \geq 0 \tag{A.1}$$

Here

$$T_n \equiv \inf\{t \geq 0 \mid N_t = n\} \tag{A.2}$$

6. for any $s \leq t, N_t - N_s \sim P(\lambda(t-s))$, the Poisson distribution with mean λ:

$$P[N_t - N_s = k] = \frac{1}{k!}\lambda^k (t-s)^k \exp\{-(t-s)\lambda\}, k \in Z^+ \tag{A.3}$$

The simple (homogeneous) Poisson process can be generalized as follows. N is called an *inhomogeneous* Poisson process with deterministic intensity function $\lambda(t)$, if the increments $N(t) - N(s)$are independent and for s, t we have

$$P[N_t - N_s = k] = \frac{1}{k!}\left(\int_s^t \lambda(u)du\right)^k \exp\left\{-\int_s^t \lambda(u)du\right\} \tag{A.4}$$

The only difference to property (6) above is that $\lambda(t-s)$ has been replaced by the integral of $\lambda(u)$ over the respective time span.

Appendix B

Summary of Credit Risk Model Features

In chapters 4 and 5, we provided an overview of structural and reduced-form models of default. The purpose of this chapter is to summarize the features of the models reviewed in chapters 4 and 5.

Credit Risk Model	Merton (1974)	LS (1995)	JLT (1997)	DS#1 (1999)
Default Process	Default occurs when firm value falls below debt value.	Default occurs when firm value falls below a stochastic boundary.	Default occurs when a firm transitions into the lowest level.	Only model hazard rate of default.
Default Probability	Determined by firm value growth and volatility.	Determined by the growth, volatility &, correlation of firm value and boundary.	Determined by a Markov Process in the firm's credit ratings.	
Recovery Process	Assumed to be value of firm at time of default.	Assumed to be a constant fraction of face value, received at maturity.	Assumed to be a constant fraction of face value, received at maturity.	Exogenously given fractional loss of market value.
Risk-Free Rate Process	Constant interest rates.	Vasicek model.	None given.	Use any interest rate model to arrive at risk-adjusted short rate.

Credit Risk Model	Merton (1974)	LS (1995)	JLT (1997)	DS#1 (1999)
Correlation Modelling	None	Between interest rates, and firm value processes.	None	None
Model Category	Structural Continuous (Closed Form) Equilibrium	Structural Continuous (Closed Form with recursion) Equilibrium	Reduced-Form Continuous Arbitrage-Free	Reduced-Form Continuous and Discrete Arbitrage-Free

Credit Risk Model	DS#2 (1999)	DS#3 (1999)
Default Process	Model mean loss rate directly.	Model default probability and loss percentage separately.
Default Probability		Estimate historically by bond class.
Recovery Process		Model using historic recovery rates.
Risk-Free Rate Process	None given.	Cox-Ingersoll-Ross for interest rates and credit spreads.

Credit Risk Model	**DS#2 (1999)**	**DS#3 (1999)**
Correlation Modelling	Between mean loss rate and interest rates.	Between interest rates and credit spreads.
Model Category	Reduced-Form Continuous Arbitrage-Free	Reduced-Form Continuous Arbitrage-Free

Appendix C

Strengths and Drawbacks of Credit Risk Models

In chapters 4 and 5, we provided an overview of structural and reduced-form models of default. The purpose of this chapter is to summarize the strengths and drawbacks of the models reviewed in chapters 4 and 5.

Model	Advantages	Disadvantages
Merton (1974)	Simple to implement.	(a) Requires inputs related to firm value. (b) Default occurs only at the maturity of debt. (c) Information in the history of defaults and credit rating changes cannot be used.
Longstaff and Schwartz (LS) (1995)	(a) Simple to implement. (b) Allows for stochastic term structure and correlation between defaults and interest rates.	(a) Requires inputs related to firm value. (b) Information in the history of defaults and credit rating changes cannot be used.
Jarrow, Lando and Turnbull (JLT) (1997)	(a) Simple to implement. (b) Can exactly match the existing prices of default-risky bond to infer risk-neutral probabilities of defaults and credit rating changes. (c) Uses the information in the history of defaults and credit rating changes.	(a) Correlation not allowed between default probabilities and the level of interest rates. (b) Credit spreads change only when credit rating changes.

Model	Advantages	Disadvantages
Duffie and Singleton (DS) (1999)	(a) Allows correlation between default probabilities and the level of interest rates. (b) Recovery ratio can be random and depend on the pre-default value of the security. (c) Any interest rate model can be accommodated and existing valuation results for risk-free term structure models can be readily used.	(a) Information in the credit history of defaults and rating changes cannot be used.

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