Chapter 4

Overview of Structural Models

4.1 Introduction

The key characteristic shared by structural models is their reliance on economic arguments for why firms default (e.g. the firm's value does not cover its obligations). These economic models provide the framework to derive a relationship between defaultable debt prices (or credit spreads) and market variables. Our investigation focuses on two structural models that are designed to price default-risky bonds: Merton (1974), and Longstaff and Schwartz (1995). We chose to investigate these two models because of their analytical tractability and the fact that the Longstaff and Schwartz model combines many distinctive features of other models. Like Merton, they assume that the firm values follows a diffusion process, as in Black and Cox (1979), they allow for early default before maturity of default-risky debt and as in Shimko, Tejima and van Deventer (1993), the riskless rate is assumed to follow the Vasicek (1977), model. Before reviewing these models, we list below the main issues that need characterization in a structural model.

1. Asset value process.

2. Issuer's capital structure.

3. Recovery process.

4. Terms and conditions of the debt issue.

5. Default-risk-free interest rate process.

6. Correlation between the default-risk-free interest rate and the asset price.

7. Correlation between interest rate risk and default risk.

4.2 Merton (1974)

Beginning with the groundbreaking Black-Scholes (1973), insight that the debt of a firm can be viewed as a contingent claim on the assets of the firm, Merton provided one of the first in-depth valuation models for default-risky bonds. The contingent claims approach requires the specification of three processes. First, a process for the total asset value process of the firm has to be explicitly modelled. Second, the bankruptcy process has to be modelled completely. That is, the "when" and "how" of bankruptcy have to be made explicit. Third, the payoffs to creditors in the event of default have to be specified in detail.

The following assumptions were made in Merton's valuation framework:

1. Riskless interest rate is constant, i.e. $r(t) = r \ \forall t \geq 0$;

2. Firm value V dynamics: $dV_t = \mu V_t dt + \sigma V_t dW_t, V_0 > 0$; μ is the instantaneous expected rate of return on the firm per unit time; σ^2 is the instantaneous variance of the return on the firm per unit time; W is a standard Gauss-Wiener process;

	Assets	Bonds	Liabilities
No default	$V_T \geq B$	B	$V_T - B$
Default	$V_T < B$	V_T	0

Table 4.1: Payoffs to the firm's liabilities at maturity

3. Firm has a single outstanding issue of debt promising B at T. Default occurs when $V_T < B$. Debt covenants grant bondholders *absolute priority*: in the event of default, bondholders get the entire firm and the shareholders get nothing.

Merton also assumes that the firm is neither allowed to repurchase shares nor to issue any new senior or equivalent claims on the firm. This assumption implies that at the debt's maturity T we have the payoffs in Table 4.1 above to the firm's liabilities.

If at time T the asset value V_T exceeds or equals the face value B of the bonds, the bondholders will receive their promised payment B and the shareholders will get the remaining $V_T - B$. However, if the value of assets V_T is less than B, the ownership of the firm will be transferred to the bondholders. Equity is then worthless (because of limited liability of equity, the shareholders cannot be forced to pay the shortfall $B - V_T$). Summarizing, the value of the default-risky bond issue $f(V_T, T)$ at time T is given by

$$f(V_T, T) = \min(B, V_T) = B - \max(0, B - V_T) \tag{4.1}$$

which is equivalent to that of a portfolio composed of a default-free loan with face value B maturing at time T and a short European put position on the assets of the firm V with strike B and maturity T. The value of the equity E_T at time T is given by

$$E_T = \max(0, V_T - B) \tag{4.2}$$

which is equivalent to the payoff of a European call option on the assets of the firm V with strike B and maturity T. With the payoff specifications just described, we are able to value corporate liabilities as contingent claims on the firm's assets.

At this point, Merton (1974), considers the formation of a zero net investment portfolio consisting of a claim whose price is the value of the assets of the firm, the debt of the firm, and the riskless debt. These are held in proportions such that the return on the portfolio is deterministic and the portfolio requires zero net investment. The expected rate on the portfolio must be zero to avoid arbitrage. This condition is sufficient to derive the PDE that the price of any contingent claim on V must satisfy. Since Merton's paper, Harrison and Kreps (1979), and Harrison and Pliska (1981), developed martingale methods for pricing derivatives. Instead of following Merton's PDE method to derive the closed form solution for the price at time t, $f(V_t, T)$ of the default-risky bond, we will follow the general martingale pricing techniques outlined in Musiela and Rutkowski (1998). The aim of introducing the martingale measure is twofold: firstly, it simplifies the explicit evaluation of arbitrage prices of derivative securities; secondly, it describes the arbitrage-free property of a given pricing model for primary securities in terms of the behaviour of relative prices.

Taking as given some risk-free short rate process r_t , we suppose that there is a security with value $\beta_t = \exp\left(\int_0^t r_s ds\right)$ at time t, which provides a riskless investment opportunity. Assuming that there are no arbitrage opportunities in the financial market, modelled by some probability space (Ω, F, P)[1], there exists a probability measure $\widetilde{Q}$, such that the processes of security prices, discounted with respect to β, are $\widetilde{Q}$-martingales (Harrison and Kreps (1979), and Harrison and Pliska (1981)). $\widetilde{Q}$ is called the equivalent martingale measure, and we let $\widetilde{E}[\cdot]$ denote the corresponding expectation operator. This gives us

$$\begin{aligned}\frac{f(V_t, T)}{e^{\int_0^t rds}} &= \widetilde{E}\left[\frac{\min(V_T, B)}{e^{\int_0^T rds}} \mid F_t\right] \\ &= \widetilde{E}\left[\frac{B - \max(B - V_T, 0)}{e^{\int_0^T rds}} \mid F_t\right]\end{aligned}$$

Therefore,

[1]It is customary in financial models to regard F_t as a model for all the information available to agents at time t.

$$
\begin{aligned}
f(V_t, T) &= \widetilde{E}\left[\frac{B - \max(B - V_T, 0)}{e^{\int_t^T r ds}} \mid F_t\right] \\
&= \widetilde{E}\left[\frac{B}{e^{\int_t^T r ds}}\right] - \widetilde{E}\left[\frac{\max(B - V_T, 0)}{e^{\int_t^T r ds}} \mid F_t\right] \\
&= Be^{-r(T-t)} - \widetilde{E}\left[\frac{\max(B - V_T, 0)}{e^{\int_t^T r ds}} \mid F_t\right] \qquad (4.3)
\end{aligned}
$$

Merton (1974), using the Black-Scholes (1973), insight that the debt of a firm can be viewed as a contingent claim on the assets of the firm, observed that

$$
\widetilde{E}\left[\frac{\max(B - V_T, 0)}{e^{\int_t^T r ds}} \mid F_t\right] \qquad (4.4)
$$

is the value of a European put option on the assets of the firm with strike B and maturity T. Thus by the Black-Scholes formula, we have

$$
\widetilde{E}\left[\frac{\max(B - V_T, 0)}{e^{\int_t^T r ds}} \mid F_t\right] = Be^{-r(T-t)}\Phi(-d_2) - V_t\Phi(-d_1) \qquad (4.5)
$$

where $\Phi(x)$=standard normal cumulative distribution function,

$$
\begin{aligned}
d_1 &= \frac{\ln(V_t/B) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\ln(V_t/B) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
&= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

Substitution of equation (4.5) into equation (4.3) gives

$$
\begin{aligned}
f(V_t, T) &= Be^{-r(T-t)} - Be^{-r(T-t)}\Phi(-d_2) + V_t\Phi(-d_1) \qquad (4.6) \\
&= Be^{-r(T-t)}\left[(1 - \Phi(-d_2)) + \Phi(-d_2)\left(\frac{V_t\Phi(-d_1)}{Be^{-r(T-t)}\Phi(-d_2)}\right)\right] \\
&= Be^{-r(T-t)}[(1 - \Phi(-d_2)) + \Phi(-d_2)\delta] \qquad (4.7)
\end{aligned}
$$

where

$$\delta = \frac{V_t\Phi(-d_1)}{Be^{-r(T-t)}\Phi(-d_2)} \tag{4.8}$$

Proposition 1 *Firm Value Dynamics*
The SDE for firm value V dynamics:

$$\begin{aligned} dV_t &= \mu V_t dt + \sigma V_t dW_t \\ V_0 &> 0 \end{aligned} \tag{4.9}$$

has a unique solution given by

$$V_t = V_0 e^{mt+\sigma W_t} \tag{4.10}$$

where

$$m = \mu - \frac{1}{2}\sigma^2 \tag{4.11}$$

Proof.
We consider the process $X_t = \mu t + \sigma W_t$. Clearly, this is a solution to $dX_t = \mu dt + \sigma dW_t$. After making the transformation $Y = e^X$, an application of Itô's lemma gives the SDE for Y, $dY_t = Y_t(\mu + \frac{1}{2}\sigma^2)dt + Y_t\sigma dW_t$. Now we consider the process $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$ and make the same transformation. Itô's lemma confirms that $dY_t = Y_t\mu dt + Y_t\sigma dW_t$.

Using equation (4.10), we can explicitly write down the actual default probability . From the definition of default,

$$\begin{aligned} p &= P[V_T < B] \\ &= P[V_0 e^{mT+\sigma W_T} < B] \\ &= P\left[mT + \sigma W_T \leq \ln\frac{B}{V_0}\right] \\ &= P\left[W_T < \left(\frac{\ln(\frac{B}{V_0}) - mT}{\sigma}\right)\right] \\ &= \Phi\left(\frac{\ln(B/V_0) - mT}{\sigma\sqrt{T}}\right) \end{aligned} \tag{4.12}$$

The result in equation (4.11) uses the fact that W_T is normally distributed with mean zero and variance T.[2] Setting $\mu = r$ in equation (4.11), gives the risk-neutral default probability $\widetilde{p}$

$$\widetilde{p} = \widetilde{P}[V_T < B] = \Phi(-d_2) = 1 - \Phi(d_2) \tag{4.13}$$

We can now interpret the equation for defaultable debt (4.7) in an intuitive way. The value of defaultable debt is the value of otherwise similar, default-risk-free debt times the risk-neutral probability of no default plus the payoff in the case of default times the risk-neutral probability of default. δ is the implied recovery rate in the Merton model.

Defining $s(V_t, T)$ as the spread above the risk-free rate at which the debt trades at t, we can rewrite equation (4.7) for $t < T$ as

$$f(V_t, T) = B\exp(-(r + s(V, t))(T - t)) \tag{4.14}$$

where

$$\begin{aligned} s(V_t, T) &= -\frac{1}{T-t} \ln\left[\frac{1}{d}\Phi(-d_1) + \Phi(d_2)\right] \\ d &= \frac{B}{V_t} e^{-r(T-t)} \end{aligned} \tag{4.15}$$

d is the discounted debt-to-asset value ratio, which can be considered as a measure of the firm's leverage. Equation (4.15) defines a term structure of credit risk[3], which depends on the time to maturity of the debt, firm's asset volatility σ (the firm's business risk), and leverage d. In Merton's model, the credit spread increases as the leverage of the firm rises. This increase in the credit spread is natural because increased leverage heightens the probability that the firm may default. Higher default probability is reflected in an increase in the credit spread. Similarly, a rise in the volatility of the firm's value increases the probability that the firm may default, thus expanding the credit spread.

Furthermore,

[2] See Section 3.1.1 for properties of the standard Brownian motion.

[3] The term structure of credit risk is also called the risk structure of interest rates, the term structure of credit spreads or the risky term structure.

$$\lim_{h \downarrow 0} s(V_t, t+h) = 0 \tag{4.16}$$

This follows from the fact that

$$\begin{aligned} \lim_{(T-t)\to 0} d_1 &= \infty \\ \lim_{(T-t)\to 0} d_2 &= \infty \end{aligned}$$

and from standard properties of the normal distribution that state that

$$\begin{aligned} \Phi(+\infty) &= 1 \\ \Phi(-\infty) &= 0 \end{aligned}$$

From equation (4.16), we see that credit spreads for maturities going to zero are zero. Zero short spreads mean that default-risky bond investors do not demand a risk premium for assuming the default risk of an issuer, as long as the time to maturity is sufficiently short. This feature is not consistent with what is observed in the market. In the market, we see non-zero credit spreads for nearly all default-risky bonds regardless of maturity.

Despite its simplicity and intuitive appeal, Merton's model has many limitations. First, the credit spreads derived from the model are significantly lower than those implied by empirical evidence (Mason, Jones and Rosenfeld (1984)). That is, Merton's model underprices credit risk. Second, in the model the firm defaults only at maturity of the debt, a scenario that is at odds with reality. Also, most firms have complicated capital structures made up of a variety of security types, as opposed to a single debt issue. The Merton framework assumes that the absolute-priority rules are actually adhered to upon default in that debts are paid off in their order of seniority. However, empirical evidence (Franks and Torous (1989),(1994)) indicates that the priority rule is often violated.

Yet another problem with the Merton model is that the value of the firm, which is an input to the valuation model is difficult to ascertain since not all the firm's assets are either tradable or observable. These real life complications make the Merton framework less useful as a tool. However, it does not decrease the intuition behind modelling the default process. Merton's framework has spawned an enormous theoretical literature on defaultable debt pricing. Some examples are Black and Cox (1979), Kim, Ramaswamy and Sundaresan (1993), Shimko, Tejima and van Deventer (1993), Leland (1994), Longstaff and Schwartz (1995), Leland and Toft (1996), and Saá-Requejo and Santa-Clara (1999). The Merton model has also been loosely implemented in a commercial package which is marketed by KMV corporation. The KMV model draws its main strength from a judicious (but not-model consistent) use of a large database of historical defaults.

4.3 Longstaff and Schwartz (1995)

Longstaff and Schwartz (hereafter, LS) provide closed form expressions for the value of risky fixed and floating rate debt. LS address some of the weaknesses of the Merton model. In a way similar to Merton, LS assumed that the value of the firm follows a diffusion process

$$dV = \mu V\,dt + \sigma V\,dZ_1 \tag{4.17}$$

where σ is a constant, μ is the rate of return on the underlying asset value and Z_1 is a standard Wiener process. In contrast to Merton's assumption of constant interest rates, LS postulated that the short-term rate follow the mean-reverting Ornstein-Uhlenbeck process first used by Vasicek (1977).

$$dr = (\gamma - \beta r)dt + \eta dZ_2 \tag{4.18}$$

where γ, β and η are constants and Z_2 is another standard Wiener process. The authors chose the Vasicek model for the short-term rate because it incorporates mean reversion and facilitates the use of closed form solutions. A more general short-term rate model would require that defaultable debt prices be solved numerically. The instantaneous correlation between Z_1 and Z_2 is ρdt, i.e.,

$$dZ_1 dZ_2 = \rho dt \tag{4.19}$$

Priority of claim	Altman Study: 1985-1991 ω	Frank and Torous Study:1983-1990 ω
Bank Debt		0.136
Secured Debt	0.395	0.199
Senior Debt	0.477	0.530
Senior Subordinated Debt	0.693	
Cash-Pay Subordinated Junior Debt	0.72	0.711
Non-Cash-Pay Subordinated Debt	0.805	

Table 4.2: Historical values of ω from various bond classes

LS also assert that strict absolute priority to claims is rarely upheld in distressed organizations, which also differs from the Merton model. The model allows for a variety of liability classes with different coupon rates, priorities and maturity dates.

LS then assumed the existence of a (constant) threshold value of the firm, K, which serves as a financial distress boundary; if the value of the assets breaches this level, default is triggered (on all outstanding obligations), some form of restructuring occurs and the remaining assets of the firm are allocated among the firm's claimants. Thus contrary to Merton's model, default can occur prior to maturity. LS simplify their analysis by postulating that it is the ratio of V to K, rather that the absolute value of which governs financial distress and call this ratio X.

If a reorganization occurs during the life of a security, the security holder receives $1-\omega$ times the face value of the security at maturity, where ω represents the write-down on a particular security and is constant over all instruments issued by the firm. This type of payoff would be consistent with a reorganization which provided new securities in exchange for old claims. The model thus avoids the dependence of the payoff on the debt on underlying asset value. Values of ω can be obtained from historical information on various classes of bonds. The authors site two such studies:(see Table 4.2 above)

For completeness, we now use the assumption of perfect, frictionless markets in which trading takes place continuously to derive the fundamental PDE that the price of any derivative asset $H(V,r,t)$must follow. We will derive this PDE using Merton's derivation of the Black-Scholes model presented in Merton (1974).

Let $P_1(r_t,t)$ and $P_2(r_t,t)$ be the prices of two zero coupon bonds with different maturities. Then by applying Itô's lemma and using equation (4.18) we have

$$\begin{aligned} dP_i(r_t,t) &= \underbrace{\left((\gamma-\beta r)\frac{\partial P_i}{\partial r}+\frac{\partial P_i}{\partial t}+\frac{1}{2}\eta^2\frac{\partial_i^P}{\partial r^2}\right)}_{\mu_{P_i}P_i}dt+\underbrace{\left(\eta\frac{\partial P_i}{\partial r}\right)}_{\sigma_{P_i}P_i}dZ_2 \\ &= \mu_{P_i}P_i dt+\sigma_{P_i}P_i dZ_2 \end{aligned} \tag{4.20}$$

We now consider forming a riskless portfolio of these bonds. Let X_1 and X_2 be our holdings of P_1 and P_2 respectively. From equation (4.20) we have

$$d[X_1P_1+X_2P_2]=(X_1\mu_{P_1}P_1+X_2\mu_{P_2}P_2)dt+(X_1\sigma_{P_1}P_1+X_2\sigma_{P_2}P_2)dZ_2 \tag{4.21}$$

To eliminate interest rate risk, we now choose X_i such that

$$X_1\sigma_{P_1}P_1+X_2\sigma_{P_2}P_2=0 \Longrightarrow -X_2P_2=\frac{X_1\sigma_{P_1P_1}}{\sigma_{P_2}} \tag{4.22}$$

We have a riskless portfolio and thus

$$\begin{aligned} X_1\mu_{P_1}P_1+X_2\mu_{P_2}P_2 &= r(X_1P_1+X_2P_2) \\ \Longrightarrow X_1\mu_{P_1}P_1-\mu_{P_2}\left(\frac{X_1\sigma_{P_1}P_1}{\sigma_{P_2}}\right) &= rX_1P_1-r\left(\frac{X_1\sigma_{P_1}P_1}{\sigma_{P_2}}\right) \\ \Longrightarrow \mu_{P_1}-\mu_{P_2}\left(\frac{\sigma_{P_1}}{\sigma_{P_2}}\right) &= r-r\left(\frac{\sigma_{P_1}}{\sigma_{P_2}}\right) \\ \Longrightarrow \frac{\mu_{P_1}-r}{\sigma_{P_1}} &= \frac{\mu_{P_2-r}}{\sigma_{P_2}} \end{aligned} \tag{4.23}$$

Therefore

$$\frac{\mu_{P_1}(t)-r}{\sigma_{P_1}(t)}=\frac{\mu_{P_2(t)-r}}{\sigma_{P_2}(t)}=\lambda(r_t,t) \tag{4.24}$$

where $\lambda(r_t,t)$ must therefore be independent of the bond's maturity. Market participants commonly refer to as the *market price of interest rate risk*.

If $H(V,r,t)$ is the price of any derivative security contingent on V,r and t, then by applying Itô's lemma to H and using equations (4.17) and (4.18) we have

$$
\begin{aligned}
dH(V,r,t) &= H_t dt + H_V dV + H_r dr \\
&\quad + \left[\frac{1}{2}H_{VV}(dV)^2 + H_{Vr}(drdV) + \frac{1}{2}H_{rr}(dr)^2\right] \\
&= H_t dt + H_V(\mu V dt + \sigma V dZ_1) + H_r((\gamma - \beta r)dt + \eta dZ_2) \\
&\quad + \left(\frac{\sigma^2}{2}V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2}H_{rr}\right)dt \\
&= \underbrace{\left(H_t + \mu V H_V + (\gamma - \beta r)H_r + \frac{\sigma^2}{2}V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2}H_{rr}\right)dt}_{\mu_H H} \\
&\quad + \sigma V H_V dZ_1 + \eta H_r dZ_2 \\
&= \mu_H H dt + \sigma V H_V dZ_1 + \eta H_r dZ_2
\end{aligned}
\tag{4.25}
$$

We now impose no-arbitrage conditions by selecting a portfolio such that the interest rate risk and asset risk are eliminated by taking positions in the underlying asset and the risk free zero coupon bond. Assume the riskless portfolio includes X_H, X_P and X_V units of the derivative security, zero coupon bond and the firm respectively. Once again by Itô we have

$$
\begin{aligned}
d[X_H H + X_P P + X_V V] &= (X_H \mu_H H + X_P \mu_P P + X_V \mu_V V)dt \\
&+ (X_H H_V + X_V)\sigma V dZ_1 \\
&+ (X_H \eta H_r + X_P \sigma_P P)dZ_2
\end{aligned}
$$

We now choose X_H, X_P and X_V such that we get a riskless portfolio

$$
\begin{aligned}
X_H H_V + X_V = 0 &\implies X_V = -X_H H_V \\
X_H \eta H_r + X_P \sigma_P P = 0 &\implies X_P P = -\frac{\eta}{\sigma_P} X_H H_r
\end{aligned}
$$

Standardizing to $X_H = 1$ gives

$$X_V = -H_V \tag{4.26}$$

and

$$X_P P = -\frac{\eta}{\sigma_P} H_r \tag{4.27}$$

Once the portfolio has been made riskless, the instantaneous return on the portfolio must equal the risk-free instantaneous interest rate. Therefore

$$\begin{aligned}
(X_H H + X_P P + X_V V)r &= X_H \mu_H H + X_P \mu_P P + X_V \mu_V V r H \\
&\quad -\frac{\eta}{\sigma_P} r H_r - r V H_V \\
&= H_t + \mu_V V H_V + (\gamma - \beta r) H_r + \frac{\sigma^2}{2} V^2 H_{VV} \\
&\quad + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2} H_{rr} - \frac{\eta}{\sigma_P} \mu_P H_r - \mu_V V H_V \\
0 &= H_t - rH + \left((\gamma - \beta r) - \eta \left(\frac{\mu_P - r}{\sigma_P} \right) \right) H_r \\
&\quad + r V H_V + \frac{\sigma^2}{2} V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2} H_{rr}
\end{aligned}$$

This leads to the following equation;

$$\frac{\sigma^2}{2} V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2} H_{rr} + r V H_V + (\alpha - \beta r) H_r - rH + H_t = 0 \tag{4.28}$$

where

$$\begin{aligned}
\alpha &= \gamma - \lambda \\
\lambda &= \eta \frac{\mu_P - r}{\sigma_P}
\end{aligned}$$

λ is the adjusted market price of interest rate risk.

Equation (4.28) is the fundamental PDE defining the price of any derivative security contingent on V, r and t. The value of the derivative security is obtained by solving equation (4.28) subject to the appropriate maturity condition.

Vasicek (1977), showed that the price of a riskless discount bond $D(r, T)$ when the dynamics of r are given by equation (4.18) is given by

$$D(r, T) = \exp(A(T) - B(T) r) \tag{4.29}$$

where

$$
\begin{aligned}
A(T) &= \left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta}\right) T + \left(\frac{\eta^2}{\beta^3} - \frac{\alpha}{\beta^2}\right)(\exp(-\beta T) - 1) \\
&\quad - \left(\frac{\eta^2}{4\beta^3}\right)\exp((-2\beta T) - 1) \\
B(T) &= \frac{1 - \exp(-\beta T)}{\beta}
\end{aligned}
$$

Given the LS framework, the value of a default-risky discount bond[4] is the solution to equation (4.28) with $H(V, r, T) = P(X, r, T)$ and $X = V/K$, subject to the following maturity payoff:

$$1 - \omega I(\tau \leq T) \tag{4.30}$$

where $I(\cdot)$ is an indicator function taking the value one if the first passage time τ of V to K is less than or equal to T, and zero otherwise. Based on the above assumptions, the value of a default-risky discount bond can be written as

$$P(X, r, T) = D(r, T)(1 - \omega Q(X, r, T)) \tag{4.31}$$

where $X = V/K$, $D(r, T)$ is the value of a riskless discount bond under the Vasicek model and $Q(X, r, T) = \widetilde{E}[I(\tau \leq T)]$ is the probability that the first passage time of $\ln X$ to zero is less than T, where the expectation is taken with respect to the risk-adjusted processes

$$
\begin{aligned}
d\ln X &= \left(r - \frac{\sigma^2}{2} - \rho\sigma\eta B(T-t)\right) dt + \sigma dZ_1 && (4.32) \\
dr &= (\alpha - \beta r - \eta^2 B(T-t))dt + \eta dZ_2 && (4.33)
\end{aligned}
$$

Unfortunately, there is no known closed-form solution for $Q(X, r, T)$ when interest rates are stochastic, so LS proposed a numerical solution that is based on an implicit formula for the first passage density due to Buonocore, Nobile and Ricciardi (1987). The first passage density of $\ln X$ to zero at time τ starting from $\ln X$ at time zero, $q(0, \tau | \ln X, 0)$, is defined implicitly by the integral equation

[4] In this dissertation, we use the terms discount bond and zero coupon bond interchangeably.

$$\Phi\left(\frac{-\ln X - M(t,T)}{S(t)}\right) = \int_0^t q(0,\tau \mid \ln X, 0)\Phi\left(\frac{M(\tau,T) - M(t,T)}{S(t) - S(T)}\right) d\tau \tag{4.34}$$

where $\tau \leq t \leq T$. To obtain an explicit formula for the first passage density, the authors discretize time into n equal intervals, and define time $t_i = \frac{iT}{n}$ for $\{i = 1, 2, \ldots, n\}$. Discretizing equation (4.34) gives the recursive system for the terms below.

$$Q(X,r,T) = \lim_{n\to\infty} Q(X,r,T,n) \tag{4.35}$$

where

$$Q(X,r,T,n) = \sum_{i=1}^{n} q_i, \tag{4.36}$$

and the q_i are defined recursively by

$$\begin{aligned} q_1 &= \Phi(a_1), \\ q_i &= \Phi(a_i) - \sum_{j=1}^{i-1} q_j \Phi(b_{ij}), i = 2, 3, \ldots, n, \end{aligned}$$

The parameters a_i and b_{ij} are now given by

$$\begin{aligned} a_i &= \frac{-\ln X - M(iT/n, T)}{\sqrt{S(iT/n)}} \\ b_{ij} &= \frac{M(jT/n,T) - M(iT/n,T)}{\sqrt{S(iT/n) - S(jT/n)}} \end{aligned}$$

Here the authors use the functions M and S which are

$$\begin{aligned} M(t,T) &= \left(\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2}\right)t \\ &+ \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3}\right)\exp(-\beta T)\exp((\beta t) - 1) \\ &+ \left(\frac{r}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta}{\beta^3}\right)(1 - \exp(\beta t)) \\ &- \left(\frac{\eta^2}{2\beta^3}\right)\exp(\beta T)(1 - \exp(\beta t)) \end{aligned}$$

and

$$S(t) = \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2\right) - \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)(1 - \exp(-\beta t)) + \left(\frac{\eta}{\beta^3}\right)(1 - \exp(-2\beta t)) \tag{4.37}$$

LS propose using $n = 200$ as approximation to the infinite sum,

$$\lim_{n\to\infty} \sum_{i=1}^{n} q_i \tag{4.38}$$

The equation for defaultable debt, (4.31), has the intuitive structure that the value of risky debt can be viewed as the difference between the riskless bond and the discount for the default risk of the bond. The term, $\omega D(r, T)$, is the present value of the write-down on the bond in the event of a default. The term, $Q(X, r, T)$, is the risk-neutral probability of default.

According to Rogers (1999), LS's derivation for the price of the default-risky bond is *flawed* because they applied the results of Buonocore, Nobile and Ricciardi (1987), concerning the first-passage distributions of one dimensional diffusions to the log of the discounted firm value, but this process is not a diffusion. Also, Collin-Dufresne and Goldstein (1999), (henceforth, CG) assert that the numerical solution to $Q(X, r, T)$ proposed by LS is only valid for one-factor Markov processes, that is, when interest rates are non-stochastic. This means the LS formula is only an approximation to the true solution to their model. CG derived what they claim to be an efficient algorithm for computing the exact solution to the LS model. They report that the difference between the LS approximation and the exact solution to their model is economically significant for typical parameter values.

In the LS model, default risk is captured by the variable X, so bonds can be valued by conditioning on X directly rather than on the default status of other bonds. This implies that coupon bonds can be valued as the sum of a series of zero coupon bonds. From this model we can see that the price of a default-risky bond is an increasing function of X and a decreasing function of ω and T. Default-risky bonds have shorter durations[5] than their risk free equivalents

[5]The sensitivity of the bond price to changes in r provides a measure of the duration of the bond.

and this property also holds for the LS model. As r increases, $D(X, r, T)$ and $Q(X, r, T)$ become smaller. $Q(X, r, T)$ becomes smaller because the increase in r causes V to drift away from K at a faster rate. The model can also display that the duration of a default-risky discount bond need not be a monotone-increasing function of its maturity. In fact, it can display how the duration can decrease with time, given a moderate level of default risk. Therefore, it is clear that while default-risky bond prices are generally decreasing with increases of r , this can be reversed for extremely defaultable debt.

Findings cited by the authors show that the LS model allows for various term structures of credit spreads for different levels of default risk. The model displays a monotone increasing term structure of credit spreads for bonds with low default risk and a hump shaped structure for bonds with high default risk. Also, the model indicates that there should be a negative relationship between credit spreads and the level of interest rates.

Another important finding of the LS model is that the effect of a firm's correlation with interest rate changes can be very significant in determining the value of its debt. Exogenously specifying the write-down variable, ω, introduces another degree of freedom into the LS model so that it could, in principle, be made to fit any given level of the default spread observed in interest rates. More, problematic, however, is the assumption that ω is a constant. The violation of the absolute priority rule may imply stochastic values of ω, contrary to this assumption. The authors state that their model can easily be extended to allow for unsystematic stochastic values of ω which are uncorrelated with both business and interest rate risks. Setting the default trigger, K, to be a constant is not a satisfactory way of capturing the events that precipitate a firm into bankruptcy. Nevertheless, it precludes the undesirable property of simple versions of Merton's model that, before maturity, firm value can fall significantly below the face value of the bond without triggering default. Ideally, the critical level K should be a function of the liabilities outstanding at each point in time; ω should be stochastic. However, incorporation of such features may sacrifice the model's tractability without providing additional insight into the valuation of defaultable debt.

4.4 The Merton (1974) and LS(1995) Models: A Comparison

The relationship between Merton (1974), and LS (1995), is important since it not only provides a foundation for empirical comparisons, but also relates to some fundamental issues of pricing default-risky bonds. This section provides a comparison of these models. The three major findings are as follows. First, although the Merton model is less general that the LS model in terms of default probability, it is more general in terms of the recovery rate. Second, in both Merton (1974) and LS (1995), the condition triggering a default is not consistent with the no-arbitrage argument. Third, both models are restrictive due to predictable arrival times of default, which implies that the term structure of credit risk has to start from zero.

In LS, default happens when firm value $V(t)$, which follows a diffusion process with a continuous sample path, reaches a constant default threshold K from above. This results in two important features of the LS model. First, it permits default before the maturity date of default-risky debt. As a result, the LS model is more general than the Merton model which permits default only at the maturity date. Second, the probability of default is predictable, i.e., a currently solvent firm cannot default on its debt in the next instantaneous moment. The consequence of this feature is that when the time to maturity goes to zero, the LS model generates a term structure of credit risk that converges to zero too. This is also a restriction of the Merton model.

For comparison purposes, the pricing formulas for default-risky discount bonds under the Merton (1974) and LS (1995) models are given below. From equation (4.7), we have for the Merton model

$$P(V_t, T) = Be^{-r(T-t)}[1 - \Phi(-d_2) + \Phi(-d_2)\delta] \tag{4.39}$$

where

$$\delta = \frac{V_t\Phi(-d_1)}{Be^{-r(T-t)}\Phi(-d_2)} \tag{4.40}$$

is the implied recovery rate. B is the face value of a riskless discount bond and $\Phi(-d_2)$ is the probability of default in a risk neutral world. From equation (4.31), we have for the LS model

$$P(X, r, T) = D(r, T)((1 - Q(X, r, T)) + (1 - \omega)Q(X, r, T)) \tag{4.41}$$

where ω is the write-down proportion of the debt value in case of default, $X = V/K$, $D(r, T)$ is the value of a riskless discount bond under the Vasicek model and $Q(X, r, T)$ is the probability of default.

It is evident from equations (4.39) and (4.41) that the pricing formula for default-risky discount bonds has the same form in both the Merton and LS models. However, there are three key differences between equations (4.39) and (4.41). First, because of the assumption of constant interest rates in the Merton model, the price of a riskless discount bond is simply the present value of the face value of the bond, $Be^{-r(T-t)}$, whereas in the LS model the price of a riskless discount bond, $D(r, T)$, is given by the Vasicek model. Second, the Merton model, has a closed-form solution for the risk neutral probability of default, $\Phi(-d_2)$, whereas in the LS model, the probability of default, $Q(X, r, T)$, can be solved iteratively. Third, in the LS model, the recovery rate of default-risk debt, $1 - \omega$, is an exogenously specified constant whereas in the Merton model the recovery rate of default-risk debt, δ, is stochastic. In other words, the LS model assumes a zero covariance between recovery rate and probability of default. Therefore, the LS model is less general than the Merton model in terms of recovery rate.