Chapter 1

Introduction

The study of turbulence either in Newtonian fluids or Non-Newtonian fluids is one of the greatest unsolved and still not well understood problems in contemporary applied sciences. For indepth coverage of the deep and fascinating investigations undertaken in this field, the abundant wealth of results obtained and remarkable advances achieved we refer to the monographs [48, 80, 88, 110] and references therein. It is also a commonly accepted fact that the rigorous understanding of turbulence is one of the most challenging task for the future development of certain fields of mathematics such as analysis and theory of partial differential equations.

The hypothesis relating the turbulence to the "randomness of the background field" is one of the motivations of the study of stochastic version of equations governing the motion of fluids flows. The introduction of random external forces of noise type reflects (small) irregularities that give birth to a new random phenomenon, makes the problem more realistic. Such approach in the understanding of the turbulence phenomenon was pioneered by Bensoussan and Temam in [10] where they studied the stochastic Navier-Stokes equations (SNSE). Since then stochastic partial differential equations and stochastic models of fluid dynamics have been the object of intense investigations which have generated several important results. We refer, for instance, to [2], [8], [16], [19], [37], [38], [44], [83], [95],[98], [101], [105], [106]. Similar investigations for Non-Newtonian fluids have almost not been undertaken except in very few work; we refer, for instance, to [59],[66], [67], [78], [92], [118] for some computational studies of stochastic models of polymeric fluids and to [14], [65], [68], [69] for their mathematical analysis. It is worth to note that (especially in the Non-Newtonian case) the study of stochastic models is relevant not only for the analytical approach to turbulent flows but also for practical needs related to the physics of the corresponding fluids [92]. As it is said in the preface of [3] "it is also motivated by physical consideration, aiming at including perturbative effects, which cannot be modeled deterministically, due too many degrees of freedom being involved, or aiming at taking into account different time scales of the components of the underlying dynamics". The models considered in the papers [59],[66], [67], [78], [118], [14], [65], [68], [69], for example, occur very naturally from the kinetic theory of polymer dynamics. Indeed they arise from the reformulation of Fokker-Planck or diffusion equations as stochastic differential equations ([92]).

In the present work, we initiate the mathematical analysis for the stochastic model of incompressible second grade fluids which is a special example of a Non-Newtonian fluid classified in the differential Rivlin-Ericksen fluids. We will give more details on this particular fluid in the forthcoming section.

1.1 Physical background of second grade fluids

This section is devoted to the physical background of second grade fluids. Most of the information covered here have been taken from [40], [41], [49], [89] and [97].
For a homogeneous incompressible fluid, the constitutive law satisfies

$$\mathbb{T} = -\tilde{p}\mathbf{1} + \hat{\mathbb{T}}(\mathbf{D}),$$

where $\mathbb{T}$ is the Cauchy stress tensor, $\tilde{p}$ is the undetermined pressure due to incompressibility condition, $\mathbf{1}$ is the identity tensor. The argument tensor $\mathbf{D}$ of the symmetric-valued function $\hat{\mathbb{T}}$ is defined through

$$\mathbf{D} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}, \quad \mathbf{L} = \operatorname{grad} u,$$

where u is the velocity field of the fluid and the T superscript denotes the matrix transpose. When $\hat{\mathbb{T}}$ is nonlinear then the fluid is said to be Non-Newtonian.

In this work, we study a particular class of Non-Newtonian fluids in which the Cauchy stress tensor depends only on a very short history of the deformation gradient $\mathbf{L}$. More precisely, we will consider Non-Newtonian fluids such that, apart from the pressure $\tilde{p}$, its stress tensor $\mathbb{T}$ is just a function of the velocity gradient $\mathbf{L}$ and its time derivatives $\mathbf{L}^{(i)}$ up to order $n-1$; $\mathbf{L}^{(i)}$ is the i-th time derivative of $\mathbf{L}$. We call these materials differential Rivlin-Ericksen fluids of complexity n or simply fluids of complexity n. For an incompressible fluid of complexity n, each $\mathbf{L}^{(i)}$, $i = 0, 1, ..., n$ is a traceless tensor and the defining constitutive equation is (see [89] and [97])

$$\mathbb{T} = -\tilde{p}\mathbf{1} + \hat{\mathbb{T}}(\mathbf{L}, \mathbf{L}^{(1)}, ..., \mathbf{L}^{(n-1)}).$$

To each fluid, we conventionally associate with it a stored energy function $\hat{\psi}$:

$$\psi = \hat{\psi}(\mathbf{L}, \mathbf{L}^{(1)}, ..., \mathbf{L}^{(m-1)}).$$

A fluid of complexity n is said to be compatible with thermodynamics if throughout all motions the following holds

$$\rho\psi^{(1)} \leq \mathbb{T} \cdot \mathbf{L}, \tag{1.1}$$

in which ρ designates the constant density of the fluid. We should notice that for isothermal and/or isentropic processes, (1.1) is exactly the same thermodynamic setting outlined in [40]. We refer to [41] for a detailed discussion about this equivalence. The integers n and m may not be the same but it was shown in [41] that for a fluid of complexity n compatible with thermodynamics $m < n$. Throughout we assume that $1 \leq m < n$.

By the frame indifference principle (see [89]), there exists two isotropic functions $\tilde{\mathbb{T}}$ and $\tilde{\psi}$ such that

$$\mathbb{T} = -\tilde{p}\mathbf{1} + \tilde{\mathbb{T}}(\mathbf{A}_1, ..., \mathbf{A}_n),$$
$$\psi = \tilde{\psi}(\mathbf{A}_1, ..., \mathbf{A}_m),$$

where the $\mathbf{A}_i$-s are the Rivlin-Ericksen tensors defined by

$$\mathbf{A}_1 = 2\mathbf{D},$$
$$\mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L}, n \geq 1.$$

The operator D/Dt denotes the material time derivative which is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla.$$

It was shown in [40] that a fluid of complexity 2 is compatible with thermodynamics if and only if

(i) the stored energy depends only on $\mathbf{L}$, that is,

$$\psi = \hat{\psi}(\mathbf{L}).$$

(ii) the reduced dissipation inequality

$$\rho\hat{\psi}^{(1)} \leq \hat{\mathbb{T}} \cdot \mathbf{L},$$

holds.

As a consequence of this, the stored energy of a fluid of complexity 2 compatible with thermodynamics has a stationary point at equilibrium, which is fully determined by $\hat{\mathbb{T}}(0, \mathbf{L}^{(1)})$. More specifically, $\hat{\psi}$ is twice differentiable at zero and

$$\frac{d\hat{\psi}}{d\mathbf{L}}(0) = 0 \text{ and } \rho\frac{d^2\hat{\psi}}{d\mathbf{L}^2}(0) \cdot (\mathbf{L}^{(1)} \otimes \mathbf{L}) = \hat{\mathbb{T}}(0, \mathbf{L}^{(1)}) \cdot \mathbf{L}.$$

We will apply these results to a special subclass of a fluid of complexity 2. We consider a second grade fluid, that is, a fluid of complexity 2 in which the function $\tilde{\mathbb{T}}$ is a polynomial in the arguments $\mathbf{A}_1, \mathbf{A}_2$. The constitutive law of a second grade fluid is explicitly given by:

$$\mathbb{T} = -\tilde{p}\mathbf{1} + \nu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$

where ν is the viscosity of the fluid, the constants α_1 and α_2 represent the normal stress moduli. We refer to [89] (see also [97]) for more details about this class of Non-Newtonian fluids.

It follows from the thermodynamical conditions (i) and (ii) that a second grade fluid is compatible with thermodynamics if and only if

(a) $\nu \geq 0$

(b) $\alpha_1 = -\alpha_2$

(c) the stored energy $\hat{\psi}$ is a quadratic function of $\mathbf{L}$.

This result was borrowed from [40]. By using the point (c) of the preceding result, Dunn and Fosdick [40] showed that the stored energy $\hat{\psi}$ has a minimum at equilibrium if and only if $\alpha_1 \geq 0$. On the basis of the analysis done in Sections 4, 6, 7, 8, and 9 of [40], this condition ensures the unique existence and boundedness of the flow of a second grade fluid. We also refer to [41] and [49] for more recent work concerning these conditions. We will assume for the rest of the work that

$$\nu > 0,$$

$$\alpha_1 = \alpha > 0,$$

$$\alpha_1 + \alpha_2 = 0.$$

These thermodynamical conditions imply that the stress tensor $\mathbb{T}$ can be written in the following form

$$\mathbb{T} = -\tilde{p}\mathbf{1} + \nu \mathbf{A}_1 + \alpha \left(\frac{D\mathbf{A}_1}{Dt} + \frac{1}{2}\mathbf{A}_1 \left(\mathbf{L} - \mathbf{L}^T\right) - \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right)\mathbf{A}_1 \right),$$

where

$$\mathbf{L} = \left(\frac{\partial u_i}{\partial x_j} \right)_{i,j}.$$

The incompressibility requires that

$$\operatorname{div} u = 0.$$

The following holds

$$\operatorname{div} \mathbb{T} = -\nabla \tilde{p} + \nu \Delta u + \alpha \frac{\partial \Delta u}{\partial t} + \alpha \left(\operatorname{curl}(\Delta u) \times u + \nabla(u.\Delta u + \frac{1}{4}|\mathbf{A}_1|^2) \right). \tag{1.2}$$

For a given external force f the dynamical equation for a second grade fluid is

$$\frac{\partial u}{\partial t} + \operatorname{curl}(u) \times u + \nabla(\frac{1}{2}|u|^2) = \operatorname{div} \mathbb{T} + f.$$

Making use of the latter equation and (1.2) we obtain the system of partial differential equations

$$\begin{cases} \frac{\partial}{\partial t}(u - \alpha \Delta u) - \nu \Delta u + \operatorname{curl}(u - \alpha \Delta u) \times u + \nabla P = f, \\ \operatorname{div} u = 0, \end{cases} \tag{1.3}$$

where

$$P = \tilde{p} - \alpha(u.\Delta u + \frac{1}{4}|\mathbf{A}_1|) + \frac{1}{2}|u|^2$$

is the modified pressure. For a given connected subset D of $\mathbb{R}^2$ and finite time horizon $[0, T]$ we complete the above system with the initial condition

$$u(0) = u_0 \quad \text{in} \quad D. \tag{1.4}$$

The interest in the investigation of problem (1.3) arises from the fact that it is an admissible model of a large class of Non-Newtonian fluids. Furthermore, once the above thermodynamical compatibility conditions are satisfied "the second grade fluid has general and pleasant properties such as boundedness, stability, and exponential decay"(see again [40]). It can also be taken as a generalization of the Navier-Stokes equations (NSE). Indeed it reduces to NSE when $\alpha = 0$; moreover recent work [60] shows that it is a good approximation of the NSE. We refer to [21], [22], [57], [58], [107], [108] for interesting discussions concerning their relationship with other models of fluids. We also should note that second grade fluids are connected to Turbulence Theory. Indeed the discussion on the relation between Non-Newtonian fluids, especially fluids of differential type, and Turbulence Theory started with the work of Rivlin [104]. It was rediscovered recently (see, for example, [46] and [30]) that the flow of second grade fluids can be used as a basis for a turbulence closure model.

Due to the above nice properties, the mathematical analysis of the second grade fluid has attracted many prominent researchers in the deterministic case. The first relevant analysis was done by Ouazar in his 1981 thesis; together with Cioranescu, they published the related results in [33] and [34]. Their method was based on the Galerkin approximation scheme involving a priori estimates for the approximating solutions using a special basis consisting of eigenfunctions corresponding to the scalar product associated with the operator curl$(u - \alpha\Delta u)$. They proved global existence and uniqueness without restriction on the initial data for the two dimensional case. Cioranescu and Girault [32], Bernard [11] extended this method to the three dimensional case; global existence was also obtained with some reasonable restrictions on the initial data. For another approach to global existence using Schauder's fixed point technics, we refer to [50], [51], [77] and some relevant references therein.

1.2 Overview of the thesis

As already mentioned, in this work we investigate a stochastic version of the problem (1.3), (1.4) under various boundary conditions (Dirichlet and periodic). More precisely, in Chapter 3 and Chapter 4 we assume that a connected and bounded open set D in $\mathbb{R}^2$ with boundary ∂D of class $\mathcal{C}^3$, a finite time horizon $[0,T]$, and a non random initial value u_0 are given. We consider the problem

$$\begin{cases} d(u-\alpha\Delta u)+(-\nu\Delta u+\mathrm{curl}(u-\alpha\Delta u)\times u+\nabla P)dt = F(u,t)\,dt+G(u,t)dW \\ \text{in } D\times(0,T], \\ \mathrm{div}\, u = 0 \text{ in } D\times(0,T], \\ u = 0 \text{ in } \partial D\times(0,T], \\ u(0)=u_0 \text{ in } D, \end{cases} \tag{1.5}$$

where $u=(u_1,u_2)$ and P represent the random velocity and pressure, respectively. The system is to be understood in the Itô sense. It is the equation of motion of an incompressible second grade fluid driven by random external forces $F(u,t)$ and $G(u,t)dW$, where W is a $\mathbb{R}^m$-valued standard Wiener process. In Chapter 3 we are concerned with the establishment of an existence and uniqueness results of the strong probabilistic solution of (1.5) under Lipschitz condition (in u) on $F(u,t)$ and $G(u,t)$. Here the term "strong" must be understood in the sense of Stochastic Analysis; that is, we look for a stochastic process u which is defined on a prescribed filtered complete probabilistic space on which W is defined. We reformulate (1.5) as an abstract problem by introducing some abstract operators defined on Hilbert spaces. Then, we derive crucial estimates for the solution of the Galerkin approximation of the problem, which is defined by means of the special basis we mentioned in the previous section. These estimates allow us to pass to the limit in the approximated abstract problem and obtain the first result of Chapter 3. In Section 4 of the very same chapter we analyze the long time behavior of the unique strong probabilistic solution of (1.5). We show that under some hypotheses on the data the solution decays exponentially in mean square. We also prove that if the deterministic part F of the external forces is time independent, then the strong solution of our model converges exponentially in mean square to the stationary solution. Here *stationary solution* is taken in the sense of (deterministic) partial differential equations.

In Chapter 4, we weaken the hypotheses by assuming that $F(u,t)$ and $G(u,t)$ are no longer Lipschitz in their first argument; we only suppose that they are continuous and have linear growth. Under this new assumptions we show that a weak probabilistic solution exists. This is achieved by proving that the law of the approximating solutions from the Galerkin scheme is tight, so that we can apply Prokhorov's Compactness Theorem and Skorohod's Embedding Theorem. The results of this chapter are the object of the published paper [99].

For the fifth chapter of the thesis, we assume that (1.5) is subjected to the periodic boundary condition. We investigate the behavior of the solution of (1.5) when the normal stress modulus α tends to zero. The main result of this chapter is that a sequence of strong probabilistic solutions of (1.5) can be constructed so that it converges in appropriate topology to the strong probabilistic solution of the stochastic Navier-Stokes equations. We refer to this result as the convergence theorem. The results of this chapter is the object of the article [100].

The second chapter of the work is intended to give some necessary preliminary results that are used throughout the thesis. We mainly recall some basic and useful results of analytic and probabilistic nature. This chapter is not intended to be exhaustive, so for the details we urge the reader to consult the specialized references cited therein.

As far as we know, this thesis is the first work dealing with the stochastic version of the equations governing the motion of a second grade fluid. Consequently, we could by no means exhaust the mathematical analysis of the problem; many questions are still opened but we hope that this pioneering work will find its applications elsewhere. It should be noted that solving the problem presented here is not easy, even in the deterministic case, the nature of the nonlinearities being one of the main difficulties in addition to the complex structure of the equations. Besides the obstacles encountered in the deterministic case, the introduction of the noise term $G(u,t)dW$ in the stochastic version induces the appearance of expressions that are very hard to control when proving some crucial estimates. Overcoming these problems will require a tour de force in the work. Nearly almost all the results and estimates obtained in the thesis are new for stochastic second grade fluids.

Chapter 2

Preliminary Results

We collect in this chapter the notations frequently used in the thesis. We also state without proof some useful well-known results from Analysis, Probability Theory and Stochastic Calculus. We do not pretend to be exhaustive so we refer the reader interested in the details to appropriate references.

2.1 Analytical preliminaries

We start with some information about some functional spaces needed in this work. Let D be an open and bounded subset of $\mathbb{R}^2$, $p \in [1, \infty)$ and k a nonnegative integer. We denote by $L^p(D)$ the space of p-integrable functions on D and by $W^{k,p}(D)$ the Sobolev space of p-integrable functions together with their derivatives up to order k. The spaces $L^p(D)$ and $W^{k,p}(D)$ respectively endowed with the norms

$$|\phi|_{L^p(D)} = \left(\int_D |\phi|^p dx \right)^{\frac{1}{p}},$$

and

$$|\phi|_{W^{k,p}(D)} = \left(\int_D \sum_{|\zeta| \le k} \left| \frac{\partial^{|\zeta|} \phi(x)}{\partial x^\zeta} \right|^p dx \right)^{\frac{1}{p}}$$

are Banach spaces. In case $p = \infty$,

$$|\phi|_{L^\infty(D)} = \operatorname*{ess\,sup}_{x \in D} |\phi(x)|.$$

We denote by $W_0^{k,p}(D)$ the closure in $W^{k,p}(D)$ of $C_c^\infty(D)$ the space of infinitely differentiable functions with compact support in D. For $p = 2$, $W_0^{k,2}(D)$ and $W^{k,2}(D)$ are Hilbert spaces that we denote respectively by $H_0^k(D)$ and $H^k(D)$. These spaces are endowed with the scalar products

$$(u,v)_{H_0^k(D)} = \sum_{|\zeta|=k}^{n} \left(\frac{\partial^{|\zeta|} u}{\partial x^{\zeta}}, \frac{\partial^{|\zeta|} v}{\partial x^{\zeta}} \right)_{L^2(D)},$$

and

$$(u,v)_{H^k(D)} = \sum_{|\zeta|\leq k}^{n} \left(\frac{\partial^{|\zeta|} u}{\partial x^{\zeta}}, \frac{\partial^{|\zeta|} v}{\partial x^{\zeta}} \right)_{L^2(D)},$$

where

$$(f,g)_{L^2(D)} = \int_D fg dx$$

is the scalar product in $L^2(D)$. In the particular case $H_0^1(D)$ we denote the scalar product by $((.,.))$ and the norm generated by this scalar product by $||.||$.
The following theorem is taken from [1].

Theorem 2.1. *If D is a bounded domain with sufficiently smooth boundary ∂D , then for any k and p such that $kp > 2$ the embedding*

$$W^{j+k,p} \subset W^{j,q},$$

is compact for any $1 \leq q \leq \infty$.

More Sobolev embedding theorems can be found in [1] and references therein.

We now touch upon Sobolev spaces of periodic functions needed in Chapter 5. Let L be a nonnegative number and $D = [0, L]^2$ a periodic square of side length L. We denote by $H^k_{per}(D)$ the space consisting of functions u that are in $H^k_{Loc}(\mathbb{R}^2)$ and are periodic with period L:

$$u(x + Lr_i) = u(x), \;\; i = 1, 2,$$

where $\{r_1, r_2\}$ represents the canonical basis of $\mathbb{R}^2$. Here the space $H^k_{Loc}(\mathbb{R}^2)$ is the space of functions u such that u restricted to $\mathcal{O}$ is an element of the Sobolev space $H^k(\mathcal{O})$ for every bounded set $\mathcal{O} \subset \mathbb{R}^2$. Functions in $H^k_{per}(D)$ can be characterized by their Fourier series expansions

$$H^k_{per}(D) = \left\{ u, u = \sum_{z\in\mathbb{Z}^2} c_z e^{2i\pi z.x/L}, c_z \in \mathbb{C}, \bar{c}_z = c_{-z}, \sum_{z\in\mathbb{Z}^2} |z|^{2k}|c_z|^2 < \infty \right\}, \tag{2.1}$$

and the norm $|u|_{H^k_{per}(D)}$ is equivalent to $(\sum_{z\in\mathbb{Z}^2}(1 + |z|^{2k})|c_z|^2)^{1/2}$. This definition holds true more generally for $k \in \mathbb{R}$. We denote by $H^k_{mp}(D)$ the set of functions $u \in H^k_{per}(D)$

such that $\int_D u(x)dx = 0$, this is a Hilbert space for the norm $(\sum_{z\in\mathbb{Z}^2} |z|^{2k}|c_z|^2)^{1/2}$, and $H^k_{mp}(D)$ and $H^{-k}_{mp}(D)$ are in duality for all $k \in \mathbb{R}$. We refer to [117] (see also [35], [48]) for more details about these spaces.

We also need the following product formula in Chapter 5, we refer to [29] for its proof in the case of the whole space (see for example [52] for the case of periodic condition).

Theorem 2.2. *Let D be a n-dimensional periodic box and let $\beta, \gamma \in \mathbb{R}$ such that $\beta+\gamma > 0$, $\beta < \frac{n}{2}$, $\gamma < \frac{n}{2}$. If $u \in H^\gamma_{per}(D)$ and $v \in H^\beta_{per}(D)$, then there exists a positive constant C such that*

$$|uv|_{H^{\gamma+\beta-\frac{n}{2}}_{per}(D)} \leq C|u|_{H^\gamma_{per}(D)}|v|_{H^\beta_{per}(D)}.$$

If $|\gamma| < \frac{n}{2}$, then

$$|uv|_{H^{-\frac{n}{2}-\varepsilon}_{per}(D)} \leq C'|u|_{H^\gamma_{per}(D)}|v|_{H^{-\gamma}_{per}(D)}, \tag{2.2}$$

for any $u \in H^\gamma_{per}(D)$, $v \in H^{-\gamma}_{per}(D)$ and $\varepsilon > 0$.

We proceed now with the definitions of additional spaces frequently used in this work. In what follows we denote by $\mathbb{X}$ the space of $\mathbb{R}^2$-valued functions such that each component belongs to X. A simply-connected bounded domain D with boundary of class $\mathcal{C}^3$ is given. We introduce the spaces

$$\begin{aligned}
\mathcal{V} &= \{u \in \mathcal{C}^\infty_c(D) \times \mathcal{C}^\infty_c(D) \text{ such that } \operatorname{div} u = 0\}\\
\mathbb{V} &= \text{ closure of } \mathcal{V} \text{ in } \mathbb{H}^1_0(D)\\
\mathbb{H} &= \text{ closure of } \mathcal{V} \text{ in } \mathbb{L}^2(D).
\end{aligned}$$

We denote by $(\cdot,\cdot)$ (resp. $|\cdot|$) the scalar product (resp. the norm) induced by the scalar product (resp. the norm) of $\mathbb{L}^2(D)$ in $\mathbb{H}$. The inner product (resp., the norm) still denoted by $((.,.))$ (resp., $||.||$) in $\mathbb{V}$ is induced by the inner product $((.,.))$ (resp., the norm $||.||$) in $\mathbb{H}^1_0(D)$.

We recall that for any $u \in \mathbb{V}$ we have the inequality of Poincaré

$$|u| \leq \mathcal{P}||u||, \tag{2.3}$$

where $\mathcal{P}$ is the so called Poincaré's constant. On $\mathbb{V}$, the norm $||.||$ is equivalent to the norm generated by the following scalar product (see for example [33])

$$(u,v)_{\mathbb{V}} = (u,v) + \alpha((u,v)), \text{ for any } u \text{ and } v \in \mathbb{V}.$$

Furthermore, we have

$$(\mathcal{P}^2+\alpha)^{-1}|v|_{\mathbb{V}}^2 \le ||v||^2 \le (\alpha)^{-1}|v|_{\mathbb{V}}^2, \text{ for any } v \in \mathbb{V}. \tag{2.4}$$

Remark 2.3. For Chapter 5 we shall use the following notations since we are studying the asymptotic behavior of the solution when the problem (1.5) is subjected to the periodic boundary condition.

$$\begin{aligned}
\mathcal{V}_{per} &= \left\{ u \in \mathcal{C}_{per}^{\infty}(D) \times \mathcal{C}_{per}^{\infty}(D) : \operatorname{div} u = 0 \text{ and } \int_D u dx = 0 \right\} \\
\mathbb{V}_{per} &= \text{ closure of } \mathcal{V}_{per} \text{ in } \mathbb{H}_{mp}^1(D) \\
\mathbb{H}_{per} &= \text{ closure of } \mathcal{V}_{per} \text{ in } \mathbb{L}_{per}^2(D),
\end{aligned}$$

Here the bounded domain D is replaced by a periodic square $D=[0,L]^2$ and the space $\mathbb{H}_{mp}^1(D)$ is the space of periodic functions which are in $\mathbb{H}^1(D)$ and with zero mean. We denote the norms and scalar products on $\mathbb{H}_{per}$, $\mathbb{V}_{per}$, $\mathbb{H}_{per}^k(D)$ with the same symbols we used for the norms and scalar products for $\mathbb{H}$, $\mathbb{V}$ and $\mathbb{H}^k(D)$.

We also introduce the following space

$$\mathbb{W} = \left\{ u \in \mathbb{V} \text{ such that } \operatorname{curl}(u-\alpha\Delta u) \in L^2(D) \right\}.$$

The following lemma tells us that the norm generated by the scalar product

$$(u,v)_{\mathbb{W}} = (u,v)_{\mathbb{V}} + (\operatorname{curl}(u-\alpha\Delta u), \operatorname{curl}(v-\alpha\Delta v)), \tag{2.5}$$

is equivalent to the usual $\mathbb{H}^3(D)$-norm on $\mathbb{W}$. Its proof can be found for example in [33] and [32].

Lemma 2.4. *The following (algebraic and topological) identity holds*

$$\mathbb{W} = \widetilde{\mathbb{W}},$$

where

$$\widetilde{\mathbb{W}} = \left\{ v \in \mathbb{H}^3(D) \textit{ such that } \operatorname{div} v = 0 \textit{ and } v_{|\partial D} = 0 \right\}.$$

Moreover, there exists a positive constant C such that

$$|v|_{\mathbb{H}^3(D)}^2 \le C(|v|_{\mathbb{V}}^2 + |\operatorname{curl}(v-\alpha\Delta v)|^2),$$

for any $v \in \widetilde{\mathbb{W}}$.

By this lemma we can endow the space $\mathbb{W}$ with norm $|.|_{\mathbb{W}}$ which is generated by the scalar product (2.5).

From now on, we identify the space $\mathbb{V}$ with its dual space $\mathbb{V}^\star$ via the Riesz representation, and we have the Gelfand chain

$$\mathbb{W} \subset \mathbb{V} \subset \mathbb{W}^\star, \tag{2.6}$$

where each space is dense in the next one and the inclusions are continuous.

The following inequalities will be used frequently:

Lemma 2.5. *For any $u \in \mathbb{W}$, $v \in \mathbb{W}$ and $w \in \mathbb{W}$ we have*

$$|(\operatorname{curl}(u - \alpha\Delta u) \times v, w)| \leq C|u|_{\mathbb{H}^3}|v|_{\mathbb{V}}|w|_{\mathbb{W}}. \tag{2.7}$$

We also have

$$|(\operatorname{curl}(u - \alpha\Delta u) \times u, w)| \leq C|u|^2_{\mathbb{V}}|w|_{\mathbb{W}}, \tag{2.8}$$

for any $u \in \mathbb{W}$ and $w \in \mathbb{W}$.

Proof. We introduce the well known trilinear form b used in the study of the Navier-Stokes equation by setting

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_D u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

We state the following identity (see for instance [11] and [33]):

$$((\operatorname{curl} \Phi) \times v, w) = b(v, \Phi, w) - b(w, \Phi, v), \tag{2.9}$$

for any smooth (solenoidal) functions Φ, v and w. We derive from (2.9) that for any $u \in \mathbb{W}$, $v \in \mathbb{W}$ and $w \in \mathbb{W}$

$$|(\operatorname{curl}(u - \alpha\Delta u) \times v, w)| \leq C|v|_{\mathbb{L}^2(D)}|\nabla(u - \alpha\Delta u)|_{\mathbb{L}^2(D)}|w|_{\mathbb{L}^\infty(D)} \tag{2.10}$$

where Hölder's inequality was used. Theorem 2.1 and the equivalence of the norms $|.|_{\mathbb{W}}$ and $|.|_{\mathbb{H}^3(D)}$ on $\mathbb{W}$ imply (2.7).

In view of (2.9) we deduce

$$(\operatorname{curl}(u - \alpha\Delta u) \times u, w) = b(u, u, w) - \alpha b(u, \Delta u, w) + \alpha b(w, \Delta u, u). \tag{2.11}$$

With the help of integration-by-parts and using the fact that u and w are elements of $\mathbb{W}$ we have that

$$b(u,\Delta u,w)=\sum_{j=1}^{2}b(\frac{\partial u}{\partial x_j},w,\frac{\partial u}{\partial x_j})+\sum_{i=1}^{2}b(u,\frac{\partial w}{\partial x_j},\frac{\partial u}{\partial x_j}) \tag{2.12}$$

$$b(w,\Delta u,u)=\sum_{j=1}^{2}b(\frac{\partial w}{\partial x_j},u,\frac{\partial u}{\partial x_j}). \tag{2.13}$$

We use these results to derive the following estimate. For any elements $u\in\mathbb{V}$ and $w\in\mathbb{L}^4(D)$, we obtain by Hölder's inequality

$$|b(u,u,w)|\leq C|u|_{\mathbb{L}^4(D)}||u||\ |w|_{\mathbb{L}^4(D)}.$$

Since the spaces $\mathbb{V}$ and $\mathbb{W}$ are, respectively, continuously embedded in $\mathbb{L}^4(D)$ and $\mathbb{V}$, then

$$|b(u,u,w)|\leq C|u|_{\mathbb{V}}^2|w|_{\mathbb{W}}. \tag{2.14}$$

We also have

$$\begin{aligned}|b(u,\Delta u,w)|\leq&|\nabla w|_{\mathbb{L}^\infty(D)}\sum_{j=1}^{2}\left|\frac{\partial u}{\partial x_j}\right|_{\mathbb{L}^2(D)}^2\\&+|u|_{\mathbb{L}^4(D)}\left(\sum_{j=1}^{2}\left|\frac{\partial w}{\partial x_j}\right|_{\mathbb{L}^4(D)}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{2}\left|\frac{\partial u}{\partial x_j}\right|_{\mathbb{L}^2(D)}^2\right)^{\frac{1}{2}}\end{aligned} \tag{2.15}$$

We derive from (2.15) and Theorem 2.1 that

$$|b(u,\Delta u,w)|\leq C|u|_{\mathbb{V}}^2|w|_{\mathbb{W}}. \tag{2.16}$$

Similarly we have

$$|b(w,\Delta u,u)|\leq C|w|_{\mathbb{W}}|u|_{\mathbb{V}}^2. \tag{2.17}$$

The estimates (2.11), (2.14), (2.16) and (2.17) yield

$$|(\mathrm{curl}(u-\alpha\Delta u)\times u,w)|\leq C|u|_{\mathbb{V}}^2|w|_{\mathbb{W}}, \tag{2.18}$$

for any $u\in\mathbb{W}$ and $w\in\mathbb{W}$. This completes the proof the lemma. □

The main objective of the third chapter of the thesis is the existence and uniqueness of the strong probabilistic solution of (1.5). The proof is nontrivial and requires the formulation of the problem in an abstract form. This is done by introducing some appropriate operators defined on Hilbert spaces. We denote by $\mathbf{P} : \mathbb{L}^2(D) \to \mathbb{H}$ the usual Helmoltz-Leray projector and by $A = -\mathbf{P}\Delta$ the well known Stokes operator with domain $D(A) = \mathbb{H}^2(D) \cap \mathbb{V}$. We have a very important consequence of Lemma 2.5.

Corollary 2.6. *There exists a bilinear operator $\hat{B} : \mathbb{W} \times \mathbb{V} \to \mathbb{W}^*$ such that*

$$< \hat{B}(u,v), w >= (\mathbf{P}(\mathrm{curl}(u - \alpha\Delta u) \times v), w) \text{ for any } (u,v,w) \in \mathbb{W} \times \mathbb{V} \times \mathbb{W}, \tag{2.19}$$

and (2.20)

$$|\hat{B}(u,v)|_{\mathbb{W}^*} \leq C|u|_{\mathbb{W}}|v|_{\mathbb{V}}, \tag{2.21}$$

$$|\hat{B}(u,u)|_{\mathbb{W}^*} \leq C_B|u|^2_{\mathbb{V}}, \tag{2.22}$$

$$< \hat{B}(u,v), v >= 0, \tag{2.23}$$

$$< \hat{B}(u,v), w >= - < \hat{B}(u,w), v > . \tag{2.24}$$

Proof. Thanks to the equivalence of the norm $|.|_{\mathbb{H}^3(D)}$ and the norm $|.|_{\mathbb{W}}$ on $\mathbb{W}$ and the fact that $\mathbf{P}$ is an self-adjoint operator, inequality (2.7) induces the existence of $\widehat{B}(u,v) \in \mathbb{W}^*$ which satisfies (2.19). The inequalities (2.21) and (2.22) follows from (2.7) and (2.8) respectively. The identity (2.24) follows from (2.23), which in turn can be checked by using (2.9) with $\Phi = \mathrm{curl}(u - \alpha\Delta u)$ and $w = v$. $\square$

As mentionned in the introduction of this thesis proving some crucial estimates require a tour de force in the work, one of the tools we frequently use is a result about the "generalized Stokes equations"

$$\begin{cases} v - \alpha\Delta v + \nabla q = f \text{ in } D \\ \mathrm{div}\, v = 0 \text{ in } D \\ v = 0 \text{ on } \partial D. \end{cases} \tag{2.25}$$

By a solution of this system we mean a function $v \in \mathbb{V}$ which satisfies

$$(v,h) + \alpha((v,h)) = (f,h),$$

for any $h \in \mathbb{V}$.

The following theorem is very crucial for the rest of the work, its proof can be derived from an adaptation of the results obtained by Solonnikov in [112, 113].

Theorem 2.7. *Let D be a connected, bounded open set of $\mathbb{R}^n (n \geq 2)$ with boundary ∂D of class $\mathcal{C}^l$ and let f be a function in $\mathbb{H}^l(D)$, $l \geq 0$. Then (2.25) has an unique solution v. Moreover, $v \in \mathbb{H}^{l+2}(D) \cap \mathbb{V}$ and the following hold:*

$$(v, h)_{\mathbb{V}} = (v, h) \text{ for any } h \in \mathbb{V}, \tag{2.26}$$

$$\text{and } |v|_{\mathbb{W}} \leq C|f|_{\mathbb{V}}, \text{ if } f \text{ is an element of } \mathbb{V}. \tag{2.27}$$

Now we turn our attention to the definitions of some operators needed in this work.

(OP1) The operator $(I + \alpha A)^{-1}$ defines an isomorphism from $\mathbb{H}^l(D) \cap \mathbb{H}$ onto $\mathbb{H}^{l+2}(D) \cap \mathbb{V}$ provided that D is of class $\mathcal{C}^l$, $l \geq 1$ (see Theorem 2.7). Moreover for any $f \in \mathbb{H}^l(D) \cap \mathbb{H}$ and any $v \in \mathbb{V}$ we have

(a) $((I + \alpha A)^{-1} f, v)_{\mathbb{V}} = (f, v)$,

(b) $|(I + \alpha A)^{-1} f|_{\mathbb{V}} \leq C|f|$.

(OP2) It follows from (OP1) that the mapping

$$\widehat{A} = (I + \alpha A)^{-1} A$$

is linear continuous from $\mathbb{H}^l(D) \cap \mathbb{V}$ onto itself, $l \geq 2$, and it satisfies

$$(\widehat{A}u, v)_{\mathbb{V}} = (Au, v) = ((u, v)),$$

for any $u \in \mathbb{W}$ and $v \in \mathbb{V}$. If $u = v \in \mathbb{W}$ then we have

$$(\widehat{A}u, u)_{\mathbb{V}} = ||u||^2. \tag{2.28}$$

In view of the fact that we deal most of the time with time-dependent functions and stochastic processes, it is necessary to give some notations about evolution spaces. For any Banach space $\mathfrak{X}$ with norm $||.||_{\mathfrak{X}}$, for any $p \geq 1$, $L^p(0, T; \mathfrak{X})$ is the space of $\mathfrak{X}$-valued measurable functions u defined on $[0, T]$ and such that

$$||u||_{L^p(0,T;\mathfrak{X})} = \left(\int_0^T ||u||_{\mathfrak{X}}^p dt \right)^{\frac{1}{p}} < \infty, \; p \in [1, \infty),$$

and

$$||u||_{L^\infty(0,T;\mathfrak{X})} = \operatorname*{ess\ sup}_{t \in (0,T)} ||u(t)||_{\mathfrak{X}} < \infty, \; p = \infty,$$

Chapter 4 of the present work is about the existence of weak probabilistic solutions of the stochastic second grade fluid. The main ingredient of the proof is a compactness method which relies on the following result known as Aubin-Lions's compactness Theorem; its proof can be found in [109].

Theorem 2.8. *Let $\mathfrak{X}, \mathfrak{B}, \mathfrak{Y}$ three Banach spaces such that the following embedding are continuous*

$$\mathfrak{X} \subset \mathfrak{B} \subset \mathfrak{Y}.$$

Moreover, assume that the embedding $\mathfrak{X} \subset \mathfrak{B}$ is compact, then the set $\mathfrak{F}$ consisting of functions $v \in L^q(0, T; \mathfrak{B})$, $1 \leq q \leq \infty$ such that

$$\sup_{0 \leq h \leq 1} \int_{t_1}^{t_2} |v(t+h) - v(t)|^p_{\mathfrak{Y}} dt \to 0, \; as\, h \to 0,$$

for any $0 < t_1 < t_2 < T$ is compact in $L^p(0, T; \mathfrak{B})$ for any p.

2.2 Some results from Probability Theory and Stochastic Calculus

In this section, we give some basic definitions and classical theorems from Probability Theory and Stochastic Analysis. We do not provide too much details since most of them are very well-known. For the details and for further reading on Probability Theory and Stochastic Analysis, we urge the reader to consult [4], [36], [53], [70], [71], [95], [96], [103], [111] among many other references.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is a set (it may be a topological vector space) with elements ω, $\mathcal{F}$ denotes the Borel σ-field of subsets of Ω, and $\mathbb{P}$ is a probability measure. Throughout we denote by $\mathbb{E}$ the mathematical expectation associated to the probability measure $\mathbb{P}$.

Definition 2.9. Let $(E, \mathcal{E})$ be a measurable set. Any measurable mapping $X : \Omega \to E$ is called E-valued random variable or a random variable in E. Let $T > 0$ and $I = [0, T]$, a stochastic process in E is any family $X_t = (X(t), t \in I))$ of random variables in E. It is said continuous if its sample paths $X_t(\omega)$ or $X(t,\omega)$ is a continuous function of t for almost all (almost everywhere) $\omega \in \Omega$. A process Y_t is a modification or a version of X_t if

$$\mathbb{P}(\omega : X_t(\omega) = Y_t(\omega)) = 1, \quad \forall t \in I.$$

Throughout this work we will make no difference between X_t and its version. The following theorem is a simple criterion for the existence of a continuous version of a real-valued process X_t. We refer to [71, 103] for its proof and some of its extensions.

Theorem 2.10 (Kolmogorov-Čentsov)**.** *Suppose that a real-valued process* $X = \{X_t, 0 \leq t \leq T\}$ *on a probability space* $(\Omega, \mathbb{P})$ *satisfies the condition*

$$\mathbb{E}|X_{t+h} - X_t|^{\gamma} \leq Ch^{1+\beta}, \; 0 \leq t, h \leq T,$$

for some positive constants $\gamma, \beta,$ *and* C*. Then there exists a continuous modification* $\tilde{X} = \{\tilde{X}_t, 0 \leq t \leq T\}$ *of* X*, which is locally Hölder-continuous with exponent* $\kappa \in (0, \frac{\beta}{\gamma})$.

A filtration $(\mathbb{F}^t)_{0 \leq t \leq T}$ is an increasing σ-fields $\mathbb{F}^t \subset \mathcal{F}$, $t \in I$.

Definition 2.11. An E-valued stochastic process X_t is *adapted* to $(\mathbb{F}^t)_{0 \leq t \leq T}$ if, for any $t \in I$, X_t is $\mathbb{F}^t$-measurable.

An $\mathbb{F}^t$-adapted stochastic process X_t taking its values in E is said to be a $\mathbb{F}^t-$martingale or simply a martingale if it is integrable (with respect to $\mathbb{P}$) such that

$$\mathbb{E}(X_t|\mathbb{F}^s) = X_s, \text{ almost surely, for any } t > s.$$

We refer to [36] (see also [95]) for the notion of integrability of a random variable and for the construction of the conditional expectation $\mathbb{E}(X_t|\mathbb{F}^s)$.
An extended real-valued random variable τ is a *stopping time* if $(\omega : \tau(\omega) \leq t) \in \mathbb{F}^t$ for any $t \in I$.

Definition 2.12. A E-valued process X_t is said to be a *local martingale* if there exists an increasing sequence of stopping times $\tau_n \nearrow \infty$ almost surely such that $X_{t\wedge\tau_n}$ is a martingale for each n. Here $a \wedge b$ means $\min(a,b)$. For $1 \leq p < \infty$, a stochastic process (and/or martingale) X_t is said p-th integrable if $||X_t||_E$ is measurable and $\mathbb{E}||X_t||_E^p < \infty$. We denote by $L^p(\Omega, \mathcal{F}, \mathbb{P})$ the space of p-th integrable stochastic processes.

Now we introduce the Itô integral of a process X with respect to a standard Brownian motion W. As we are dealing with stochastic evolution equations, then we first need to give some notations about probabilistic evolution spaces that are frequently used in this work. Let $(\Omega, \mathcal{F}, (\mathbb{F}^t)_{0\leq t\leq T}, \mathbb{P})$ be a stochastic basis and let $\mathcal{H}$ be a Banach space. For any $1 \leq r, p < \infty$ we denote by $L^p(\Omega, \mathbb{P}; L^r(0,T;\mathcal{H}))$ the space of processes $u = u(\omega, t)$ with values in $\mathcal{H}$ defined on $\Omega \times [0,T]$ such that:

1. u is measurable with respect to (ω, t) and for each t, $u(.,t)$ is $\mathbb{F}^t$-measurable. We call such a stochastic process a progressively measurable process.

2. $u(t,\omega) \in \mathcal{H}$ for almost all (ω, t) and

$$||u||_{L^p(\Omega,\mathbb{P};L^r(0,T;\mathcal{H}))} = \left(\mathbb{E}\left(\int_0^T ||u||^r_{\mathcal{H}} dt\right)^{\frac{p}{r}}\right)^{\frac{1}{p}} < \infty$$

   where $\mathbb{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$.

When $r = \infty$, we write

$$||u||_{L^p(\Omega,\mathbb{P};L^\infty(0,T;\mathcal{H}))} = \left(\mathbb{E} \text{ ess} \sup_{0\leq t\leq T} ||u||^p_{\mathcal{H}}\right)^{\frac{1}{p}} < \infty.$$

Assume that $\mathcal{H}$ is a Hilbert space and let X be a $\mathcal{H}$-valued process such that

(a) X_t is $\mathbb{F}^t$-measurable for each t,

(b) $\mathbb{E}\int_0^T ||X(.,t)||^2_{\mathcal{H}} dt < \infty$,

that is $X_t \in L^2(\Omega, \mathbb{P}; L^2(0,T;\mathcal{H}))$. For such process we can define the integral

$$I(T)X = \int_0^T X(t)dW(t),$$

where W is a standard one dimensional Wiener process, as the limit in probability of the sums

$$\sum_{k=1}^n X_{t_k}(W_{t_{k+1}} - W_{t_k}),$$

as $|\Delta^n| \to 0$. Here $|\Delta^n| = \max_{1\leq k\leq n} |t_k - t_{k-1}|$ is the mesh (or modulus) of the partition $\Delta^n = \{0 = t_0 < t_1 < ... < t_n = T\}$ of $I = [0,T]$. This idea was initially developed by Itô

[63] and extended by several authors to stochastic integral with respect to a wide class of stochastic processes (see for examples [4], [13], [36], [53], [64], [70], [71], [82], [95], [96], [103]).

We state an important property of the stochastic integral. See, for example, [53] and [103] for its proof.

Theorem 2.13. *For process X_t satisfying (a) and (b) the stochastic process $\int_0^t X(s)dW(s)$ is an $\mathcal{H}$-valued continuous martingale. Moreover, we have*

$$\mathbb{E}\int_0^{\cdot} X(s)dW(s) = 0. \tag{2.29}$$

We note that for those $\mathbb{F}^t$-adapted stochastic process X_t such that

$$\int_0^T ||X(.,t)||^2_{\mathcal{H}}dt < \infty \text{ almost surely,}$$

$I(t)X$ is no longer a martingale but a continuous local martingale.

Now let Z_t be a $\mathbb{R}$-valued Itô's integral with respect to a standard Brownian motion in $\mathbb{R}^m$ defined by

$$Z_t = \sum_{j=1}^m \int_0^t g_j(s)dW^j(s),$$

where $(g_j(s))_{1\leq j\leq m}$ are $\mathbb{F}^t$-adapted process such that

$$\int_0^T g_j^2(s)ds < \infty \text{ almost surely,}$$

for $1 \leq j \leq m$. The corresponding Itô integrals exist and they are local martingales. We have the following result known as Itô's formula (see for example [53], [103]):

Theorem 2.14. *Let X_t be a stochastic process given by*

$$X_t = \int_0^t b(s)ds + \sum_{j=1}^m \int_0^t g_j(s)dW^j(s),$$

where $b(s)$ is an adapted integrable process over $[0,T]$ in $\mathbb{R}$.

Suppose that $\phi : \mathbb{R}\times[0,T] \to \mathbb{R}$ is a continuous function such that $\phi(x,t)$ is continuously differentiable twice in x and once in t. Then, the following holds

$$\begin{aligned}\phi(X_t,t) = \phi(X_0,0) &+ \int_0^t \frac{\partial\phi(X_s,s)}{\partial s}ds + \int_0^t \frac{\partial\phi(X_s,s)}{\partial x}b(s)ds \\ &+ \sum_{j=1}^m \int_0^t \frac{\partial\phi(X_s,s)}{\partial x}g_j(s)dW^j(s) \\ &+ \frac{1}{2}\sum_{j=1}^m \int_0^t \frac{\partial^2\phi(X_s,s)}{\partial x^2}g_j^2(s)ds.\end{aligned}$$

In what follows we quote the famous Burkhölder-Davis-Gundy inequality (cf. e.g [95], [103]).

Theorem 2.15. *If Z_t is the Itô's integral in $\mathcal{H}$ given by*

$$Z_t = \int_0^t X(s)dW(s),$$

then for any $p > 0$ there exists a constant K_p ($K_2 = 3$) such that

$$\mathbb{E} \sup_{0\le t\le T} \left|\left| \int_0^t X(s)dW(s) \right|\right|_{\mathcal{H}}^p \le K_p \mathbb{E} \left(\int_0^T ||X(s)||^2_{\mathcal{H}} ds \right)^{\frac{p}{2}},$$

provided that

$$\mathbb{E} \left(\int_0^T ||X(s)||^2_{\mathcal{H}} ds \right)^{\frac{p}{2}} < \infty.$$

Now we turn our attention to the weak convergence topology in the space of Borel probability measures on topological spaces (see [70]). For a topological space $\mathfrak{X}$ we denote by $\mathcal{P}(\mathfrak{X})$ the space of Borel probability measures on $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$, $\mathcal{B}(\mathfrak{X})$ is the Borel σ-field of $\mathfrak{X}$.

Definition 2.16. Let $\mathfrak{X}$ be a topological space.

(i) A family $\mathfrak{P}_k$ of probability measures on $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$ is relatively compact if every sequence of elements of $\mathfrak{P}_k$ contains a subsequence $\mathfrak{P}_{k_j}$ which converges weakly to a probability measure $\mathfrak{P}$, that is, for any ϕ bounded and continuous function on $\mathfrak{X}$,

$$\lim_{k_j\to\infty} \int_{\mathfrak{X}} \phi(\mathfrak{x}) \mathfrak{P}_{k_j}(d\mathfrak{x}) = \int_{\mathfrak{X}} \phi(\mathfrak{x}) \mathfrak{P}(d\mathfrak{x}).$$

(ii) The family $\mathfrak{P}_k$ is said to be tight if for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathfrak{X}$ such that $\mathbb{P}(K_\varepsilon) \ge 1 - \varepsilon$, for every $\mathbb{P} \in \mathfrak{P}_k$.

For a Polish space $\mathfrak{X}$ (that is, a separable and complete metric space), the following theorem due to Prokhorov gives a sufficient and necessary condition for a sequence of probability measures on $\mathfrak{X}$ to be weakly (or relatively) compact. We refer to [36] (see also [70]) for its proof.

Theorem 2.17 (Prokhorov)**.** *The family $\mathfrak{P}_k$ is relatively compact if and only if it is tight.*

Next we present the relationship between convergence in distribution and convergence almost surely of random variables (see [36], [70]).

Theorem 2.18 (Skorokhod)**.** *For any sequence of probability measures $\mathfrak{P}_k$ on Ω which converges to a probability measure $\mathfrak{P}$, there exist a probability space $(\Omega', \mathbb{F}', \mathbb{P}')$ and random variables X_k, X with values in Ω such that the probability law of X_k (resp., X) is $\mathfrak{P}_k$(resp., $\mathfrak{P}$) and $\lim_{k\longrightarrow\infty} X_k = X$ $\mathbb{P}'$-almost surely*

To close this section we present a result relating the convergence in measure (or almost surely) of random variable to the convergence in mean of order 1.

Definition 2.19. A family of random variables $(X_n)_{n\in\mathbb{N}}$ is said to be uniformly integrable if and only if

$$\lim_{A\to\infty} \int_{|X_n|>A} |X_n| d\mathbb{P} = 0$$

uniformly in $n \in \mathbb{N}$.

The following result gives a sufficient and necessary condition for a family $(X_n)_{n\in\mathbb{N}}$ to be uniform integrable.

Theorem 2.20. *(see [45]) Let $(X_n)_{n\in\mathbb{N}}$ be a bounded subset of $L^1(\Omega, \mathcal{F}, \mathbb{P})$. That is $\int |X_n| d\mathbb{P}$ is bounded. Then, the following propositions are equivalent*

1. *$(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.*
2. *there exists an increasing function $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(x)/|x| \to \infty$ as $|x| \to \infty$ and $\sup_n \int \phi(X_n) d\mathbb{P} < \infty$.*

We quote Vitali's Convergence Theorem which is a generalization of Lebesgue's Dominated Convergence for finite measure. For its proof we refer to [45].

Theorem 2.21. *Let $0 < r < \infty$, $X_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $X_n \to X$ in probability. Then, the following three propositions are equivalent*

1. *$(|X_n|)_n$ is uniformly integrable,*
2. *$X_n \to X$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$,*
3. *$\mathbb{E}|X_n| \to \mathbb{E}|X|$.*

Chapter 3

Existence, uniqueness and long time behavior of the strong probabilistic solution

3.1 Introduction

The study of stochastic partial differential equations (SPDEs) in a Hilbert space goes back to Backlan [5] but the existence and uniqueness of strong probabilistic solution were first established by Bensoussan and Temam in [9], [10] for some classes of stochastic nonlinear evolution equations including stochastic Navier-Stokes equations. These results were further extended for more general SPDEs by Pardoux [94], Krylov and Rozovskii [73] among many others.

Unlike most of work dealing with strong probabilistic solution of SPDEs (see for examples [73], [81], [94],...), we prove in the first section of this chapter an existence theorem of strong probabilistic solution for the stochastic model of the bidimensional second grade fluids without using the monotonicity method. The idea of the proof is to show that the Galerkin approximating solutions of problem (1.5) converges strongly in $L^2(\Omega, \mathbb{P}; L^2(0, \tau_M; \mathbb{V}))$, where $(\tau_M)_{M\geq 0}$ is an increasing sequence of stopping times which converges to T as $M \to \infty$. The original idea goes back to Pardoux [94], and it was extensively used in [15], [28], and [39].

In Section 2 we prove the pathwise uniqueness of the solution. The proof follows from the Itô's formula for the square of the norm in $\mathbb{V}$ of the difference of two solutions defined on the same filtered probability space and starting with the same initial condition.

As the long time behavior of the flow is very interesting and very important in the theory of fluid dynamics (see for example [76] and [116] ...), we will study the time asymptotic stability and the decay of the solution in the last section of the current chapter. For more details and indepth coverage on the long time behavior of the solutions of some examples of SPDEs we refer to [26], [27], [31] and relevant references therein.

3.2 Existence of the strong probabilistic solution

In this part we investigate the existence of the strong probabilistic solution of problem (1.5). This section contains two subsections. The first one is devoted to the formulation of the hypotheses and the statement of the existence theorem while the second consists of the full proof of the result.

3.2.1 Hypotheses and statement of the existence theorem

We start by stating some hypotheses relevant for most part of the chapter. First we shall assume throughout that we endow the prescribed complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\mathbb{F}^t$, $0 \leq t \leq T$, which is the σ-field generated by the random variables $\{W(s), 0 \leq s \leq t\}$ and the null sets of $\mathcal{F}$. The functions F and G introduced in (1.5) induce the following mappings denoted by the same symbols.

(F) The mapping

$$F : \mathbb{V} \times [0, T] \to \mathbb{V}$$

is measurable in the second variable and

(a) for any $t \in [0, T]$,

$$F(0, t) = 0, \tag{3.1}$$

(b) for any $t \in [0, T]$ and any $(u_1, u_2) \in \mathbb{V} \times \mathbb{V}$, we have

$$|F(u_1, t) - F(u_2, t)|_{\mathbb{V}} \leq C|u_1 - u_2|_{\mathbb{V}}. \tag{3.2}$$

From now on we set

$$\mathfrak{X}^{\otimes m} = \underbrace{\mathfrak{X} \times \cdots \times \mathfrak{X}}_{m \text{ times}},$$

for any Banach space $\mathfrak{X}$.

(G) The mapping

$$G : \mathbb{V} \times [0,T] \to \mathbb{V}^{\otimes m}$$

is measurable in the second variable and

(a) for any $t \in [0,T]$

$$G(0,t) = 0, \tag{3.3}$$

(b) for any $t \in [0,T]$ and any $u_1, u_2 \in \mathbb{V}$

$$|G(u_1,t) - G(u_2,t)|_{\mathbb{V}^{\otimes m}} \leq C|u_1 - u_2|_{\mathbb{V}}. \tag{3.4}$$

We can define on $\mathbb{V} \times [0,T]$ two operators $\widehat{F}$ and $\widehat{G}$ taking values in $\mathbb{W}$ and $\mathbb{W}^{\otimes m}$, respectively, by setting

$$\widehat{F}(u,t) = (I + \alpha A)^{-1} F(u,t),$$

and

$$\widehat{G}(u,t) = (I + \alpha A)^{-1} G(u,t).$$

Thanks to the features of $(I + \alpha A)^{-1}$ properties such as measurability in t of F and G are verified by $\widehat{F}$ and $\widehat{G}$. In particular we have

$$|\widehat{F}(u_1,t) - \widehat{F}(u_2,t)|_{\mathbb{V}} \leq C_F|u_1 - u_2|_{\mathbb{V}}, \tag{3.5}$$

$$|\widehat{G}(u_1,t) - \widehat{G}(u_2,t)|_{\mathbb{V}^{\otimes m}} \leq C_G|u_1 - u_2|_{\mathbb{V}}. \tag{3.6}$$

Remark 3.1. The condition (3.1) is required without loss of generality in order to simplify the computations. We can assume that $F(0,t) \neq 0$, but then

$$\int_0^T |F(0,t)|_{\mathbb{V}}^4 dt < \infty,$$

should hold. The same remark applies to the operator G.

Alongside (1.5), we consider the abstract evolution stochastic problem

$$\begin{cases} du(t) + \nu\widehat{A}u(t)dt + \widehat{B}(u(t), u(t))dt = \widehat{F}(u(t), t)dt + \widehat{G}(u(t), t)dW(t), \\ u_0 = u(0), \end{cases} \tag{3.7}$$

which holds in $\mathbb{W}^*$. With the properties of the operators involved we can prove that the stochastic process u satisfies (3.7) if and only if it verifies (1.5) in the weak sense of PDEs.

Now we introduce the concept of solution of problem (1.5) relevant here.

Definition 3.2. By a strong probabilistic solution of the system (1.5), we mean a stochastic process u such that

1. $u \in L^p(\Omega, \mathbb{P}; L^2(0, T; \mathbb{W}))$ with $1 \le p < \infty$,
2. For all t, $u(t)$ is $\mathbb{F}^t$-measurable,
3. $\mathbb{P}$-almost surely the following integral identity holds

$$(u(t) - u(0), v)_{\mathbb{V}} + \int_0^t [\nu((u, v)) + (\text{curl}(u(s) - \alpha\Delta u(s)) \times u, v)]\, ds$$
$$= \int_0^t (F(u(s), s), v)ds + \int_0^t (G(u(s), s), v)dW(s),$$

for any $t \in [0, T]$ and $v \in \mathbb{W}$. Or equivalently, the following equation

$$u(t) + \int_0^t (\nu\widehat{A}u(s) + \widehat{B}(u(s), u(s)))ds = u_0 + \int_0^t \widehat{F}(u(s), s)ds + \int_0^t \widehat{G}(u(s), s)dW(s),$$

holds in $\mathbb{W}^\star$ $\mathbb{P}$-almost surely for any $t \in [0, T]$.

Remark 3.3. In the above definition the quantity $\int_0^t (G(u(s), s), v)dW(s)$ should be understood as

$$\int_0^t (G(u(s), s), v)dW(s) = \sum_{k=1}^m \int_0^t (G_k(u(s), s), v)dW_k(s),$$

where G_k and W_k denote the k-th component of G and W, respectively.

Now we are ready to formulate the main result of this section.

Theorem 3.4. *Assume that $u_0 \in \mathbb{W}$ is non-random and that all the assumptions, namely (3.1)- (3.4), on the operators F and G are satisfied, then the problem* (1.5) *has a solution in the sense of Definition 3.2. Moreover, almost surely the stochastic process u has $\mathbb{W}$ (resp., $\mathbb{V}$)-valued weak (resp., strong) continuous paths..*

Remark 3.5. The theorem still holds if we assume that u_0 is $\mathbb{F}^0$-measurable and satisfies

$$\mathbb{P}\{\omega : |u_0(\omega)|_{\mathbb{W}} < \infty\} = 1.$$

This result remains also valid if one considers measurable Lipschitz mappings $F : \Omega \times \mathbb{V} \times [0,T] \to \mathbb{V}$ and $G : \Omega \times \mathbb{V} \times [0,T] \to \mathbb{V}^{\otimes m}$.

3.2.2 Proof of the existence result

This subsection is devoted to the proof of the existence result stated in the preceding subsection. We split the proof into two parts.

Part I: The approximate solution and its a priori estimates

In this part we introduce the Galerkin approximation scheme for problem (1.5) and establish crucial a priori estimates for the corresponding approximating solution. They will serve as a toolkit for the proof of Theorem 3.4.

The following statement is a consequence of a spectral theorem for self-adjoint compact operator stated in [102]: The injection of $\mathbb{W}$ into $\mathbb{V}$ is compact. Let I be the isomorphism of $\mathbb{W}^\star$ onto $\mathbb{W}$, then the restriction of I to $\mathbb{V}$ is a continuous compact operator into itself. Thus, there exists a sequence (e_i) of elements of $\mathbb{W}$ which forms an orthonormal basis in $\mathbb{W}$, and an orthogonal basis in $\mathbb{V}$. This sequence verifies:

$$\text{for any } v \in \mathbb{W} \quad (v,e_i)_{\mathbb{W}} = \lambda_i (v,e_i)_{\mathbb{V}}, \tag{3.8}$$

where $\lambda_{i+1} > \lambda_i > 0$, $i = 1, 2, ...$

We have the following important result established in [32] concerning the regularity of the eigenfunctions e_i.

Lemma 3.6. *Let D be a bounded, simply-connected open set of $\mathbb{R}^2$ with a boundary of class $\mathcal{C}^3$, then the functions e_i belong to $\mathbb{H}^4(D)$.*

We now introduce the Galerkin approximation scheme for the problem (1.5). We consider the subset $\mathbb{W}_N = \text{Span}(e_1, \dots, e_N) \subset \mathbb{W}$ and look for a finite-dimensional approximation of a solution of our problem as a vector $u^N \in \mathbb{W}_N$ that can be written as a Fourier series:

$$u^N(t) = \sum_{i=1}^{N} c_{iN}(t) e_i(x). \tag{3.9}$$

We require u^N to satisfy the following system

$$\begin{aligned} & d(u^N, e_i)_{\mathbb{V}} + \nu((u^N, e_i))dt + b(u^N, u^N, e_i)dt - \alpha b(u^N, \Delta u^N, e_i)dt + \alpha b(e_i, \Delta u^N, u^N)dt \\ & = (F(u^N, t), e_i)dt + (G(u^N, t), e_i)dW, i \in \{1, ..., N\}, \end{aligned} \tag{3.10}$$

or equivalently

$$d(u^N, e_i)_{\mathbb{V}} + \nu(\widehat{A}u^N, e_i)_{\mathbb{V}}dt + < \widehat{B}(u^N, u^N), e_i > dt = (\widehat{F}(u^N, t), e_i)_{\mathbb{V}}dt + (\widehat{G}(u^N, t), e_i)_{\mathbb{V}}dW, \tag{3.11}$$

for any $i \in \{1, .., N\}$. The function u_0^N is the orthogonal projection of $u(0)$ onto the space $\mathbb{W}_N$, and

$$u_0^N (\text{or } u^N(0)) \to u(0) \text{ strongly in } \mathbb{V} \text{ as } N \to \infty.$$

The Fourier coefficients c_{iN} in (3.9) are solutions of a system of stochastic ordinary differential equations (SODEs) with locally Lipschitz coefficients. By well known existence and uniqueness theorem on SODEs (see for example [71], [111]), a sequence of continuous functions u^N exists at least on a short interval $[0, T_N]$. Global existence will follow from a priori estimates for u^N.

From now on, we denote by C any constant depending only on the data, and which may change from one line to the next. We start by proving the following result.

Lemma 3.7. *For any $N \geq 1$ we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} |u^N(t)|_{\mathbb{V}}^2 + \mathbb{E} \int_0^T |u^N(t)|_{\mathbb{V}}^2 dt < +\infty. \tag{3.12}$$

We also have

$$\mathbb{E} \sup_{0 \leq t \leq T} |u^N(t)|_{\mathbb{W}}{}^2 + \mathbb{E} \int_o^T |u^N(t)|_{\mathbb{W}}{}^2 dt < +\infty. \tag{3.13}$$

Proof. From now we denote by $|v|_*$ the quantity $|\operatorname{curl}(v - \alpha \Delta v)|$ for any $v \in \mathbb{W}$. For any integer $M \geq 1$ we introduce the stopping times

$$\tau_M = \begin{cases} \inf \left\{0 \leq t; |u^N(t)|_{\mathbb{V}} + |u^N(t)|_* \geq M \right\} \\ T \text{ if } \quad \left\{0 \leq t; |u^N(t)|_{\mathbb{V}} + |u^N(t)|_* \geq M \right\} = \emptyset \end{cases}$$

We shall use a modification of the argument used in [2].

For any $0 \leq s \leq t \wedge \tau_M$, $t \in [0,T]$, we may apply Itô's formula (see Theorem 2.14) with $\phi((u^N(s), e_i)_{\mathbb{V}}) = (u^N(s), e_i)^2_{\mathbb{V}}$ to equation (3.10) and obtain

$$\begin{aligned}
&(u^N(s), e_i)^2_{\mathbb{V}} + 2\int_0^s (u^N(r), e_i)_{\mathbb{V}} \left[\nu((u^N(r), e_i)) + b(u^N(r), u^N(r) - \alpha\Delta u^N(r), e_i)\right] dr \\
=&2\int_0^s (u^N(r), e_i)_{\mathbb{V}} \left[-\alpha b(e_i, \Delta u^N(r), u^N(r)) + (F(r, u^N), e_i)\right] dr + \int_0^s (G(r, u^N), e_i) dW \\
&+ \int_0^s (u^N(r), e_i)_{\mathbb{V}} (G(r, u^N), e_i)^2 dr
\end{aligned}$$

We note that $|u^N|^2_{\mathbb{V}} = \sum_{i=1}^N \lambda_i (u^N, e_i)^2_{\mathbb{V}}$. Multiplying the above equation by λ_i and summing over i from 1 to N give us

$$\begin{aligned}
|u^N(s)|^2_{\mathbb{V}} + 2\nu\int_0^s ||u^N||^2 dr = |u^N_0|^2_{\mathbb{V}} + 2\int_0^s (F(r, u^N), u^N) dr + \sum_{i=1}^N \lambda_i \int_0^s (G(r, u^N), e_i)^2 dr \\
+2\int_0^s (G(r, u^N), u^N) dW,
\end{aligned} \tag{3.14}$$

where we have used the fact that $b(u^N, u^N, u^N) = 0$. In view of Remark 3.3 here and in the sequel we make the convention

$$(G(u^N, t), e_i)^2 = \sum_{k=1}^m (G_k(u^N, t), e_i)^2.$$

We obtain from (3.14) that

$$\begin{aligned}
|u^N(s)|^2_{\mathbb{V}} + 2\nu\int_0^s ((u^N(r), u^N(r))) dr \leq |u^N_0|^2_{\mathbb{V}} + \sum_{i=1}^N \lambda_i \int_0^s (G(u^N(r), r), e_i)^2 dr \\
+ 2\int_0^s |(F(u^N(r), r), u^N(r))| dr \\
+ \left| 2\int_0^s (G(u^N(r), r), u^N(r)) dW \right|,
\end{aligned} \tag{3.15}$$

for any $0 \leq s \leq t \wedge \tau_M$, $t \in [0,T]$.

Poincaré's inequality (2.3) implies that

$$|(F(u^N(s), s), u^N)| \leq \mathcal{P}^2 ||u^N||\ ||F(u^N(s), s)||.$$

From this estimate and (2.4) we find that

$$|(F(u^N(s), s), u^N(s))| \leq 2C\frac{\mathcal{P}^2}{\alpha}(1 + |u^N(s)|^2_{\mathbb{V}}). \tag{3.16}$$

Finding uniform estimate for the corrector term $\sum_{i=1}^{N} \lambda_i \left(G(u^N(s), e_i)\right)^2$ is not straightforward; this is one of the difficulties already mentioned in the introduction. Since the corrector term is explicitly written as a function depending on the scalar product (in $\mathbb{L}^2(D)$) $(.,.)$ and the e_i-s form an orthonormal basis (resp. orthogonal basis) of $\mathbb{W}$ (resp. $\mathbb{V}$), then the usual Bessel's inequality (see for example [8]) does not apply anymore. To circumvent this difficulty we consider the following generalized Stokes problem

$$\begin{cases} \tilde{G} - \alpha\Delta\tilde{G} + \nabla q = G(u^N(s), s) \text{ in } D \\ \operatorname{div} \tilde{G} = 0 \text{ in } D \\ \tilde{G} = 0 \text{ on } \partial D, \end{cases} \tag{3.17}$$

for any $s \in [0, T]$. By Theorem 2.7 the equation (3.17) has a solution $\tilde{G}$ in $\mathbb{W}^{\otimes m}$ when ∂D is of class $\mathcal{C}^3$ and $G(u^N(s), s) \in \mathbb{V}^{\otimes m}$. Moreover, there exists a positive constant C_0 such that

$$|\tilde{G}|_{[\mathbb{H}^3(D)]^{\otimes m}} \leq C_0 |G(u^N(s), s)|_{\mathbb{V}^{\otimes m}},$$

and $(\tilde{G}_k, e_i)_{\mathbb{V}} = (G_k(u^N(s), s), e_i)$ for any $i \geq 1$.
Since the norms $|.|_{\mathbb{H}^3(D)}$ and $|.|_{\mathbb{W}}$ are equivalent on $\mathbb{W}$, then there exists another positive constant C_* such that

$$|\tilde{G}|_{\mathbb{W}^{\otimes m}} \leq C_* C_0 |G(u^N(s), s)|_{\mathbb{V}^{\otimes m}}. \tag{3.18}$$

The equation (3.18) implies that $\tilde{G}$ depends continuously on the data $G(u^N(s), s)$. Therefore, we denote the above $\tilde{G}$ as $\widehat{G}(u^N(s), s)$. We find from (2.26) and (3.8) that

$$\begin{aligned} \sum_{i=1}^{N} \lambda_i (G(u^N(s), s), e_i)^2 &= \sum_{i=1}^{N} \lambda_i (\widehat{G}(u^N(s), s), e_i)_{\mathbb{V}}^2, \\ &= \sum_{i=1}^{N} \frac{1}{\lambda_i} (\widehat{G}(u^N(s), s), e_i)_{\mathbb{W}}^2. \end{aligned}$$

We deduce from this that

$$\sum_{i=1}^{N} \lambda_i (G(u^N(s), s), e_i)^2 \leq \frac{1}{\lambda_1} |\widehat{G}(u^N(s), s)|_{\mathbb{W}^{\otimes m}}^2.$$

By (3.18) and the assumption on G, we have

$$\sum_{i=1}^{N} \lambda_i (G(u^N(s), s), e_i)^2 \leq C(1 + |u^N(s)|_{\mathbb{V}}^2). \tag{3.19}$$

Collecting these information, we obtain from (3.15) that

$$\begin{aligned}&|u^N(s)|^2_{\mathbb{V}} + 2\nu \int_0^s ||u^N(r)||^2 dr \\ &\leq C + C\int_0^s |u^N(r)|^2_{\mathbb{V}} dr + 2\left|\int_0^s (G(u^N(r), r), u^N(r)) dW\right|.\end{aligned} \tag{3.20}$$

Taking the sup over $0 \leq s \leq t \wedge \tau_M$ in both sides of this inequality and passing to the mathematical expectation in the resulting relation and finally applying the Burkhölder-Davis-Gundy's inequality (cf. Theorem 2.15) to the stochastic term, we get

$$\begin{aligned}&\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|^2_{\mathbb{V}} + 2\nu\mathbb{E}\int_0^{t\wedge\tau_M} ||u^N(s)||^2 ds \\ \leq& C + C\mathbb{E}\int_0^{t\wedge\tau_M} |u^N(s)|^2_{\mathbb{V}} ds \\ &+ 2C_1\mathbb{E}\left(\int_0^{t\wedge\tau_M} (G(u^N(s), s), u^N(s))^2 ds\right)^{\frac{1}{2}}.\end{aligned} \tag{3.21}$$

Now we are interested in estimating

$$\gamma = \mathbb{E}\left(\int_0^{t\wedge\tau_M} (G(u^N(s), s), u^N(s))^2 ds\right)^{\frac{1}{2}}.$$

Using the same argument as in the estimate of $|(F(u^N(s), s), u^N(s))|$, we have

$$\gamma \leq C\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|_{\mathbb{V}} \left(\int_0^{t\wedge\tau_M} |G(u^N(s), s)|^2_{\mathbb{V}} ds\right)^{\frac{1}{2}}\right].$$

By ε-Young's inequality

$$\gamma \leq C\varepsilon\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|^2_{\mathbb{V}} + C_\varepsilon\mathbb{E}\int_0^{t\wedge\tau_M} |G(u^N(s), s)|^2_{\mathbb{V}} ds.$$

Using the assumption on G one has

$$\gamma \leq C\varepsilon\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|^2_{\mathbb{V}} + C_\varepsilon\mathbb{E}\int_0^{t\wedge\tau_M} (1 + |u^N(s)|^2_{\mathbb{V}}). \tag{3.22}$$

A convenient choice of ε with the estimates (3.21) and (3.22) gives us

$$\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|^2_{\mathbb{V}} + 4\nu\mathbb{E}\int_0^{t\wedge\tau_M} ||u^N(s)||^2 ds \leq C + C\mathbb{E}\int_0^{t\wedge\tau_M} |u^N(s)|^2_{\mathbb{V}} ds.$$

We derive from this last inequality and Gronwall's inequality that

$$\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|^2_{\mathbb{V}} + \frac{4\nu}{\mathcal{P}^2 + \alpha}\mathbb{E}\int_0^{t\wedge\tau_M} |u^N(s)|^2_{\mathbb{V}} ds \leq C. \tag{3.23}$$

We recall the following relation which is very important in the sequel

$$\lambda_i(G(u^N(s),s),e_i) = (\widehat{G}(u^N(s),s),e_i)_{\mathbb{W}}, i \geq 1, \tag{3.24}$$

where $\widehat{G}(u^N(s),s)$ is the solution in $\mathbb{W}$ of (3.17). For sake of simplicity we only write u^N when we mean $u^N(.)$. Let us set

$$\phi(u^N) = -\nu\Delta u^N + \operatorname{curl}(u^N - \alpha\Delta u^N) \times u^N - F(u^N,t).$$

By Lemma 3.6, $\phi(u^N) \in \mathbb{H}^1(D)$ and we have

$$d(u^N,e_i)_{\mathbb{V}} + (\phi(u^N),e_i)dt = (G(u^N,t),e_i)dW.$$

By Theorem 2.7, a solution $v^N \in \mathbb{W}$ of the following system

$$\begin{cases} v^N - \alpha\Delta v^N + \nabla q = \phi(u^N) \text{ in } D, \\ \operatorname{div} v^N = 0 \text{ in } D, \\ v^N = 0 \text{ on } \partial D, \end{cases}$$

exists. Moreover,

$$(v^N,e_i)_{\mathbb{V}} = (\phi(u^N),e_i),$$

for any i. Thus,

$$\begin{aligned} d(u^N,e_i)_{\mathbb{V}} + (\phi(u^N),e_i)dt &= d(u^N,e_i)_{\mathbb{V}} + (v^N,e_i)_{\mathbb{V}}dt \\ &= (G(u^N,t),e_i)dW. \end{aligned}$$

Multiplying the latter equation by λ_i and by making use of the relationship (3.8), we have

$$d(u^N,e_i)_{\mathbb{W}} + (v^N,e_i)_{\mathbb{W}}dt = \lambda_i(G(u^N,t),e_i)dW.$$

Recalling (3.24), we obtain

$$d(u^N,e_i)_{\mathbb{W}} + (v^N,e_i)_{\mathbb{W}}dt = (\widehat{G}(u^N,t),e_i)_{\mathbb{W}}dW.$$

We argue as before by considering the stopping times τ_M. By applying Itô's formula to $\varphi((u^N,e_i)_{\mathbb{W}}) = (u^N,e_i)^2_{\mathbb{W}}$, we have

$$d(u^N,e_i)^2_{\mathbb{W}} + 2(u^N,e_i)_{\mathbb{W}}(v^N,e_i)_{\mathbb{W}}dt = (\widehat{G}(u^N,t),e_i)^2_{\mathbb{W}}dt + 2(u^N,e_i)_{\mathbb{W}}(\widehat{G}(u^N,t),e_i)_{\mathbb{W}}dW.$$

Summing both sides of the last equation from 1 to N yields

$$d|u^N|^2_{\mathbb{W}} + 2(u^N, v^N)_{\mathbb{W}}dt = \sum_{i=1}^{N}(\widehat{G}(u^N,t), e_i)^2_{\mathbb{W}}dt + 2(\widehat{G}(u^N,t), u^N)_{\mathbb{W}}dW.$$

Using the definition of $|.|_{\mathbb{W}}$ and the scalar product $(.,.)_{\mathbb{W}}$, we can rewrite the above equation in the form

$$\begin{aligned}
&d[|u^N|^2_{\mathbb{V}} + |u^N|^2_*] + 2[(v^N, u^N)_{\mathbb{V}} + (\mathrm{curl}(v^N - \alpha\Delta v^N), \mathrm{curl}(u^N - \alpha\Delta u^N))]dt \\
&= 2(\mathrm{curl}(\widehat{G}(u^N,t) - \alpha\Delta\widehat{G}(u^N,t)), \mathrm{curl}(u^N - \alpha\Delta u^N))dW \\
&+ \sum_{i=1}^{N}\lambda_i^2(\widehat{G}(u^N,t), e_i)^2_{\mathbb{V}}dt + 2(\widehat{G}(u^N,t), u^N)_{\mathbb{V}}dW.
\end{aligned}$$

In view of the Remark 3.3, we agree that in the sequel

$$(\mathrm{curl}(G(u^N,t)), \mathrm{curl}(u^N - \alpha\Delta u^N))\frac{dW}{dt} = \sum_{k=1}^{m}(\mathrm{curl}(G_k(u^N,t)), \mathrm{curl}(u^N - \alpha\Delta u^N))\frac{dW_k}{dt}.$$

Using the definition of v^N and $\widehat{G}$, we obtain

$$\begin{aligned}
&d[|u^N|^2_{\mathbb{V}} + |u^N|^2_*] + 2[(\phi(u^N), u^N) + (\mathrm{curl}(\phi(u^N)), \mathrm{curl}(u^N - \alpha\Delta u^N))]dt \\
&= \sum_{i=1}^{N}\lambda_i^2(G(u^N,t), e_i)^2dt + 2(G(u^N,t), u^N)dW \\
&+ 2(\mathrm{curl}(G(u^N,t), \mathrm{curl}(u^N - \alpha\Delta u^N))dW.
\end{aligned}$$

With the help of (3.14) the latter equation can be rewritten in the following way

$$\begin{aligned}
d|u^N|^2_* + 2(\mathrm{curl}\,\phi(u^N), \mathrm{curl}(u^N - \alpha\Delta u^N))dt &= 2(\mathrm{curl}(G(u^N,t)), \mathrm{curl}(u^N - \alpha\Delta u^N))dW \\
&+ \sum_{i=1}^{N}(\lambda_i + \lambda_i^2)(G(u^N,t), e_i)^2dt.
\end{aligned} \tag{3.25}$$

We infer from the definition of $\phi(u^N)$ that

$$\mathrm{curl}\,\phi(u^N) = -\nu\,\mathrm{curl}(\Delta u^N + F(u^N,t)) + \mathrm{curl}\left(\mathrm{curl}(u^N - \alpha\Delta u^N) \times u^N\right).$$

Using the two-dimensional nature of our problem, we have

$$\mathrm{curl}\left(\mathrm{curl}(u^N - \alpha\Delta u^N) \times u^N\right) = (u^N.\nabla)(\mathrm{curl}(u^N - \alpha\Delta u^N)).$$

This yields

$$\begin{aligned}&((u^N.\nabla)(\operatorname{curl}(u^N-\alpha\Delta u^N))-\nu\operatorname{curl}(\Delta u^N+F(u^N,t)),\operatorname{curl}(u^N-\alpha\Delta u^N))\\&=(\operatorname{curl}\phi(u^N),\operatorname{curl}(u^N-\alpha\Delta u^N)).\end{aligned}$$

Owing to Lemma 3.6 we can check that

$$((u^N.\nabla)\beta,\beta)=0,$$

where $\beta=\operatorname{curl}(u^N-\alpha\Delta u^N)$. Thus

$$\begin{aligned}&\frac{\nu}{\alpha}|u^N|_*-\frac{\nu}{\alpha}(\operatorname{curl}u^N+\frac{\alpha}{\nu}\operatorname{curl}(F(u^N,t)),\operatorname{curl}(u^N-\alpha\Delta u^N))\\&=(\operatorname{curl}\phi(u^N),\operatorname{curl}(u^N-\alpha\Delta u^N)).\end{aligned}\tag{3.26}$$

We derive from (3.25) and (3.26) that

$$\begin{aligned}&\frac{d}{dt}|u^N|_*^2+\frac{2\nu}{\alpha}|u^N|_*^2-\frac{2\nu}{\alpha}(\operatorname{curl}u^N+\frac{\alpha}{\nu}\operatorname{curl}(F(u^N,t)),\operatorname{curl}(u^N-\alpha\Delta u^N))\\&=\sum_{i=1}^N(\lambda_i+\lambda_i^2)(G(u^N,t),e_i)^2+2\left(\operatorname{curl}(G(u^N,t)),\operatorname{curl}(u^N-\alpha\Delta u^N)\right)\frac{dW}{dt}.\end{aligned}\tag{3.27}$$

We infer from (3.27) that

$$\begin{aligned}&|u^N(s)|_*^2+\int_0^s\left(\frac{2\nu}{\alpha}|u^N(r)|_*^2-\sum_{i=1}^N(\lambda_i+\lambda_i^2)(G(u^N(r),r),e_i)^2\right)dr\\&=\int_0^s\frac{2\nu}{\alpha}\left[\left(\operatorname{curl}(u^N(r))-\frac{\alpha}{\nu}\operatorname{curl}(F(u^N(r),r)),\operatorname{curl}(u^N(r)-\alpha\Delta u^N(r))\right)\right]dr\\&+2\int_0^s(\operatorname{curl}(G(u^N(r),r)),\operatorname{curl}(u^N(r)-\alpha\Delta u^N(r)))dW.\end{aligned}$$

Hence,

$$\begin{aligned}&|u^N(s)|_*^2+\int_0^s\frac{2\nu}{\alpha}|u^N(r)|_*^2dr-\sum_{i=1}^N(\lambda_i+\lambda_i^2)\int_0^s(G(u^N(r),r),e_i)^2dr\\&\leq|u_0^N|_*^2+\int_0^s\frac{2\nu}{\alpha}|\operatorname{curl}(u^N(r))||u^N(r)|_*+\int_0^s2|\operatorname{curl}(F(u^N(r),r))||u^N(r)|_*dr\\&+2\left|\int_0^s\left(\operatorname{curl}(G(u^N(r),r)),\operatorname{curl}(u^N(r)-\alpha\Delta u^N(r))\right)dW\right|.\end{aligned}\tag{3.28}$$

Taking the supremum over $0\leq s\leq t\wedge\tau_M$ in (3.28), and passing to the mathematical expectation yield

$$\begin{aligned}&\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M}|u^N(s)|_*^2+\mathbb{E}\int_0^{t\wedge\tau_M}\frac{2\nu}{\alpha}|u^N(s)|_*^2ds-\sum_{i=1}^N(\lambda_i+\lambda_i^2)\mathbb{E}\int_0^{t\wedge\tau_M}(G(u^N(s),s),e_i)^2ds\\&\leq|u_0^N|_*^2+\mathbb{E}\int_0^{t\wedge\tau_M}\frac{2\nu}{\alpha}|\operatorname{curl}(u^N(s))||u^N(s)|_*+\mathbb{E}\int_0^{t\wedge\tau_M}2|\operatorname{curl}(F(u^N(s),s))||u^N(s)|_*ds\\&+2\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M}\left|\int_0^{s\wedge\tau_M}\left(\operatorname{curl}(G(u^N(s),s)),\operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)dW\right|.\end{aligned}$$

For any $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$, we have

$$\begin{aligned}
&\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|_*^2 + \mathbb{E}\int_0^{t\wedge\tau_M} \frac{2\nu}{\alpha}|u^N(s)|_*^2 ds - \sum_{i=1}^N (\lambda_i+\lambda_i^2)\mathbb{E}\int_0^{t\wedge\tau_M} (G(u^N(s),s),e_i)^2 ds \\
&\leq |u_0^N|_*^2 + \mathbb{E}\int_0^{t\wedge\tau_M}\left(\frac{2\nu}{\alpha\varepsilon_1}|\operatorname{curl}(u^N(s))|^2 + \frac{2}{\varepsilon_2}|\operatorname{curl}(F(u^N(s),s))|^2\right) ds \\
&+ 2\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M}\left|\int_0^{s\wedge\tau_M}\left(\operatorname{curl}(G(u^N(s),s)),\operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right) dW\right| \\
&+ (\frac{2\nu\varepsilon_1}{\alpha} + 2\varepsilon_2)\int_0^{t\wedge\tau_M} |u^N(s)|_*^2 ds.
\end{aligned}$$

We choose $\varepsilon_1 = 1/4$, $\varepsilon_2 = \nu/4\alpha$ and we deduce from the last inequality the following estimate,

$$\begin{aligned}
&\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|_*^2 + \mathbb{E}\int_0^{t\wedge\tau_M} \frac{\nu}{\alpha}|u^N(s)|_*^2 ds - \sum_{i=1}^N (\lambda_i+\lambda_i^2)\mathbb{E}\int_0^{t\wedge\tau_M} (G(u^N(s),s),e_i)^2 ds \\
&\leq |u_0^N|_*^2 + \mathbb{E}\int_0^{t\wedge\tau_M}\left(\frac{8\nu}{\alpha\varepsilon_1}|\operatorname{curl}(u^N(s))|^2 + \frac{2\alpha}{\nu}|\operatorname{curl}(F(u^N(s),s))|^2\right) ds \\
&+ 2\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M}\left|\int_0^{s\wedge\tau_M}\left(\operatorname{curl}(G(u^N(s),s)),\operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right) dW\right|.
\end{aligned} \tag{3.29}$$

Thanks to (3.19), (3.24) and (3.18) we see that

$$\sum_{i=1}^N (\lambda_i+\lambda_i^2)\mathbb{E}\int_0^{t\wedge\tau_M} (G(u^N(s),s),e_i)^2 ds \leq C + C\mathbb{E}\int_0^{s\wedge\tau_M} |u^N(s)|_{\mathbb{V}}^2 ds. \tag{3.30}$$

Now let us estimate

$$\gamma = 2\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M}\left|\int_0^{t\wedge\tau_M}\left(\operatorname{curl}(G(u^N(s),s)),\operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right) dW\right|.$$

By Fubini's Theorem and Burkhölder-Davis-Gundy's inequality (see Theorem 2.15) we obtain

$$\begin{aligned}
\gamma &\leq 6\mathbb{E}\left(\int_0^{t\wedge\tau_M}\left(\operatorname{curl}(G(u^N(s),s)),\operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2 ds\right)^{\frac{1}{2}}, \\
&\leq 6\mathbb{E}\left(\sup_{0\leq s\leq t\wedge\tau_M}|u^N(s)|_*\left(\int_0^{t\wedge\tau_M}|\operatorname{curl}(G(u^N(s),s))|^2 ds\right)^{\frac{1}{2}}\right).
\end{aligned}$$

By making use of Young's inequality, the following holds

$$\gamma \leq 6\varepsilon\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M}|u^N(s)|_*^2 + \frac{6}{\varepsilon}\mathbb{E}\int_0^{t\wedge\tau_M}|\operatorname{curl}(G(u^N(s),s))|^2 ds.$$

Choosing $\varepsilon = 1/12$, we have

$$\gamma \leq \frac{1}{2}\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|_*^2 + 72\mathbb{E}\int_0^{t\wedge\tau_M} |\operatorname{curl}(G(u^N(s),s))|^2 ds. \tag{3.31}$$

Gathering (3.29), (3.30) and (3.31), we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|_*^2 + \mathbb{E}\int_0^{t\wedge\tau_M} \frac{\nu}{\alpha}|u^N(s)|_*^2 ds \\
&\leq C\mathbb{E}\int_0^{t\wedge\tau_M} |\operatorname{curl}(F(u^N(s),s))|^2 + C\mathbb{E}\int_0^{t\wedge\tau_M} |\operatorname{curl}(G(u^N(s),s))|^2 ds \\
&C + |u_0^N|_*^2 + C\mathbb{E}\int_0^{t\wedge\tau_M} |u^N(s)|_{\mathbb{V}}^2 ds + C\mathbb{E}\int_0^{t\wedge\tau_M} |\operatorname{curl}(u^N(s))|^2 ds.
\end{aligned} \tag{3.32}$$

Since

$$|\operatorname{curl}(\phi)|^2 \leq \frac{2}{\alpha}|\phi|_{\mathbb{V}}^2 \text{ for any } \phi \in \mathbb{V},$$

the assumptions on F and G, and the relations (3.32), (3.23) imply that

$$\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|_*^2 + \mathbb{E}\int_0^{t\wedge\tau_M} \frac{\nu}{\alpha}|u^N(s)|_*^2 ds \leq C. \tag{3.33}$$

Now we need to show that $T_N = T$ almost surely. This is equivalent to prove that $\lim_{M\to\infty} \tau_M = T$ almost surely. We argue exactly as in [2]. For any $t \in [0,T]$ we have

$$\begin{aligned}
\mathbb{E}|u^N(t\wedge\tau_M|_{\mathbb{W}}^2 &= \mathbb{E}[|u^N(t\wedge\tau_M)|_{\mathbb{W}}^2 1_{\tau_M<t}] + \mathbb{E}[|u^N(t\wedge\tau_M)|_{\mathbb{W}}^2 1_{\tau_M\geq t}] \\
&= \mathbb{E}[|u^N(\tau_M)|_{\mathbb{W}}^2 1_{\tau_M<t}] + \mathbb{E}[|u^N(t)|_{\mathbb{W}}^2 1_{\tau_M\geq t}]
\end{aligned}$$

Since u^N is continuous on the finite dimensional space $\mathbb{W}_n$, then we have

$$|u^N(\tau_M)|_* + |u^N(\tau_M)|_{\mathbb{V}} \geq M,$$

which implies that

$$|u^N(\tau_M)|_{\mathbb{W}}^2 = |u^N(\tau_M)|_*^2 + |u^N(\tau_M)|_{\mathbb{V}}^2 \geq \frac{M^2}{2}.$$

Notice that $\mathbb{E}1_{(\tau_M<t)} = \mathbb{P}(\tau_M < t)$, therefore

$$\begin{aligned}
\mathbb{E}|u^N(t\wedge\tau_M)|_{\mathbb{W}}^2 &= \mathbb{E}[|u^N(\tau_M)|_{\mathbb{W}}^2 1_{\tau_M<t}] + \mathbb{E}[|u^N(t)|_{\mathbb{W}}^2 1_{\tau_M\geq t}] \\
&\geq \frac{M^2}{2}\mathbb{P}(\tau_M < t).
\end{aligned}$$

Hence, for all $M \geq 1$

$$\begin{aligned}\mathbb{P}(\tau_M < t) &\leq \frac{2}{M^2}\mathbb{E}|u^N(t\wedge\tau_M)|^2_{\mathbb{W}}\\ &\leq \frac{2C}{M^2}.\end{aligned}$$

Since C is independent of N and M, we infer that

$$\lim_{M\to\infty}\mathbb{P}(\tau_M < t) = 0,\ \forall t\in[0,T].$$

That is, $\tau_M \to T$ in probability. Thus, we can extract from τ_M a subsequence τ_{M_j} such that $\tau_{M_j}\to T$ almost surely. We deduce from the monotonity of $(\tau_M)_M$ (this sequence is increasing) that $\tau_M \nearrow T$ almost surely. This yields that $T_N = T$ almost surely.

Since the constant C is independent of N, the estimates (3.23), (3.33) and the Dominated Lebesgue's Convergence Theorem complete the proof of the lemma. □

Lemma 3.8. *For any $4\leq p<\infty$ we have*

$$\mathbb{E}\sup_{0\leq s\leq T}(|u^N(s)|_{\mathbb{V}})^p + \mathbb{E}\left(\int_0^T |u^N(s)|_{\mathbb{V}}{}^2\right)^{\frac{p}{2}} < \infty \tag{3.34}$$

and

$$\mathbb{E}\sup_{0\leq s\leq T}(|u^N(s)|_{\mathbb{W}})^p + \mathbb{E}\left(\int_0^T |u^N(s)|_{\mathbb{W}}{}^2\right)^{\frac{p}{2}} < \infty. \tag{3.35}$$

Proof. We recall that

$$\begin{aligned}&d|u^N(t)|^2_{\mathbb{V}} + 2\nu||u^N(t)||^2 dt - 2(F(u^N(t),t),u^N(t))dt\\ &= \sum_{i=1}^N \lambda_i (G(u^N(t),t),e_i)^2 dt + 2(G(u^N(t),t),u^N(t))dW.\end{aligned}$$

For a fixed $p\geq 4$ Itô's formula to the function $\phi(|u^N(t)|^2_{\mathbb{V}}) = |u^N(t)|^{2\frac{p}{4}}_{\mathbb{V}}$ (cf. Theorem 2.14) yields

$$\begin{aligned}|u^N(t)|^{\frac{p}{2}}_{\mathbb{V}} = |u_0^N|^{\frac{p}{2}}_{\mathbb{V}} &+ \frac{p}{2}\int_0^t |u^N(s)|^{\frac{p}{2}-2}_{\mathbb{V}}\Big[-\nu||u^N(s)||^2 + (F(u^N(s),s),u^N(s))\\ &+ \frac{1}{2}\sum_{i=1}^N (G(u^N(s),s),e_i)^2 + \frac{p-4}{4}\frac{(G(u^N(s),s),u^N(s))^2}{|u^N(s)|^2_{\mathbb{V}}}\Big]ds\\ &+ \frac{p}{2}\int_0^t |u^N(s)|^{\frac{p}{2}-2}_{\mathbb{V}}(G(u^N(s),s),u^N(s))dW,\end{aligned}$$

for any $t \in [0,T]$. By squaring both sides of the last equation and by making use of some elementary inequalities we obtain

$$
\begin{aligned}
|u^N(t)|_{\mathbb{V}}^p \leq C|u_0^N|_{\mathbb{V}}^p + C\Big(\int_0^t |u^N(s)|_{\mathbb{V}}^{\frac{p}{2}-2}\Big[-\nu||u^N(s)||^2 + (F(u^N(s),s),u^N(s)) \\
+ \frac{1}{2}\sum_{i=1}^N (G(u^N(s),s),e_i)^2 + \frac{p-4}{4}\frac{(G(u^N(s),s),u^N(s))^2}{|u^N(s)|_{\mathbb{V}}^2}\Big] ds\Big)^2 \\
+ C\left(\int_0^t |u^N(s)|_{\mathbb{V}}^{\frac{p}{2}-2}(G(u^N(s),s),u^N(s))dW\right)^2.
\end{aligned}
$$

We deduce from this inequality along with (3.16) and (3.19) that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}|u^N(s)|_{\mathbb{V}}^p \leq C|u_0^N|_{\mathbb{V}}^p + C\mathbb{E}\int_0^t |u^N(s)|_{\mathbb{V}}{}^{p-4}(1+|u^N(s)|_{\mathbb{V}})^4 ds \\
+ C\mathbb{E}\sup_{0\leq s\leq t}\left(\int_0^s |u^N(r)|_{\mathbb{V}}{}^{\frac{p}{2}-2}(G(u^N(r),r),u^N(r))dW\right)^2.
\end{aligned}
\tag{3.36}
$$

This estimate together with arguments similar to those used in the proof of Lemma 3.7 lead to

$$
\mathbb{E}\sup_{0\leq s\leq T}|u^N(s)|_{\mathbb{V}}^p < \infty. \tag{3.37}
$$

We now proceed to the proof of an important estimate concerning $|u^N|_{\mathbb{W}}^p$. We rewrite the equation (3.27) in the form

$$
\begin{aligned}
d|u^N(s)|_*^2 = \sum_{i=1}^N(\lambda_i+\lambda_i^2)(G(u^N(s),s),e_i)^2 + \frac{2\nu}{\alpha}\left(\operatorname{curl} u^N(s), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)\Big) ds \\
+ \Big(-\frac{2\nu}{\alpha}|u^N(s)|_*^2 + 2(\operatorname{curl}(F(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))) \\
+ 2(\operatorname{curl}(G(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s)))dW.
\end{aligned}
$$

Applying Itô's formula to the function $\varphi(|u^N(s)|_*^2) = |u^N(s)|_*^{2\frac{p}{4}}$ we have

$$
\begin{aligned}
d|u^N(s)|_*^{\frac{p}{2}} - \frac{p}{2}|u^N(s)|_*^{\frac{p}{2}-2}\Big(2(\operatorname{curl}(F(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))) \\
= \frac{1}{2}\sum_{i=1}^N(\lambda_i+\lambda_i^2)(G(u^N(s),s),e_i)^2 + \frac{2\nu}{\alpha}\left(\operatorname{curl} u^N(s), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right) \\
- \frac{2\nu}{\alpha}|u^N(s)|_*^2 + \frac{p-4}{4}\frac{\left(\operatorname{curl}(G(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2}{|u^N(s)|_*^2}\Big) ds \\
+ \frac{p}{2}|u^N(s)|_*^{\frac{p}{2}-2}\left(\operatorname{curl}(G(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right) dW.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|u^N(t)|_*^{\frac{p}{2}} = |u_0^N|_*^{\frac{p}{2}} + \frac{p}{2}\int_0^t |u^N(s)|_*^{\frac{p}{2}-2}\Big(2(\operatorname{curl}(F(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s)))\\
+\frac{1}{2}\sum_{i=1}^{N}(\lambda_i+\lambda_i^2)(G(u^N(s),s),e_i)^2 + \frac{2\nu}{\alpha}\left(\operatorname{curl} u^N(s), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)\\
-\frac{2\nu}{\alpha}|u^N(s)|_*^2 + \frac{p-4}{4}\frac{\left(\operatorname{curl}(G(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2}{|u^N(s)|_*^2}\Big)ds\\
+\frac{p}{2}\int_0^t |u^N(s)|_*^{\frac{p}{2}-2}\left(\operatorname{curl}(G(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)dW,
\end{aligned}
$$

for any $t \in [0,T]$. By squaring both sides of the last inequality, we obtain

$$
\begin{aligned}
|u^N(t)|_*^{p} \leq C|u_0^N|_*^{p} + C\Big[\int_0^t |u^N(s)|_*^{\frac{p}{2}-2}\Big(2(\operatorname{curl}(F(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s)))\\
+\frac{1}{2}\sum_{i=1}^{N}(\lambda_i+\lambda_i^2)(G(u^N(s),s),e_i)^2 + \frac{2\nu}{\alpha}\left(\operatorname{curl} u^N(s), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)\\
-\frac{2\nu}{\alpha}|u^N(s)|_*^2 + \frac{p-4}{4}\frac{\left(\operatorname{curl}(G(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2}{|u^N(s)|_*^2}\Big)ds\Big]^2\\
+C\left(\int_0^t |u^N(s)|_*^{\frac{p}{2}-2}\left(\operatorname{curl}(G(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)dW\right)^2.
\end{aligned}
\tag{3.38}
$$

We note that

$$
|(\operatorname{curl} u^N(s), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s)))| \leq C(1+|u^N(s)|_{\mathbb{V}})(1+|u^N(s)|_{\mathbb{W}}), \forall s\in[0,T].
$$

We also check that

$$
|(\operatorname{curl}(F(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s)))| \leq C(1+|u^N(s)|_{\mathbb{V}})(1+|u^N(s)|_{\mathbb{W}}), \tag{3.39}
$$

and

$$
\left|\frac{\left(\operatorname{curl}(G(u^N(s),s)), \operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2}{|u^N(s)|_*^2}\right| \leq C(1+|u^N(s)|_{\mathbb{V}})^2.
$$

Thanks to the continuous injection of $\mathbb{W}$ into $\mathbb{V}$, all the above estimates still hold with $|u^N(.)|_{\mathbb{V}}$ replaced by $|u^N(.)|_{\mathbb{W}}$. Hence we can derive from (3.38) that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}|u^N(s)|_*^p \leq |u_0^N|_*^p + C\mathbb{E}\left(\int_0^t |u^N(s)|_*^{\frac{p}{2}-2}(1+|u^N(s)|_{\mathbb{W}})^2 ds\right)^2\\
+C\mathbb{E}\sup_{0\leq s\leq t}\left(\int_0^s |u^N(r)|_*^{\frac{p}{2}-2}\left(\operatorname{curl}(G(u^N,r), \operatorname{curl}(u^N-\alpha\Delta u^N)\right)dW\right)^2.
\end{aligned}
\tag{3.40}
$$

Applying the Martingale inequality (see Theorem 2.15) and Hölder's inequality in the last estimate we obtain

$$\begin{aligned}\mathbb{E}\sup_{0\leq s\leq t}|u^N(s)|_*^p &\leq |u_0^N|_*^p + C\mathbb{E}\int_0^t |u^N(s)|_*^{p-4}(1+|u^N(s)|_{\mathbb{W}})^4 ds \\ &\quad + C\mathbb{E}\int_0^t |u^N(s)|_*^{p-4}\left(\operatorname{curl}(G(u^N(s),s),\operatorname{curl}(u^N(s)-\alpha\Delta u^N(s))\right)^2 ds.\end{aligned} \tag{3.41}$$

We can use the same idea as that used to find (3.39) to get an upper-bound of the form $C(1+|u^N(s)|_{\mathbb{W}})^4$ for $|(\operatorname{curl}(G(u^N(s),s)),\operatorname{curl}(u^N(s)-\alpha\Delta u^N(s)))^2|$. Then, we derive from (3.41) that

$$\mathbb{E}\sup_{0\leq s\leq t}|u^N(s)|_*^p \leq C|u_0^N|_*^p + C\int_0^t (1+|u^N(s)|_{\mathbb{W}})^p ds. \tag{3.42}$$

We obviously have

$$|u^N(s)|_{\mathbb{W}}^p \leq C(|u^N(s)|_{\mathbb{V}}^p + |u^N(s)|_*^p).$$

Finally, using (3.37), (3.42) and Gronwall's inequality we obtain

$$\mathbb{E}\sup_{0\leq s\leq T}|u^N(s)|_*^p < \infty. \tag{3.43}$$

It is not hard to see that Lemma 3.8 follows by using the estimates (3.20), (3.28) along with (3.37) and (3.43). □

Remark 3.9. Lemmas 3.7 and 3.8 imply in particular that

$$\mathbb{E}\sup_{0\leq s\leq T}|u^N(s)|_{\mathbb{V}}{}^p + \mathbb{E}\left(\int_0^T |u^N(s)|_{\mathbb{V}}{}^2\right)^{\frac{p}{2}} < \infty,$$

and

$$\mathbb{E}\sup_{0\leq s\leq T}|u^N(s)|_{\mathbb{W}}{}^p + \mathbb{E}\left(\int_0^T |u^N(s)|_{\mathbb{W}}{}^2\right)^{\frac{p}{2}} < \infty.$$

for any $1\leq p<\infty$.

Part II: Passage to the limit and proof of the continuity of the paths of u

To complete the proof of Theorem 3.4 we need to pass to the limit in the Galerkin approximation.

Thanks to Lemma 3.7 and Lemma 3.8 (see also Remark 3.9) we can extract by a diagonal process a subsequence u^{N_μ} such that

$$u^{N_\mu} \rightharpoonup u \text{ weakly star in } L^2(\Omega, \mathbb{P}, L^\infty(0,T,\mathbb{V})) \tag{3.44}$$

$$u^{N_\mu} \rightharpoonup u \text{ weakly star in } L^2(\Omega, \mathbb{P}, L^\infty(0,T,\mathbb{W}))$$

$$u^{N_\mu} \rightharpoonup u \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V}))$$

$$u^{N_\mu} \rightharpoonup u \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{W})), \tag{3.45}$$

as $N_\mu \to \infty$. Owing to the properties of the operators A, $\widehat{B}$, F and G together with Lemma 3.7 and Lemma 3.8 there exist three operators $\widehat{B}^*(s)$, $F^*(s)$ and $G^*(s)$ such that

$$Au^{N_\mu} \rightharpoonup Au \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V})),$$

$$\widehat{B}(u^{N_\mu}, u^{N_\mu}) \rightharpoonup \widehat{B}^*(s) \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{W}^*)),$$

$$F(u^{N_\mu}, s) \rightharpoonup F^*(s) \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V})),$$

$$G(u^{N_\mu}, s) \rightharpoonup G^*(s) \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V}^{\otimes m})).$$

It follows from the properties of the linear operator $(I+\alpha A)^{-1}$ that

$$\widehat{A}u^{N_\mu} \rightharpoonup \widehat{A}u \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V})) \tag{3.46}$$

$$\widehat{F}(u^{N_\mu}, s) \rightharpoonup \widehat{F}^*(s) \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V})),$$

$$\widehat{G}(u^{N_\mu}, s) \rightharpoonup \widehat{G}^*(s) \text{ weakly in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V}^{\otimes m})), \tag{3.47}$$

where $\widehat{F}^*$ (resp., $\widehat{G}^*$) denotes $(I+\alpha A)^{-1}F^*$(resp., $(I+\alpha A)^{-1}G^*$). With these convergences at hand we see from (3.11) that the following holds

$$\begin{aligned}(u(t), e_i)_{\mathbb{V}} + \int_0^t (\nu(\widehat{A}u(s), e_i)_{\mathbb{V}} + < \widehat{B}^*(s), e_i >)ds = (u_0, e_i)_{\mathbb{V}} + \int_0^t (\widehat{F}^*(s), e_i)_{\mathbb{V}} ds \\ + \int_0^t (\widehat{G}^*(s), e_i)_{\mathbb{V}} dW(s),\end{aligned} \tag{3.48}$$

for any $t \in [0,T]$, $i \in \{1,..,N\}$, and for almost all $\omega \in \Omega$.

Arguing as in [94] (see Chapitre 2, Page 42) we deduce from the latter equation that almost all paths of u are $\mathbb{V}$-valued continuous functions. Since

$$u \in L^2(\Omega, \mathbb{P}; L^\infty(0,T;\mathbb{V})),$$

(see for example (3.44)), then we can use the argument in [115] (Chap. 3, Section 1) to assert that almost all paths of u are weakly continuous as functions taking values in $\mathbb{W}$. To complete the proof of Theorem 3.4 we need the following result.

Lemma 3.10. *We have the following identities*

$$\widehat{B}^*(s) = \widehat{B}(u,u) \text{ in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{W}^*)), \tag{3.49}$$

$$\widehat{F}^*(s) = \widehat{F}(u,s) \text{ in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V})), \tag{3.50}$$

$$\widehat{G}^*(s) = \widehat{G}(u,s) \text{ in } L^2(\Omega, \mathbb{P}, L^2(0,T,\mathbb{V}^{\otimes m})). \tag{3.51}$$

The proof of this lemma requires the following result.

Lemma 3.11. *For any $M \geq 1$ we have that*

$$1_{[0,\tau_M]}(u^{N_\mu} - u) \to 0 \text{ strongly in } L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{V})), \tag{3.52}$$

as $N_\mu \to \infty$. Here τ_M is defined by

$$\tau_M = \begin{cases} \inf\left\{0 \leq t \leq T; |u^N(t)|_{\mathbb{V}} + |u^N(t)|_* \geq M\right\} \\ T \text{ if } \left\{0 \leq t \leq T; |u^N(t)|_{\mathbb{V}} + |u^N(t)|_* \geq M\right\} = \emptyset. \end{cases}$$

Proof. We follow some arguments used in [15]. From now on we set $N := N_\mu$, $\widehat{F}(v,t) := \widehat{F}(v)$, and $\widehat{G}(v,t) := \widehat{G}(v)$ for any $v \in \mathbb{V}$ and $t \in [0,T]$. Let $P^N : \mathbb{W} \to \mathbb{W}_N$ be the orthogonal projection defined by

$$P^N v = \sum_{i=1}^{N} (v, e_i)_{\mathbb{W}} e_i = \sum_{i=0}^{N} \lambda_i (v, e_i)_{\mathbb{V}} e_i, \; v \in \mathbb{W}.$$

For any $v \in L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{W}))$ we have

$$P^N v(\omega,t) \to v(\omega,t) \text{ strongly in } \mathbb{W} \text{ for almost all } (\omega,t) \in \Omega \times [0,T],$$

and

$$|P^N v|^2_{\mathbb{W}} \leq C|v|^2_{\mathbb{W}}.$$

Therefore, by the Dominated Convergence Theorem we see that

$$P^N v \to v \text{ strongly in } L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{W})). \tag{3.53}$$

Thanks to the continuous embedding $L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{W})) \hookrightarrow L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{V}))$, we also have

$$P^N v \to v \text{ strongly in } L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{V})). \tag{3.54}$$

We can show through some calculations that

$$(P^N v, e_i)_{\mathbb{V}} = (v, e_i)_{\mathbb{V}},$$

for any $v \in \mathbb{W}$. It follows from this and (3.48) that

$$(P^N u(t), e_i)_{\mathbb{V}} + \int_0^t (\nu(\widehat{A}u(s), e_i)_{\mathbb{V}} + < \widehat{B}^*(s), e_i >) ds - (u_0^N, e_i)_{\mathbb{V}}$$
$$= \int_0^t (\widehat{F}^*(s), e_i)_{\mathbb{V}} ds + \int_0^t (\widehat{G}^*(s), e_i)_{\mathbb{V}} dW(s),$$

for any $t \in [0,T]$, $i \in \{1, .., N\}$. This equation with (3.11) imply that

$$(P^N u(t) - u^N(t), e_i)_{\mathbb{V}} + \int_0^t \nu(\widehat{A}(u(s) - u^N(s)), e_i)_{\mathbb{V}}$$
$$= \int_0^t < \widehat{B}(u^N(s), u^N(s)) - \widehat{B}^*(s), e_i > ds + \int_0^t (\widehat{F}^*(s) - \widehat{F}(u^N), e_i)_{\mathbb{V}} ds$$
$$+ \int_0^t (\widehat{G}^*(s) - \widehat{G}(u^N), e_i)_{\mathbb{V}} dW(s).$$

We apply Itô's formula (see again Theorem 2.14) to the function $(P^N u(t) - u^N(t), e_i)^2_{\mathbb{V}}$, then we multiply the resulting equation by λ_i and sum over i from 1 to N to obtain

$$|P^N u - u^N|^2_{\mathbb{V}} + 2\nu \int_0^t (\widehat{A}(u - u^N), P^N u - u^N)_{\mathbb{V}}$$
$$= 2 \int_0^t < \widehat{B}(u^N, u^N) - \widehat{B}^*(s), P^N u - u^N > ds + 2 \int_0^t (\widehat{F}^*(s) - \widehat{F}(u^N), P^N u - u^N)_{\mathbb{V}} ds$$
$$+ \sum_{i=1}^N \lambda_i \int_0^t (\widehat{G}^*(s) - \widehat{G}(u^N), e_i)^2_{\mathbb{V}} ds + 2 \int_0^t (\widehat{G}^*(s) - \widehat{G}(u^N), P^N u - u^N)_{\mathbb{V}} dW(s).$$

Here we have set $u(s) = u$, $u^N(s) = u^N$; $s \in [0,T]$. Now let

$$\sigma(t) = \exp(-\eta_1 t - \eta_2 \int_0^t |u(s)|^2_{\mathbb{W}} ds),$$

where η_1 and η_2 are two positive constants to be defined later. We obtain by application of Itô's formula to the process $\sigma(t)|P^N u - u^N|^2_{\mathbb{V}}$ that

$$\begin{aligned}
&\sigma(t)|P^N u - u^N|^2_{\mathbb{V}} + 2\nu \int_0^t \sigma(s)(\widehat{A}(u-u^N), P^N u - u^N)_{\mathbb{V}} \\
=&\sum_{i=1}^N \lambda_i \int_0^t \sigma(s)(\widehat{G}^*(s) - \widehat{G}(u^N), e_i)^2_{\mathbb{V}} ds + 2\int_0^t \sigma(s)(\widehat{G}^*(s) - \widehat{G}(u^N), P^N u - u^N)_{\mathbb{V}} dW(s) \\
&- \eta_1 \int_0^t \sigma(s)|P^N u - u^N|^2_{\mathbb{V}} ds - \eta_2 \int_0^t \sigma(s)|P^N u - u^N|^2_{\mathbb{V}} |u(s)|^2_{\mathbb{W}} ds \\
&+ 2\int_0^t \sigma(s) < \widehat{B}(u^N, u^N) - \widehat{B}^*(s), P^N u - u^N > ds \\
&+ 2\int_0^t \sigma(s)(\widehat{F}^*(s) - \widehat{F}(u^N), P^N u - u^N)_{\mathbb{V}} ds.
\end{aligned} \tag{3.55}$$

Let us analyze each term of (3.55). Firstly we have that

$$(\widehat{A}(u-u^N), P^N u - u^N)_{\mathbb{V}} = (\widehat{A}(u-u^N), u - u^N)_{\mathbb{V}} - (\widehat{A}(u-u^N), u - P^N u)_{\mathbb{V}}. \tag{3.56}$$

By the definition of P^N we have $P^N u^N = u^N$ and

$$|P^N(u-u^N)|_{\mathbb{V}} \le |u-u^N|_{\mathbb{V}}.$$

Then we deduce from (2.28), (2.4) together with the above properties of P^N that

$$\frac{1}{\mathcal{P}^2+\alpha}|P^N u - u^N|^2_{\mathbb{V}} \le (\widehat{A}(u-u^N), u-u^N)_{\mathbb{V}}. \tag{3.57}$$

Secondly, by the properties of $\widehat{F}(.,t)$ the following holds

$$\begin{aligned}
2(\widehat{F}^*(s) - \widehat{F}(u^N), P^N u - u^N)_{\mathbb{V}} \le& \frac{4\mathcal{P}^2 C_F}{\alpha}|u^N - P^N u|^2_{\mathbb{V}} + \frac{2\mathcal{P}^2 C_F}{\alpha}|P^N u - u|^2_{\mathbb{V}} \\
&+ 2(\widehat{F}^*(s) - \widehat{F}(u), P^N u - u^N)_{\mathbb{V}}.
\end{aligned} \tag{3.58}$$

Thirdly, let us set

$$\mathcal{G} = \sum_{i=0}^N \lambda_i (\widehat{G}(u^N) - \widehat{G}^*(s), e_i)^2_{\mathbb{V}}.$$

We have that

$$\mathcal{G} = |P^N[\widehat{G}(u^N) - \widehat{G}^*(s)]|^2_{\mathbb{V}}.$$

We derive from this relation that

$$\begin{aligned}
\mathcal{G} =& |P^N[\widehat{G}(u) - \widehat{G}(u^N)]|^2_{\mathbb{V}^{\otimes m}} - |P^N[\widehat{G}(u) - \widehat{G}^*(s)]|^2_{\mathbb{V}^{\otimes m}} \\
&+ 2(P^N[\widehat{G}^*(s) - \widehat{G}(u^N)], P^N[\widehat{G}^*(s) - \widehat{G}(u)])_{\mathbb{V}}.
\end{aligned}$$

It follows from the properties of $\widehat{G}$ that

$$\begin{aligned}\mathcal{G} \leq 2C_G|P^N u-u|_{\mathbb{V}}^2 + 2C_G|P^N u - u^N|_{\mathbb{V}}^2 - |P^N[\widehat{G}(u) - \widehat{G}^*(s)]|_{\mathbb{V}^{\otimes m}}^2 \\ + 2(P^N[\widehat{G}^*(s) - \widehat{G}(u^N)], P^N[\widehat{G}^*(s) - \widehat{G}(u)])_{\mathbb{V}}.\end{aligned} \tag{3.59}$$

Fourthly, we have that

$$\begin{aligned}< \widehat{B}(u^N, u^N), P^N u - u^N > = < \widehat{B}(P^N u - u^N, P^N u - u^N), P^N u > \\ + < \widehat{B}(P^N u, P^N u), P^N u - u^N > .\end{aligned} \tag{3.60}$$

Indeed, the property (2.23) implies that

$$\begin{aligned}< \widehat{B}(P^N u - u^N, P^N u - u^N), P^N u > \\ = < \widehat{B}(u^N, u^N), P^N u - u^N > - < \widehat{B}(P^N u, u^N), P^N u >,\end{aligned} \tag{3.61}$$

and

$$- < \widehat{B}(P^N u, u^N), P^N u > = < \widehat{B}(P^N u, P^N u - u^N), P^N u > . \tag{3.62}$$

Gathering (3.61) and (3.62), and applying (2.24) yield (3.60). We deduce from (3.60) along with property (2.8) that

$$\begin{aligned}2 < \widehat{B}(u^N, u^N), P^N u - u^N > \leq 2C_B|P^N u - u^N|_{\mathbb{V}}^2 |P^N u|_{\mathbb{W}} \\ + 2 < \widehat{B}(P^N u, P^N u), P^N u - u^N > .\end{aligned}$$

Therefore

$$\begin{aligned}2 < \widehat{B}(u^N, u^N) - \widehat{B}^*(s), P^N u - u^N > \leq C_B^2|P^N u - u^N|_{\mathbb{V}}^2 |P^N u|_{\mathbb{W}}^2 + |u^N - P^N u|_{\mathbb{V}}^2 \\ + 2 < \widehat{B}(P^N u, P^N u) - \widehat{B}^*(s), P^N u - u^N > .\end{aligned} \tag{3.63}$$

Using (3.56) and (3.57) in (3.55) one gets

$$\begin{aligned}\sigma(t)|P^N u(t) - u^N(t)|_{\mathbb{V}}^2 + \frac{2\nu}{\mathcal{P}^2 + \alpha}\int_0^t \sigma(s)|P^N u - u^N|_{\mathbb{V}}^2 ds \\ \leq \text{ Right Hand Side of (3.55) } + 2\nu\int_0^t \sigma(s)(\widehat{A}(u - u^N), u - P^N u)_{\mathbb{V}} ds.\end{aligned} \tag{3.64}$$

Now we can choose

$$\begin{aligned}\eta_1 &= \frac{4\mathcal{P}^2 C_F}{\alpha} + 2C_G + 1, \\ \eta_2 &= C_B^2,\end{aligned}$$

and we can infer from (3.64), (3.63), (3.59) and (3.58) that

$$\begin{aligned}
&\sigma(t)|P^N u(t) - u^N(t)|_{\mathbb{V}}^2 + \int_0^t \sigma(s)\left(\frac{2\nu}{\mathcal{P}^2+\alpha}|P^N u - u^N|_{\mathbb{V}}^2 + |P^N[\widehat{G}(u) - \widehat{G}^*(s)]|_{\mathbb{V}^{\otimes m}}^2\right) ds \\
\leq & 2\nu \int_0^t \sigma(s)(\widehat{A}(u-u^n), u - P^N u)_{\mathbb{V}} ds + C\int_0^t \sigma(s)|P^N u - u|_{\mathbb{V}}^2 ds \\
&+ 2\int_0^t \sigma(s) < \widehat{B}(P^N u, P^N u) - \widehat{B}^*(s), P^N u - u^N > ds \\
&+ 2\int_0^t \sigma(s)(\widehat{F}^*(s) - \widehat{F}(u), P^N u - u^N)_{\mathbb{V}} ds \\
&+ 2\int_0^t \sigma(s)(P^N[\widehat{G}^*(s) - \widehat{G}(u^N)], P^N[\widehat{G}^*(s) - \widehat{G}(u)])_{\mathbb{V}} ds \\
&+ 2\int_0^t \sigma(s)(\widehat{G}^*(s) - \widehat{G}(u^N), P^N u - u^N)_{\mathbb{V}} dW.
\end{aligned} \tag{3.65}$$

Replacing t by τ_M in the estimate (3.65), and taking the mathematical expectation yield

$$\begin{aligned}
&\mathbb{E}\sigma(\tau_M)|P^N u(\tau_M) - u^N(\tau_M)|_{\mathbb{V}}^2 + C\mathbb{E}\int_0^{\tau_M} \sigma(s)\left(|P^N u - u^N|_{\mathbb{V}}^2 - |P^N u - u|_{\mathbb{V}}^2\right) ds \\
\leq & -\mathbb{E}\int_0^{\tau_M} \sigma(s)|P^N[\widehat{G}(u) - \widehat{G}^*(s)]|_{\mathbb{V}^{\otimes m}}^2 ds + 2\nu\mathbb{E}\int_0^{\tau_M} \sigma(s)(\widehat{A}(u-u^n), u - P^N u)_{\mathbb{V}} ds \\
&+ 2\mathbb{E}\int_0^{\tau_M} \sigma(s)(P^N[\widehat{G}^*(s) - \widehat{G}(u^N)], P^N[\widehat{G}^*(s) - \widehat{G}(u)])_{\mathbb{V}} ds \\
&+ 2\mathbb{E}\int_0^{\tau_M} \sigma(s) < \widehat{B}(P^N u, P^N u) - \widehat{B}^*(s), P^N u - u^N > ds \\
&+ 2\mathbb{E}\int_0^{\tau_M} \sigma(s)(\widehat{F}^*(s) - \widehat{F}(u), P^N u - u^N)_{\mathbb{V}} ds.
\end{aligned}$$

Since

$$\begin{aligned}
\left|1_{[0,\tau_M]}\sigma(t)[\widehat{B}(P^N u(t), P^N u(t)) - \widehat{B}(u(t), u(t))]\right|_{\mathbb{W}^*} \leq & 1_{[0,\tau_M]} C_B |u(t)|_{\mathbb{W}} |P^N u(t) - u(t)|_{\mathbb{W}} \\
&+ 1_{[0,\tau_M]} C_B |u(t)|_{\mathbb{W}} |P^N u(t) - u(t)|_{\mathbb{V}},
\end{aligned}$$

then

$$\left|1_{[0,\tau_M]}\sigma(t)[\widehat{B}(P^N u(t), P^N u(t)) - \widehat{B}(u(t), u(t))]\right|_{\mathbb{W}^*} \to 0 \text{ almost all } (\omega, t) \in \Omega \times [0,T], \tag{3.66}$$

as $N \to \infty$. Furthermore

$$\left|1_{[0,\tau_M]}\sigma(t)[\widehat{B}(P^N u(t), P^N u(t)) - \widehat{B}(u(t), u(t))]\right|_{\mathbb{W}^*} \leq 2MC_B C^* |u(t)|_{\mathbb{W}}^2 \in L^2(\Omega \times [0,T], \mathbb{R}), \tag{3.67}$$

where C^* is the constant in the continuous embedding $\mathbb{W} \hookrightarrow \mathbb{V}$. By the convergences (3.45), (3.53) we have

$$P^N u - u^N \rightharpoonup 0 \text{ weakly in } L^2(\Omega, \mathbb{P}; L^2(0, T; \mathbb{W})).$$

We derive from this, (3.66) and (3.67) that

$$\mathbb{E} \int_0^{\tau_M} \sigma(s) < \widehat{B}(P^N u, P^N u) - \widehat{B}(u, u), P^N u - u^N > ds \to 0$$

as $N \to \infty$. Hence

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E} \int_0^{\tau_M} \sigma(s) < \widehat{B}(P^N u, P^N u) - \widehat{B}^*(s), P^N u - u^N > ds \\
=& \lim_{N\to\infty} \mathbb{E} \int_0^{\tau_M} \sigma(s) < \widehat{B}(P^N u, P^N u) - \widehat{B}(u, u), P^N u - u^N > ds \\
&+ \lim_{N\to\infty} \mathbb{E} \int_0^{\tau_M} \sigma(s) < \widehat{B}(u, u) - \widehat{B}^*(s), P^N u - u^N > ds \\
=&0
\end{aligned} \tag{3.68}$$

Owing to (3.46) the sequence $\widehat{A}(u - u^N)$ is bounded in $L^2(\Omega, \mathbb{P}; L^2(0, T; \mathbb{V}))$, thus

$$\mathbb{E} \int_0^{\tau_M} \sigma(s)(\widehat{A}(u - u^N), P^N u - u)_{\mathbb{V}} ds \to 0 \text{ as } N \to \infty. \tag{3.69}$$

We can also show that

$$\mathbb{E} \int_0^{\tau_M} \sigma(s)(\widehat{F}^*(s) - \widehat{F}(u), P^N u - u^N)_{\mathbb{V}} ds \to 0 \text{ as } N \to \infty.$$

The convergence (3.54) implies that

$$\lim_{N\to 0} \mathbb{E} \int_0^{\tau_M} \sigma(s)|P^N u - u|_{\mathbb{V}}^2 ds = 0. \tag{3.70}$$

Since

$$1_{[0,\tau_M]}\sigma(s) \left(\widehat{G}^*(s) - \widehat{G}(u)\right) \in L^2(\Omega, \mathbb{P}; L^2(0, T; \mathbb{V}))$$

then we can derive from (3.47) that

$$\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_M} \sigma(s)(P^N[\widehat{G}^*(s) - \widehat{G}(u^N)], P^N[\widehat{G}^*(s) - \widehat{G}(u)])_{\mathbb{V}} ds = 0. \tag{3.71}$$

The convergences (3.68)-(3.71) imply that

$$\begin{aligned}
&\mathbb{E}\sigma(\tau_M)|P^N u(\tau_M) - u^N(\tau_M)|_{\mathbb{V}}^2 + C\mathbb{E} \int_0^{\tau_M} \sigma(s)|P^N u - u^N|_{\mathbb{V}}^2 ds + \\
&+ \mathbb{E} \int_0^{\tau_M} \sigma(s)|P^N[\widehat{G}(u) - \widehat{G}^*(s)]|_{\mathbb{V}^{\otimes m}}^2 \to 0,
\end{aligned} \tag{3.72}$$

as $N \to \infty$. The proof of the lemma follows from (3.54) and (3.72). □

Let us now prove Lemma 3.10.

Proof of Lemma 3.10. Now we are going to complete the proof of the identities (3.49)-(3.51). First note that for any $w \in \mathbb{W}$

$$\begin{aligned} S &=< \widehat{B}(u^N, u^N) - \widehat{B}(u,u), w > \\ &=< \widehat{B}(u^N - u, u^N), w > + < \widehat{B}(u, u^N - u), w > . \end{aligned}$$

We also have

$$< \widehat{B}(u^N - u, u^N), w >=< \widehat{B}(u^N, u^N), w > - < \widehat{B}(u, u^N), w >,$$

and

$$< \widehat{B}(u^N, u - u^N), w >=< \widehat{B}(u^N, u), w > - < \widehat{B}(u^N, u^N), w > .$$

Therefore

$$S =< \widehat{B}(u, u-u^N), w > - < \widehat{B}(u^N, u-u^N), w > + < \widehat{B}(u^N, u), w > - < \widehat{B}(u, u^N), w > . \tag{3.73}$$

Thanks to (2.22) the operator

$$\begin{aligned} \widehat{B}_{a,\cdot} :& \mathbb{V} \to \mathbb{W}^* \\ & v \mapsto \widehat{B}_{a,\cdot}(v) = \widehat{B}(a, v) \end{aligned}$$

is linear continuous for any fixed $a \in \mathbb{W}$. Hence, if $u^N \rightharpoonup u$ in $L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{V}))$, then we have

$$\widehat{B}(u, u^N) \rightharpoonup \widehat{B}(u,u) \text{ weakly in } L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{W}^*)). \tag{3.74}$$

By a similar argument, we have

$$\widehat{B}(u^N, u) \rightharpoonup \widehat{B}(u,u) \text{ weakly in } L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{W}^*)). \tag{3.75}$$

Now let w be an element of $L^\infty(\Omega, \mathbb{P}; L^\infty(0,T;\mathbb{W}))$. We deduce from the properties of $\widehat{B}$ that

$$\begin{aligned} &\left| \mathbb{E} \int_0^T 1_{[0,\tau_M]} < \widehat{B}(u, u - u^N), w(s) > - < \widehat{B}(u^N, u - u^N), w(s) > ds \right| \\ &\leq C\mathbb{E} \int_0^{\tau_M} |u|_{\mathbb{W}} |u^N - u|_{\mathbb{V}} ds + C\mathbb{E} \int_0^{\tau_M} |u^N|_{\mathbb{W}} |u^N - u|_{\mathbb{V}} ds. \end{aligned}$$

We derive from (3.52) that

$$\lim_{N\to\infty} \mathbb{E}\int_0^T 1_{[0,\tau_M]} < \widehat{B}(u, u-u^N), w(s) > - < \widehat{B}(u^N, u-u^N), w(s) > ds = 0 \tag{3.76}$$

As $\tau_M \nearrow T$ almost surely, $L^\infty(\Omega, \mathbb{P}; L^\infty(0,T;\mathbb{W}))$ is dense in $L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{W}))$, then we deduce from (3.73)-(3.76) that

$$\widehat{B}(u(.), u(.)) = \widehat{B}^* \text{ in } L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{W}^*)). \tag{3.77}$$

Next, thanks to the property of $\widehat{F}$ we see that

$$\mathbb{E}\int_0^{\tau_M} |\widehat{F}(u^N) - \widehat{F}(u)|^2_{\mathbb{V}} ds \leq C_F \mathbb{E}\int_0^{\tau_M} |u^N - u|^2_{\mathbb{V}} ds.$$

Thanks to (3.52) and the fact that $\tau_M \nearrow T$ almost surely as $M \to \infty$, we obtain that

$$\widehat{F}(u^N) \to \widehat{F}(u) \text{ strongly in } L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{V})). \tag{3.78}$$

And from (3.72) we see that

$$\widehat{G}^*(.) = \widehat{G}(u(.)) \tag{3.79}$$

as an element of $L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{V}^{\otimes m}))$. And this completes the proof of Lemma 3.10. □

It follows from (3.77), (3.78), (3.79), and (3.48) that u is a solution of the stochastic model for the second grade fluid. And this ends the proof of Theorem 3.4.

3.3 On the uniqueness of the strong probabilistic solution

As already mentioned in the introduction we also discuss the pathwise uniqueness of solution. More precisely we prove the following result.

Theorem 3.12. *Let u_1 and u_2 be two strong solutions defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^t, P)$ of the problem (1.5). If we set $v = u_1 - u_2$, then we have $v = 0$ almost surely.*

Proof. Let u_1 and u_2 be two solutions with the same initial condition u_0 and let $v = u_1 - u_2$. It can be shown that the process v satisfies the equation

$$d(v(t) + \alpha A v(t)) + \nu A v(t) dt - F(u_1(t), t) - F(u_2(t), t) dt + G(u_1(t), t) - G(u_2(t), t) dW$$
$$= -\mathbb{P}(\text{curl}(u_1(t) - \alpha\Delta u_1(t)) \times u_1(t) - \text{curl}(u_2(t) - \alpha\Delta u_2(t)) \times u_2(t)) dt,$$

and $v(0) = 0$. We obtain by multiplying this equation by $(I + \alpha A)^{-1}$ and by applying Itô's formula to $|v|_{\mathbb{V}}^2$ (see [73] or [94] for the infinite dimensional version of Theorem 2.14) that

$$\begin{aligned}
&|v(t)|_{\mathbb{V}}^2 + 2\int_0^t \big(\nu||v(s)||^2 - (\mathrm{curl}(v(s) - \alpha\Delta v(s)) \times v(s), u_2(s))\big)\, ds \\
&= \int_0^t \Big(2(\widehat{F}(u_1(s), s) - \widehat{F}(u_2(s), s), v(s))_{\mathbb{V}} + |\widehat{G}(u_1(s), s) - \widehat{G}(u_2(s), s)|_{\mathbb{V}^{\otimes m}}^2\Big)\, ds \\
&+ 2\int_0^t (\widehat{G}(u_1(s), s) - \widehat{G}(u_2(s), s), v(s))_{\mathbb{V}} dW.
\end{aligned}$$

We recall that the operator $(I + \alpha A)^{-1}$ is linear and bijective from $\mathbb{H}$ into $\mathbb{H}^2(D) \cap \mathbb{V}$. Moreover for any $f \in \mathbb{H}$ we have

$$|(I + \alpha A)^{-1} f|_{\mathbb{V}} \leq C|f|. \tag{3.80}$$

For $t \in [0, T]$ let us consider the real process $\sigma(t) = \exp(-\int_0^t |u_2(s)|_{\mathbb{W}}^2 ds)$. We apply Itô's formula to the $\sigma(t)|v(t)|_{\mathbb{V}}^2$ and find that

$$\begin{aligned}
&\sigma(t)|v(t)|_{\mathbb{V}}^2 + 2\int_0^t \big(\sigma(s)(\nu||v(s)||^2 - (\mathrm{curl}(v(s) - \alpha\Delta v(s)) \times v(s), u_2(s))\big)\, ds \\
&= \int_0^t \Big(\sigma(s)(2(\widehat{F}(u_1(s), s) - \widehat{F}(u_2(s), s), v(s))_{\mathbb{V}} + |\widehat{G}(u_1(s), s) - \widehat{G}(u_2(s), s)|_{\mathbb{V}^{\otimes m}}^2\Big)\, ds \\
&- \int_0^t \sigma(s)|u_2(s)|_{\mathbb{W}}^2 |v(s)|_{\mathbb{V}}^2 ds + 2\int_0^t \sigma(s)(\widehat{G}(u_1(s), s) - \widehat{G}(u_2(s), s), v(s))_{\mathbb{V}} dW.
\end{aligned}$$

Using the assumptions on F and G together with (2.22) and (3.80), we derive from the last estimate that

$$\begin{aligned}
\sigma(t)|v(t)|_{\mathbb{V}}^2 + \nu\int_0^t \sigma(s)||v(s)||^2 ds \leq C\int_0^t \sigma(s)|v(s)|_{\mathbb{V}}^2 ds + 2C\int_0^t \sigma(s)|v(s)|_{\mathbb{V}}^2 |u_2(s)|_{\mathbb{W}} ds \\
+2\int_0^t \sigma(s)(\widehat{G}(u_1(s), s) - \widehat{G}(u_2(s), s), v(s))_{\mathbb{V}} dW \\
- \int_0^t \sigma(s)|u_2(s)|_{\mathbb{W}}^2 |v(s)|_{\mathbb{V}}^2 ds.
\end{aligned} \tag{3.81}$$

For any $\varepsilon > 0$, we have

$$C|v(s)|_{\mathbb{V}}^2 |u_2(s)|_{\mathbb{W}} \leq C_\varepsilon |v(s)|_{\mathbb{V}}^2 + C^2\varepsilon |v(s)|_{\mathbb{V}}^2 |u_2(s)|_{\mathbb{W}}^2,$$

for $0 \leq s \leq t$. We infer from this inequality and (3.81) that

$$\sigma(t)|v(t)|_{\mathbb{V}}^2 + \nu \int_0^t \sigma(s)||v(s)||^2 ds \leq C \int_0^t \sigma(s)|v(s)|_{\mathbb{V}}^2 ds + 2\varepsilon C^2 \int_0^t \sigma(s)|v(s)|_{\mathbb{V}}^2 |u_2(s)|_{\mathbb{W}}^2 ds$$
$$+2\int_0^t \sigma(s)(\widehat{G}(u_1(s),s) - \widehat{G}(u_2(s),s), v(s))_{\mathbb{V}} dW$$
$$-\int_0^t \sigma(s)|u_2(s)|_{\mathbb{W}}^2 |v(s)|_{\mathbb{V}}^2 ds.$$

Choosing $\varepsilon = 2/C^2$, we obtain

$$\sigma(t)|v(t)|_{\mathbb{V}}^2 \leq C \int_0^t \sigma(s)|v(s)|_{\mathbb{V}}^2 ds + 2\int_0^t \sigma(s)(\widehat{G}(u_1(s),s) - \widehat{G}(u_2(s),s), v(s))_{\mathbb{V}} dW.$$

We get rid of the stochastic term by taking the mathematical expectation (see equation (2.29)). This enables us to write

$$\mathbb{E}\sigma(t)|v(t)|_{\mathbb{V}}^2 \leq C \int_0^t \mathbb{E}\sigma(s)|v(s)|_{\mathbb{V}}^2 ds,$$

and the proof of Theorem 3.12 follows from the application of Gronwall's inequality. □

3.4 Analysis of the asymptotic behavior of the solutions

In this last section we establish some results on the long time behavior of the strong solutions of the stochastic model for the second grade fluids. We consider two subsections. In the first, we study the exponential decay in mean square of the strong solution. In the second, we strengthen the assumption on the external force F and investigate the exponential stability in mean square of the non-trivial stationary solution.

3.4.1 The exponential decay of the strong probabilistic solution

We require that u_0 (resp., F and G) satisfies the assumptions in Remark 3.5 (resp., Remark 3.1). We suppose furthermore that there exist $\theta > 0$, $M_\beta > 0$ such that

$$0 < |F(0,t)|_{\mathbb{V}} \leq M_\beta e^{-\theta t}, \quad \forall t > 0,$$

and

$$|F(u,t) - F(v,t)|_{\mathbb{V}} \leq (C_F + \beta(t))|u - v|_{\mathbb{V}}, \quad \forall t > 0, (u,v) \in \mathbb{V} \times \mathbb{V}.$$

Here $\beta(t)$ is an integrable function such that $\beta(t) = O(e^{-\theta t})$. We also assume that

$$0 < |G(0,t)|_{\mathbb{V}^{\otimes m}} \le M_\delta e^{-\theta t}, \ \ \forall t > 0,$$

and

$$|G(u,t) - G(v,t)|_{\mathbb{V}^{\otimes m}} \le (\eta + \delta(t))|u - v|_{\mathbb{V}}, \quad \forall t > 0, (u,v) \in \mathbb{V} \times \mathbb{V}.$$

Here $\delta(t)$ is an integrable function such that $\delta(t) = O(e^{-\theta t})$. Before we formulate and prove the main result of this subsection we introduce the following definition which is borrowed from [31].

Definition 3.13. The strong probabilistic solution u of (3.7) is said to decay exponentially in mean square if there exist positive constants a and b such that

$$\mathbb{E}|u(t)|^2_{\mathbb{V}} \le be^{-at},$$

for t large enough.

We have

Theorem 3.14. *Assume that*

$$\frac{2\nu}{\mathcal{P}^2 + \alpha} > \frac{2\mathcal{P}^2 C_F}{\alpha} + \frac{2\mathcal{P}^4\eta}{\alpha^2} + 1,$$

then there exists $a \in (0,\theta)$ such that

$$\mathbb{E}|u(t)|^2_{\mathbb{V}} \le \left(\mathbb{E}|u_0|^2_{\mathbb{V}} + \frac{4\alpha^2 M_\beta + 2\mathcal{P}^4 M_\delta}{\alpha^2(\theta - a)}\right) e^{-at}, \ \ \forall t > 0.$$

Proof. Since

$$\frac{2\nu}{\mathcal{P}^2 + \alpha} > \frac{2\mathcal{P}^2 C_F}{\alpha} + \frac{2\mathcal{P}^4\eta}{\alpha^2} + 1$$

we can choose $a \in (0,\theta)$ such that

$$\frac{2\nu}{\mathcal{P}^2 + \alpha} \ge a + \frac{2\mathcal{P}^2 C_F}{\alpha} + \frac{2\mathcal{P}^4\eta}{\alpha^2} + 1.$$

Thanks to Itô's formula, the following holds

$$e^{at}|u(t)|^2_{\mathbb{V}} = |u_0|^2_{\mathbb{V}} - 2\int_0^t e^{as}\nu||u||^2 + 2\int_0^t e^{as}(\widehat{F}(u,s),u)_{\mathbb{V}}ds + \int_0^t e^{as}|\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}}ds$$
$$+ a\int_0^t e^{as}|u(s)|^2_{\mathbb{V}}ds + 2\int_0^t (\widehat{G}(u,s),u)_{\mathbb{V}}dW.$$

Taking the mathematical expectation and using the property (2.29) of the stochastic integral and Cauchy-Schwarz's inequality we get

$$e^{at}\mathbb{E}|u(t)|^2_{\mathbb{V}} \leq \mathbb{E}|u_0|^2_{\mathbb{V}} - \frac{2\nu}{\mathcal{P}^2+\alpha}\int_0^t e^{as}\mathbb{E}|u(s)|^2_{\mathbb{V}}ds + 2\int_0^t e^{as}\mathbb{E}|\widehat{F}(u,s)|_{\mathbb{V}}|u|_{\mathbb{V}}ds$$
$$+\int_0^t e^{as}\mathbb{E}|\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}}ds + a\int_0^t e^{as}\mathbb{E}|u(s)|^2_{\mathbb{V}}ds.$$

Invoking the estimate (3.86) and making use of the assumptions on F and G yield

$$e^{at}\mathbb{E}|u(t)|^2_{\mathbb{V}} \leq \mathbb{E}|u_0|^2_{\mathbb{V}} + (a - \frac{2\nu}{\mathcal{P}^2+\alpha} + 2\frac{\mathcal{P}^2}{\alpha} + 2\frac{\mathcal{P}^4}{\alpha^2} + 1)\int_0^t e^{as}\mathbb{E}|u(s)|^2_{\mathbb{V}}ds$$
$$+\int_0^t (2\frac{\mathcal{P}^2}{\alpha}\beta(s) + 2\frac{\mathcal{P}^4}{\alpha^2}\delta(s))e^{as}\mathbb{E}|u(s)|^2_{\mathbb{V}}ds$$
$$+\int_0^t \left(4M_\beta + 2\frac{\mathcal{P}^4}{\alpha^2}M_\delta\right)e^{(a-\theta)s}ds.$$

Application of Gronwall's inequality to this estimate implies

$$e^{at}\mathbb{E}|u(t)|^2_{\mathbb{V}} \leq \exp\left\{(1 + a + 2\frac{\mathcal{P}^2}{\alpha} + 2\frac{\mathcal{P}^4}{\alpha^2} - \frac{2\nu}{\mathcal{P}^2+\alpha})t\right\}\exp\left\{\int_0^t (2\frac{\mathcal{P}^2}{\alpha}\beta(s) + 2\frac{\mathcal{P}^4}{\alpha^2}\delta(s))ds\right\}$$
$$\times\left(\mathbb{E}|u_0|^2_{\mathbb{V}} + \frac{4\alpha^2 M_\beta + 2\mathcal{P}^4 M_\delta}{\alpha^2(a-\theta)}(e^{(a-\theta)t} - 1)\right).$$

With the choice of a and the assumptions on $\beta(s)$ and $\delta(s)$ we obtain that

$$\mathbb{E}|u(t)|^2_{\mathbb{V}} \leq \mathbb{E}|u_0|^2_{\mathbb{V}}e^{-at} + \frac{4\alpha^2 M_\beta + 2\mathcal{P}^4 M_\delta}{\alpha^2(\theta - a)}e^{-at}, \ \ \forall t > 0.$$

This completes the proof of the Theorem. □

3.4.2 The stability of the stationary solution

In this subsection we strengthen the assumptions on F by requiring that $F(u,t) = F(u)$ for any $t \in [0,T]$ and

$$F(0) \neq 0,$$
$$||F(u) - F(v)|| \leq C_F||u - v||,$$

We associate to (3.7) the deterministic equation

$$u(t) + \int_0^t (\nu\widehat{A}u + \widehat{B}(u))ds = u_0 + \int_0^t \widehat{F}(u)ds. \tag{3.82}$$

Definition 3.15. The function u_∞ is said to be a stationary solution of (3.82) if it satisfies the time-independent problem

$$\begin{cases} -\nu\Delta u_\infty + \text{curl}(u_\infty - \alpha\Delta u_\infty)\times u_\infty + \nabla\mathfrak{P} = F(u_\infty) \text{ in } D, \\ \text{div}\, u_\infty = 0 \text{ in } D, \\ u_\infty = 0 \text{ on } \partial D. \end{cases} \tag{3.83}$$

Or equivalently, it verifies the following

$$\nu\widehat{A}u_\infty + \widehat{B}(u_\infty, u_\infty) = \widehat{F}(u_\infty)$$

as an equation in $\mathbb{W}^*$.

The analysis of (3.83) was done in [12] for the 3-dimensional case. The method in [12], which is a combination of the Galerkin method and the Brouwer fixed point method yields the following existence result for the stationary problem (3.83).

Theorem 3.16. *For given $\alpha > 0$, $\nu > 0$ such that $\nu - \mathcal{P}C_F > 0$ there exists a solution u_∞ of (3.83). This function u_∞ verifies*

$$||u_\infty|| \leq \frac{\mathcal{P}^2}{\nu - \mathcal{P}^2 C_F}||F(0)||,$$

and

$$|u_\infty|_{\mathbb{W}} \leq K(\alpha, \nu, F),$$

where $K(\alpha, \nu, F)$ is equal to

$$\sqrt{2}||F(0)||\left[\frac{\mathcal{P}^2}{\nu - \mathcal{P}^2 C_F}\left(\frac{\alpha C_F}{\nu} + 1\right) + 1\right].$$

Furthermore, this solution is unique provided that $\nu - C_B K(\alpha, \nu, F) - \mathcal{P}^2 C_f > 0$.

Definition 3.17. The stationary solution u_∞ of (3.82) is said to be exponentialy stable in mean square if there exist positive constants a and b such that

$$\mathbb{E}|u(t) - u_\infty|^2_{\mathbb{V}} \leq be^{-at},$$

for any $t > 0$.

In this definition, which is from [31], $u(t)$ is the strong solution of (3.7) and the term strong solution should be understood in the sense of Definition 3.2.

We prove the following

Theorem 3.18. *For any* $\theta > 0$, *assume that* G *satisfies*

$$|G(u_\infty, t)|^2_{\mathbb{V}^{\otimes m}} \leq \eta e^{-\theta t},$$

and

$$|G(u,t) - G(v,t)|^2_{\mathbb{V}^{\otimes m}} \leq \delta(t)|u - v|^2_{\mathbb{V}}, \ \forall (u,v) \in \mathbb{V} \times \mathbb{V},$$

where $\delta(t)$ *is a positive integrable function such that* $\delta(t) \leq C_\delta e^{-\theta t}$, $C_\delta > 0$. *Assume also that*

$$\nu > 2C_B|u_\infty|_{\mathbb{W}} + \frac{\mathcal{P}^2}{\alpha}.$$

Then there exists a constant $a \in (0, \theta)$ *such that the following holds*

$$\mathbb{E}|u(t) - u_\infty|^2_{\mathbb{V}} \leq \left(\mathbb{E}|u_0 - u_\infty|^2_{\mathbb{V}} + \frac{\eta}{(\theta - a)}\right) e^{-at}, \ \forall t > 0.$$

Proof. Thanks to Itô's formula, we have

$$\begin{aligned} |u(t) - u_\infty|^2_{\mathbb{V}} = |u_0 - u_\infty|^2_{\mathbb{V}} &- 2\int_0^t [(\nu Au, u - u_\infty) + < \widehat{B}(u,u), u - u_\infty >]ds \\ &+ 2\int_0^t (F(u), u - u_\infty)ds + \int_0^t |\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}} ds \\ &+ 2\int_0^t (\widehat{G}(u,s), u - u_\infty)_{\mathbb{V}} dW. \end{aligned}$$

Similarly

$$\begin{aligned} e^{at}|u(t) - u_\infty|^2_{\mathbb{V}} = |u_0 - u_\infty|^2_{\mathbb{V}} &- 2\int_0^t e^{as}[(\nu Au, u - u_\infty) + < \widehat{B}(u,u), u - u_\infty >]ds \\ &+ 2\int_0^t e^{as}(F(u), u - u_\infty)ds + a\int_0^t e^{as}|u - u_\infty|^2_{\mathbb{V}} ds \\ &+ \int_0^t e^{as}|\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}} ds + 2\int_0^t e^{as}(\widehat{G}(u,s), u - u_\infty)_{\mathbb{V}} dW, \end{aligned}$$

for $a > 0$. Taking the mathematical expectation in both sides of this equation, and using the property of stochastic integral (2.29) we obtain

$$\begin{aligned} e^{at}\mathbb{E}|u(t) - u_\infty|^2_{\mathbb{V}} = \mathbb{E}|u_0 - u_\infty|^2_{\mathbb{V}} &- 2\int_0^t e^{as}\mathbb{E}[(\nu Au, u - u_\infty) + < \widehat{B}(u,u), u - u_\infty >]ds \\ &+ 2\int_0^t e^{as}\mathbb{E}(F(u), u - u_\infty)ds + \int_0^t e^{as}\mathbb{E}|\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}} ds \\ &+ a\int_0^t e^{as}\mathbb{E}|u - u_\infty|^2_{\mathbb{V}} ds. \end{aligned}$$

Since u_∞ is a solution of (3.83), then we have

$$\int_0^t e^{as}\mathbb{E}[(\nu A u_\infty, u-u_\infty)+ < \widehat{B}(u_\infty,u_\infty), u-u_\infty >]ds = \int_0^t e^{as}\mathbb{E}(F(u_\infty), u-u_\infty)ds.$$

We deduce from the last two equations that

$$\begin{aligned} e^{at}\mathbb{E}|u(t)-u_\infty|^2_{\mathbb{V}} = \mathbb{E}|u_0-u_\infty|^2_{\mathbb{V}} + a\int_0^t e^{as}\mathbb{E}|u-u_\infty|^2_{\mathbb{V}}ds - 2\nu\int_0^t e^{as}\mathbb{E}||u-u_\infty||^2 ds \\ -2\int_0^t e^{as} < \widehat{B}(u,u)-\widehat{B}(u_\infty,u_\infty), u-u_\infty > ds + \int_0^t e^{as}\mathbb{E}|\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}}ds \\ +2\int_0^t e^{as}\mathbb{E}(F(u)-F(u_\infty), u-u_\infty)ds. \end{aligned}$$

By Cauchy-Schwarz's inequality we find from this relation that

$$\begin{aligned} e^{at}\mathbb{E}|u(t)-u_\infty|^2_{\mathbb{V}} \le \mathbb{E}|u_0-u_\infty|^2_{\mathbb{V}} + a\int_0^t e^{as}\mathbb{E}|u-u_\infty|^2_{\mathbb{V}}ds - 2\nu\int_0^t e^{as}\mathbb{E}||u-u_\infty||^2 ds \\ -2\int_0^t e^{as} < \widehat{B}(u,u)-\widehat{B}(u_\infty,u_\infty), u-u_\infty > ds + \int_0^t e^{as}\mathbb{E}|\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}}ds \\ +2\int_0^t e^{as}\mathbb{E}[|F(u)-F(u_\infty||u-u_\infty|]ds. \end{aligned} \tag{3.84}$$

But

$$< \widehat{B}(u,u)-\widehat{B}(u_\infty,u_\infty), u-u_\infty > = < \widehat{B}(u-u_\infty,u), u-u_\infty >,$$

and

$$< \widehat{B}(u,u)-\widehat{B}(u_\infty,u_\infty), u-u_\infty > = - < \widehat{B}(u-u_\infty,u-u_\infty), u_\infty >,$$

then we can infer from (3.84) that

$$\begin{aligned} e^{at}\mathbb{E}|u(t)-u_\infty|^2_{\mathbb{V}} \le \mathbb{E}|u_0-u_\infty|^2_{\mathbb{V}} + a\int_0^t e^{as}\mathbb{E}|u-u_\infty|^2_{\mathbb{V}}ds - 2\nu\int_0^t e^{as}\mathbb{E}||u-u_\infty||^2 ds \\ +2\int_0^t e^{as} < \widehat{B}(u-u_\infty,u-u_\infty), u_\infty > ds + \int_0^t e^{as}\mathbb{E}|\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}}ds \\ +2\int_0^t e^{as}\mathbb{E}[|F(u)-F(u_\infty||u-u_\infty|]ds. \end{aligned}$$

By (2.22) and the assumption on F we see from the previous estimate that

$$\begin{aligned} e^{at}\mathbb{E}|u(t)-u_\infty|^2_{\mathbb{V}} \le \mathbb{E}|u_0-u_\infty|^2_{\mathbb{V}} + a\int_0^t e^{as}\mathbb{E}|u-u_\infty|^2_{\mathbb{V}}ds - \frac{2\nu}{\mathcal{P}^2+\alpha}\int_0^t e^{as}\mathbb{E}|u-u_\infty|^2_{\mathbb{V}}ds \\ +2C_B\int_0^t e^{as}\mathbb{E}[|u-u_\infty|^2_{\mathbb{V}}|u_\infty|_{\mathbb{W}}]ds + \int_0^t e^{as}\mathbb{E}|\widehat{G}(u,s)|_{\mathbb{V}^{\otimes m}}ds \\ +2\frac{\mathcal{P}^2C_F}{\alpha}\int_0^t e^{as}\mathbb{E}|u-u_\infty|^2_{\mathbb{V}}ds. \end{aligned} \tag{3.85}$$

It is not hard to show that

$$|(I+\alpha A)^{-1}v|_{\mathbb{V}}^2 \leq \frac{\mathcal{P}^2}{\alpha}|v|^2 \leq \frac{\mathcal{P}^4}{\alpha^2}|v|_{\mathbb{V}}^2, \tag{3.86}$$

for any $v \in \mathbb{V}$. We plug (3.86) in (3.85) and obtain

$$\begin{aligned} e^{at}\mathbb{E}|u(t)-u_\infty|_{\mathbb{V}}^2 \leq \mathbb{E}|u_0-u_\infty|_{\mathbb{V}}^2 + (a+2C_B|u_\infty|_{\mathbb{W}} + \frac{\mathcal{P}^2C_F}{\alpha} - 2\nu)\int_0^t e^{as}\mathbb{E}|u-u_\infty|_{\mathbb{V}}^2 ds \\ + \frac{\mathcal{P}^4}{\alpha^2}\int_0^t e^{as}\mathbb{E}|G(u,s)|_{\mathbb{V}}^2 ds. \end{aligned}$$

The assumptions on G gives

$$\begin{aligned} e^{at}\mathbb{E}|u(t)-u_\infty|_{\mathbb{V}}^2 \leq \mathbb{E}|u_0-u_\infty|_{\mathbb{V}}^2 + \int_0^t (a+2C_B|u_\infty|_{\mathbb{W}} + \frac{\mathcal{P}^2C_F}{\alpha} - 2\nu)e^{as}\mathbb{E}|u-u_\infty|_{\mathbb{V}}^2 ds \\ + 2\frac{\mathcal{P}^4}{\alpha^2}\int_0^t \delta(s))e^{as}\mathbb{E}|u-u_\infty|_{\mathbb{V}}^2 ds \\ + 2\frac{\mathcal{P}^4}{\alpha^2}\eta\int_0^t e^{(a-\theta)s}ds. \end{aligned} \tag{3.87}$$

Using Gronwall's inequality in (3.87), we obtain

$$\begin{aligned} e^{at}\mathbb{E}|u(t)-u_\infty|_{\mathbb{V}}^2 \leq \exp\left\{(a+2C_B|u_\infty|_{\mathbb{W}} + \frac{\mathcal{P}^2C_F}{\alpha} - 2\nu)t\right\}\exp\left\{2\frac{\mathcal{P}^4}{\alpha^2}\int_0^t \delta(s)ds\right\} \\ \times\left(\mathbb{E}|u_0-u_\infty|_{\mathbb{V}}^2 + \frac{\eta}{a-\theta}(e^{(a-\theta)t}-1)ds\right), \end{aligned}$$

for any $t>0$. Since

$$2C_B|u_\infty|_{\mathbb{W}} + \frac{\mathcal{P}^2C_F}{\alpha} < 2\nu,$$

then we may choose $a \in (0,\theta)$ such that

$$a + 2C_B|u_\infty|_{\mathbb{W}} + \frac{\mathcal{P}^2C_F}{\alpha} \leq 2\nu.$$

With this choice and the assumption on δ we have

$$\mathbb{E}|u(t)-u_\infty|_{\mathbb{V}}^2 \leq \left(\mathbb{E}|u_0-u_\infty|_{\mathbb{V}}^2 + \frac{\eta}{(\theta-a)}\right)e^{-at}, \;\; \forall t>0.$$

And this completes the proof of the theorem. $\square$

To close this chapter we notice that the results we have in this part of the thesis are also valid for the case of periodic boundary conditions.

Chapter 4

Existence of weak probabilistic solutions

4.1 Introduction

In many cases of interest the Lipschitz condition on the forcing terms $F(v,t)$ and $G(v,t)$ no longer holds. In this chapter we consider such situation. More specifically, we suppose that $F(v,t)$ ang $G(v,t)$ are only continuous with respect to the variable v. The appropriate notion solution in this case is that of weak probabilistic solutions known as well as martingale solutions. Here we prove the existence of such solutions. To do so we mainly use a compactness method which seems to have been initially introduced by Bensoussn [8], [7]. In contrast to most work dealing with martingale solutions of SPDEs ([20], [17], [25], [43], [44], [72]...), we do not use the martingale representation. The current chapter is organized as follows. Section 2 is devoted to the formulation of the hypotheses and the main result. We introduce a Galerkin approximation of the problem and derive crucial a priori estimates for its solutions in Section 3; a compactness result is also derived. We prove the main result in Section 4.

4.2 Hypotheses and the main result

We impose on our problem the following assumptions.

4.2.1 Hypotheses

1. We assume that

$$F : \mathbb{V} \times [0,T] \to \mathbb{V}$$

is continuous in the first variable, and mesurable for the second. We also assume that there exists a constant $C > 0$ such that for any $t \in [0,T]$ and any $v \in \mathbb{V}$

$$|F(v,t)|_{\mathbb{V}} \leq C(1 + |v|_{\mathbb{V}}). \tag{4.1}$$

2. We also assume that $G(v,t)$ is continuous in its first argument, and measurable in the second variable. We suppose that there exists a constant $C > 0$ such that for any $t \in [0,T]$, $G(v,t)$ satisfies

$$|G(v,t)|_{\mathbb{V}^{\otimes m}} \leq C(1 + |v|_{\mathbb{V}}). \tag{4.2}$$

4.2.2 Statement of the existence theorem of weak probabilistic solutions

We introduce the concept of solution of problem (1.5) that is of interest to us.

Definition 4.1. By a weak probabilistic solution of the problem (1.5), we mean a system

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}^t, W, u),$$

where

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\mathbb{F}^t$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$,
2. $W(t)$ is a m-dimensional $\mathbb{F}^t$-standard Wiener process,
3. $u \in L^p(\Omega, \mathbb{P}; L^\infty(0,T;\mathbb{W}))$, $2 \leq p < \infty$,
4. For all t, $u(t)$ is $\mathbb{F}^t$-measurable,
5. $\mathbb{P}$-almost surely the following integral equation of Itô type holds

$$\begin{aligned}(u(t) - u(0), v)_{\mathbb{V}} + \int_0^t \left[\nu((u,v)) + (\operatorname{curl}(u(s) - \alpha\Delta u(s)) \times u, v)\right] ds \\ = \int_0^t (F(u(s),s),v)ds + \int_0^t (G(u(s),s),v)dW(s)\end{aligned} \tag{4.3}$$

for any $t \in [0,T]$ and $v \in \mathbb{W}$.

Remark 4.2. In the above definition the quantity $\int_0^t (G(u(s),s),v)dW(s)$ should be understood as

$$\int_0^t (G(u(s),s),v)dW(s) = \sum_{k=1}^m \int_0^t (G_k(u(s),s),v)dW_k(s),$$

where G_k and W_k denote the k-th component of G and W, respectively.

Now we state our main result.

Theorem 4.3. *Assume that $u_0 \in \mathbb{W}$, assume also that the assumptions in Subsection 4.2.1 on the operators F, G are satisfied, then the problem* (1.5) *has a solution in the sense of Definition 4.1. Moreover, almost surely the paths of the process u are $\mathbb{W}$-valued weakly continuous.*

4.3 Auxiliary results

In this part we introduce the Galerkin approximation scheme for problem (1.5) and establish crucial a priori estimates for the corresponding approximating solution.

4.3.1 The approximate solution

We make use of the Galerkin basis $\{e_i \in \mathbb{W}, i \in \mathbb{N}\}$ introduced in Chapter 3. We consider the subset $\mathbb{W}_N = \text{Span}(e_1, \dots, e_N) \subset \mathbb{W}$ and look for a finite-dimensional approximation of a solution of our problem as a vector $u^N \in \mathbb{W}_N$ that can be written as:

$$u^N(t) = \sum_{i=1}^N c_{iN}(t)e_i(x). \tag{4.4}$$

Let us consider a complete probability space system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ on which a m-dimensional standard Wiener process $\bar{W}$ taking values in $\mathbb{R}^m$ is defined and an increasing filtration $\bar{\mathbb{F}}^t$ is generated by $\bar{W}$. We require u^N to satisfy the following system

$$\begin{aligned} &d(u^N, e_i)_{\mathbb{V}} + \nu((u^N, e_i))dt + b(u^N, u^N, e_i)dt - \alpha b(u^N, \Delta u^N, e_i)dt + \alpha b(e_i, \Delta u^N, u^N)dt \\ &= (F(t, u^N), e_i)dt + (G(t, u^N), e_i)d\bar{W}, i \in \{1, 2, ..., N\}, \end{aligned} \tag{4.5}$$

where u_0^N as the orthogonal projection of $u(0)$ on the space $\mathbb{W}_N$, and

$$u_0^N (\text{or } u^N(0)) \to u(0) \text{ strongly in } \mathbb{V} \text{ as } N \to \infty.$$

The Fourier coefficients c_{iN} in (4.4) are solutions of a system of stochastic ordinary differential equations which satisfies the conditions of the existence theorem of Skorokhod (see, for instance, page 59 of [111] or Theorem 4.22 of [71]). Therefore the sequence of functions u^N exists at least on a short interval $[0, T_N]$. Global existence will follow from a priori estimates for u^N.

4.3.2 A priori estimates

The following lemmas are proved exactly as in Chapter 3 (see Lemma 3.7 and Lemma 3.8).

Lemma 4.4. *For any $N \geq 1$ we have*

$$\bar{\mathbb{E}} \sup_{0 \leq t \leq T} |u^N(t)|^2_{\mathbb{V}} < +\infty.$$

We also have

$$\bar{\mathbb{E}} \sup_{0 \leq t \leq T} |u^N(t)|^2_{\mathbb{W}} < +\infty.$$

Lemma 4.5. *For any $4 \leq p < \infty$ we have*

$$\bar{\mathbb{E}} \sup_{0 \leq s \leq T} |u^N(s)|^p_{\mathbb{V}} < \infty$$

and

$$\bar{\mathbb{E}} \sup_{0 \leq s \leq T} |u^N(s)|^p_{\mathbb{W}} < \infty.$$

Throughout $\bar{\mathbb{E}}$ denotes the mathematical expectation with respect to the probability measure $\bar{\mathbb{P}}$.

Remark 4.6. Lemmas 4.4 and 4.5 imply in particular that

$$\bar{\mathbb{E}} \sup_{0 \leq t \leq T} |u^N(t)|^p_{\mathbb{V}} < \infty,$$

$$\bar{\mathbb{E}} \sup_{0 \leq t \leq T} |u^N(t)|^p_{\mathbb{W}} < \infty,$$

for any $1 \leq p < \infty$.

The following result is central in the proof of the forthcoming crucial estimates of the finite difference of our approximating solution.

Lemma 4.7. *Let $s, t \in [0, T]$ such that $s \leq t$. For a fixed $t \in [0, T]$, let*

$$v^N(t) = \sum_{i=1}^{N} \lambda_i (v^N(t), e_i)_{\mathbb{V}} e_i$$

be an element of $\mathbb{W}_N$ which satisfies Lemma 4.4 and Lemma 4.5. The following holds

$$\begin{aligned}
&|u^N(t) - v^N(s)|_{\mathbb{V}}^2 - |u^N(s) - v^N(s)|_{\mathbb{V}}^2 + 2\int_s^t \nu \left[||u^N(r)||^2 - \left(\left(u^N(r), v^N(r)\right)\right) \right] dr \\
&= 2\int_s^t \left(G(u^N(r), r), u^N(r) - v^N(s)\right) d\bar{W} + \sum_{i=1}^{N} \lambda_i \int_s^t \left(G(u^N(r), r), e_i\right)^2 dr \\
&+ 2\int_s^t b(v^N(s), \Delta u^N(r), u^N(r)) dr + 2\int_s^t \left(F(u^N(r), r), u^N(r) - v^N(s)\right) dr \\
&- 2\int_s^t b(u^N(r), u^N(r), v^N(s)) dr - 2\int_s^t b(u^N(r), \Delta u^N(r), v^N(s)) dr.
\end{aligned}$$

Proof. For v^N, for any s, t such that $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(u^N(t) - v^N(s), e_i\right)_{\mathbb{V}} + \nu\left(\left(u^N(t), e_i\right)\right) + b(u^N(t), u^N(t), e_i) \\
&- \alpha b(u^N(t), \Delta u^N(t), e_i) + \alpha b(e_i, \Delta u^N(t), u^N(t)) \\
&= \left(F(u^N(t), t), e_i\right) + \left(G(u^N(t), t), e_i\right) \frac{d\bar{W}}{dt}, \ 1 \leq i \leq N.
\end{aligned}$$

This relation can be rewritten as an Itô equation of the form

$$\begin{aligned}
&d\left(u^N(t) - v^N(s), e_i\right)_{\mathbb{V}} + \nu\left(\left(u^N(t), e_i\right)\right) dt + b(u^N(t), u^N(t), e_i) dt \\
&- \alpha b(u^N(t), \Delta u^N(t), e_i) dt + \alpha b(e_i, \Delta u^N(t), u^N(t)) dt \\
&= \left(F(u^N(t), t), e_i\right) dt + \left(G(u^N(t), t), e_i\right) d\bar{W}.
\end{aligned}$$

Applying Itô's formula to the function $(u^N(t), v^N(s), e_i)_{\mathbb{V}}^2$, multiplying the result by λ_i, and then summing over i from 1 to N yield

$$\begin{aligned}
&d|w|_{\mathbb{V}}^2 + 2\nu((u^N(t), w)) dt + 2b(u^N(t), u^N(t), w) dt - 2\alpha b(u^N(t), \Delta u^N(t), w) dt \\
&+ 2\alpha b(w, \Delta u^N(t), u^N(t)) dt - 2(F(u^N(t), t), w) dt \\
&= \sum_{i=1}^{N} \lambda_i (G(u^N(t), t), e_i)^2 + 2(G(u^N(t), t), w) d\bar{W},
\end{aligned} \tag{4.6}$$

where $w = u^N(t) - v^N(s)$. Using the trilinearity of b and the well-known identity $b(u, u, u) = 0$, $u \in \mathbb{V}$, we find that

$$\begin{aligned}
&b(u^N(t), u^N(t), w) - \alpha b(u^N(t), \Delta u^N(t), w) + \alpha b(w, \Delta u^N(t), u^N(t)) \\
&= b(u^N(t), u^N(t), v^N(s)) - \alpha b(u^N(t), \Delta u^N(t), v^N(s)) + \alpha b(v^N(s), \Delta u^N(t), u^N(t))
\end{aligned}$$

The lemma follows by combining this relation with (4.6), and integrating the resulting equation between s and t. $\square$

The following result can be proved by a similar argument used in [44], but to be self-contained we prefer to give our own proof which is interesting in itself.

Lemma 4.8. *There exists a positive constant $C > 0$ such that for all $0 \leq \delta < 1$ and $N \in \mathbb{N}$, the following inequality holds*

$$\bar{\mathbb{E}} \sup_{|\theta|\leq\delta} \int_0^{T-\delta} |u^N(s+\theta) - u^N(s)|^2_{\mathbb{W}^*} ds \leq C\delta^{\frac{1}{2}}.$$

Proof. Since $u^N(s) \in \mathbb{W}_N$, $s \in (0,T)$ and it satisfies the Lemma 4.4 and lemma 4.5 the we can take $v^N(s) = u^N(s)$ and $t = s+\theta$, $0 \leq \theta \leq \delta \leq 1$ and apply Lemma 4.7. We obtain

$$\begin{aligned}
&|u^N(s+\theta) - u^N(s)|^2_{\mathbb{V}} + 2\int_s^{s+\theta} \nu\left[||u^N(r)||^2 - \left(\left(u^N(r), u^N(r)\right)\right)\right] dr \\
&= 2\int_s^{s+\theta} \left(G(u^N(r),r), u^N(r) - u^N(s)\right) d\bar{W} + \sum_{i=1}^N \lambda_i \int_s^{s+\theta} \left(G(u^N(r),r), e_i\right)^2 dr \\
&+ 2\int_s^{s+\theta} b(u^N(s), \Delta u^N(r), u^N(r)) dr + 2\int_s^{s+\theta} \left(F(u^N(r),r), u^N(r) - u^N(s)\right) dr \\
&- 2\int_s^{s+\theta} b(u^N(r), u^N(r), u^N(s)) dr - 2\int_s^{s+\theta} b(u^N(r), \Delta u^N(r), u^N(s)) dr.
\end{aligned}$$

We derive that

$$\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\leq\theta\leq\delta} |u^N(s+\theta) - u^N(s)|^2_{\mathbb{V}} \leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,$$

where

$$\begin{aligned}
I_1 &= 2\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\leq\theta\leq\delta} \left| \int_s^{s+\theta} \left(G(u^N(r),r), u^N(r) - u^N(s)\right) d\bar{W} \right| \\
I_2 &= \bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\leq\theta\leq\delta} \left| \sum_{i=1}^N \lambda_i \int_s^{s+\theta} \left(G(u^N(r),r), e_i\right)^2 dr \right| \\
I_3 &= \bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\leq\theta\leq\delta} \left| 2\int_s^{s+\theta} \nu\left[||u^N(r)|| + \left(\left(u^N(r), u^N(r)\right)\right)\right] dr \right| \\
I_4 &= 2\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\leq\theta\leq\delta} \left| \int_s^{s+\theta} b(u^N(s), \Delta u^N(r), u^N(r)) dr \right|
\end{aligned}$$

$$I_5 = 2\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\le\theta\le\delta} \left| \int_s^{s+\theta} b(u^N(r), \Delta u^N(r), u^N(s))dr \right|$$
$$I_6 = 2\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\le\theta\le\delta} \left| \int_s^{s+\theta} b(u^N(r), u^N(r), u^N(s))dr \right|$$
$$I_7 = 2\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\le\theta\le\delta} \left| \int_s^{s+\theta} \left(F(u^N(r), r), u^N(r) - u^N(s)\right) dr \right|.$$

The proof of the lemma will consist of the following five steps.

`Step 1: Estimate of` I_3. Owing to the equivalence of the two norms $||.||$ and $|.|_{\mathbb{V}}$ we see that I_3 is dominated by

$$2C\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\le\theta\le\delta} |u^N(s+\theta) - u^N(s)|_{\mathbb{V}} \left(\bar{\mathbb{E}} \int_s^{s+\delta} |u^N(r)|_{\mathbb{V}} dr \right) ds.$$

We find from this and by a successive application of Cauchy-Schwarz's inequality that

$$I_3 \le C\delta \left(\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\le\theta\le\delta} |u^N(s+\theta) - u^N(s)|^2_{\mathbb{V}} ds \right)^{\frac{1}{2}} \left(\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\le r\le T} |u^N(r)|^2_{\mathbb{V}} dr \right)^{\frac{1}{2}}$$

Since $s+\theta \in [0,T]$ for any $s \in [0, T-\delta]$ and any $0 \le \theta \le \delta$, we get using the triangle inequality and Lemma 4.4 that

$$I_3 \le C\delta. \tag{4.7}$$

`Step 2: Estimates for` I_4, I_5 `and` I_6. The estimate (2.8) and a successive application of Cauchy-Schwarz's inequality yield

$$I_4 \le C\delta^{\frac{1}{2}} \left(\bar{\mathbb{E}} \int_0^{T-\delta} |u^N(s)|^2_{\mathbb{W}} ds \right)^{\frac{1}{2}} \left(\bar{\mathbb{E}} \int_0^{T-\delta} \int_s^{s+\delta} |u^N(r)|^4_{\mathbb{V}} dr ds \right)^{\frac{1}{2}}.$$

By Cauchy's inequality, we have

$$I_4 \le C\delta^{\frac{1}{2}} \left(\bar{\mathbb{E}} \int_0^{T-\delta} |u^N(s)|^2_{\mathbb{W}} ds + \bar{\mathbb{E}} \int_0^{T-\delta} \int_s^{s+\delta} |u^N(r)|^4_{\mathbb{V}} dr ds \right).$$

Thanks to Lemmas 4.4 and 4.5, we derive from the latter estimate that

$$I_4 \le C\delta^{\frac{1}{2}}.$$

Similar estimates hold for I_5 and I_6.

`Step 3: Estimate for` I_7. Thanks to the idea used in the proof of the estimate (4.7), we have that the quantity

$$C\delta^{\frac{1}{2}} \left(\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0\le\theta\le\delta} |u^N(s+\theta) - u^N(s)|^2_{\mathbb{V}} ds \right)^{\frac{1}{2}} \left(\bar{\mathbb{E}} \int_0^{T-\delta} \int_s^{s+\delta} |F(u^N(r), r)|^2_{\mathbb{V}} dr ds \right)^{\frac{1}{2}}.$$

dominates I_7. By the assumption on F and the argument used in deriving (4.7) we have

$$I_7 \leq C\delta.$$

`Step 4: Estimate for` I_2. We use the same argument as used in the proof of Lemma 4.4 to get an estimate of the form

$$\sum_{i=1}^{N} \lambda_i (G(u^N(r), r), e_i) \leq C(1 + |u^N(r)|_{\mathbb{V}}^2).$$

We derive from the definition of I_2, the latter estimate and Lemma 4.4 that

$$I_2 \leq C\delta.$$

`Step 5: Estimate for` I_1. Thanks to Fubini's Theorem and the Burkhölder-Davis-Gundy inequality (see Theorem 2.15) we have

$$I_1 \leq 6 \int_0^{T-\delta} \bar{\mathbb{E}} \left(\int_s^{s+\delta} (G(u^N(r), r), u^N(r) - u^N(s))^2 dr \right)^{\frac{1}{2}} ds$$

By a sequence of Cauchy-Schwarz's inequality we find that I_1 is bounded from above by

$$C \left(\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0 \leq \theta \leq \delta} |u^N(s+\theta) - u^N(s)|_{\mathbb{V}}^2 ds \right)^{\frac{1}{2}} \left(\bar{\mathbb{E}} \int_0^{T-\delta} \int_s^{s+\delta} |G(u^N(r), r)|_{\mathbb{V}}^2 dr ds \right)^{\frac{1}{2}}.$$

By the assumption on G and by Lemma 4.4 we get from the latter equation that

$$I_1 \leq C\delta^{\frac{1}{2}}.$$

Combining all the estimates in Step 1-Step 5 we get

$$\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0 \leq \theta \leq \delta} |u^N(s+\theta) - u^N(s)|_{\mathbb{V}}^2 ds \leq C\delta^{\frac{1}{2}}.$$

Since $\mathbb{V}$ is continuously embedded in $\mathbb{W}^*$

$$\begin{aligned} \bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0 \leq \theta \leq \delta} \left|u^N(s+\theta) - u^N(s)\right|_{\mathbb{W}^*}^2 ds &\leq C\bar{\mathbb{E}} \int_0^{T-\delta} \sup_{0 \leq \theta \leq \delta} |u^N(s+\theta) - u^N(s)|_{\mathbb{V}}^2 ds \\ &\leq C\delta^{\frac{1}{2}}. \end{aligned}$$

The lemma follows readily from this last inequality and noting that a similar argument can be carried out to find a similar estimate for negative values of θ. □

4.3.3 Tightness property and application of Prokhorov's and Skorokhod's theorems

We denote by $\mathfrak{Z}$ the following subset of $L^2(0,T;\mathbb{V})$:

$$\mathfrak{Z} = \left\{ z \in L^\infty(0,T;\mathbb{W}) \cap L^\infty(0,T;\mathbb{V}); \sup_{|\theta|\leq \mu_M} \int_0^{T-\mu_M} |z(t+\theta) - z(t)|^2_{\mathbb{W}^*} \leq C\nu_M \right\},$$

for any sequences ν_M, μ_M such that $\nu_M, \mu_M \to 0$ as $M \to \infty$. The following result is a version of Theorem 2.8 due to Bensoussan [7].

Lemma 4.9. *The set $\mathfrak{Z}$ is compact in $L^2(0,T;\mathbb{V})$.*

Next we consider the space $\mathfrak{S} = C(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{V})$ endowed with its Borel σ-algebra $\mathcal{B}(\mathfrak{S})$ and the family of probability measures $\mathfrak{P}^N$ on $\mathfrak{S}$, which is the probability measure induced by the following mapping:

$$\phi : \omega \mapsto (\bar{W}(\omega,.), u^N(\omega,.)),$$

that is, for any $A \in \mathcal{B}(\mathfrak{S})$, $\mathfrak{P}^N(A) = \bar{\mathbb{P}}(\phi^{-1}(A))$.
We have the following lemma.

Lemma 4.10. *The family $(\mathfrak{P}^N)_{N\geq 1}$ is tight.*

Proof. For any $\varepsilon > 0$ and $M \geq 1$, we claim that there exists a compact subset $\mathfrak{K}_\varepsilon$ of $\mathfrak{S}$ such that $\mathfrak{P}^N(\mathfrak{K}_\varepsilon) \geq 1 - \varepsilon$. To prove our claim we define the sets

$$\mathfrak{W}_\varepsilon = \left\{ W : \sup_{\substack{t,s\in[0,T] \\ |t-s|<\frac{T}{2^M}}} 2^{\frac{M}{8}} |W(t) - W(s)| \leq J_\varepsilon, \forall M \right\}$$

and

$$\mathfrak{Z}_\varepsilon = \Big\{ z; \sup_{t\leq T} |z(t)|^2_{\mathbb{V}} \leq K_\varepsilon, \quad \sup_{t\leq T} |z(t)|^2_{\mathbb{W}} \leq L_\varepsilon,$$
$$\sup_{|\theta|\leq\mu_M} \int_0^{T-\mu_M} |z(t+\theta) - z(t)|^2_{\mathbb{W}^*} \leq R_\varepsilon \nu_M \Big\}$$

where the sequences ν_M and μ_M are chosen so that they are independent of ε, $\nu_M, \mu_M \to 0$ as $M \to \infty$ and $\sum_M \frac{\sqrt{\mu_M}}{\nu_M} < \infty$. It is clear by Ascoli-Arzela's Theorem that $\mathfrak{W}_\varepsilon$ is a compact subset of $C(0,T;\mathbb{R}^m)$, and by Lemma 4.9 $\mathfrak{Z}_\varepsilon$ is a compact subset of $L^2(0,T;\mathbb{V})$. We have to show that $\mathfrak{A}_\varepsilon = \mathfrak{P}^N\left((\bar{W},u^N)\notin \mathfrak{W}_\varepsilon\times\mathfrak{Z}_\varepsilon\right) < \varepsilon$. Indeed, we have

$$\begin{aligned}\mathfrak{A}_\varepsilon \leq \bar{\mathbb{P}}&\left[\bigcup_{M=1}^{\infty}\bigcup_{j=1}^{2^M}\left(\sup_{t,s\in I_j}|\bar{W}(t)-\bar{W}(s)|\geq J_\varepsilon\frac{1}{2^{\frac{M}{8}}}\right)\right]+\bar{\mathbb{P}}\left(\sup_{t\leq T}|u^N(t)|^2_{\mathbb{V}}\geq K_\varepsilon\right)\\ &+\bar{\mathbb{P}}\left(\bigcup_M\left\{\sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}|u^N(t+\theta)-u^N(t)|^2_{\mathbb{W}^*}\geq R_\varepsilon\nu_M\right\}\right)\\ &+\bar{\mathbb{P}}\left(\sup_{t\leq T}|u^N(t)|^2_{\mathbb{W}}\geq L_\varepsilon\right),\end{aligned}$$

where $\{I_j : 1\leq j\leq 2^M\}$ is a family of intervals of length $\frac{T}{2^M}$ which forms a partition of the interval $[0,T]$. It is well known that for any Wiener process B

$$\bar{\mathbb{E}}|B(t)-B(s)|^{2m} = C_m|t-s|^m \quad \text{for any } m\geq 1,$$

where C_m is a constant depending only on m. From this and the Markov's Inequality

$$\bar{\mathbb{P}}(\omega:\zeta(w)\geq\alpha)\leq\frac{1}{\alpha^k}\bar{\mathbb{E}}(|\zeta(\omega)|^k),$$

where $\zeta(\omega)$ is a random variable on $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$ and k and α are positive numbers , we obtain

$$\begin{aligned}\mathfrak{A}_\varepsilon &\leq \sum_{M=1}^{\infty}\sum_{j=1}^{2^M}C_m\left(2^{\frac{M}{8}}\right)^{2m}\frac{1}{J_\varepsilon^{2m}}\left(\frac{T}{2^M}\right)^m+\frac{1}{K_\varepsilon}\bar{\mathbb{E}}\sup_{t\leq T}|u^N(t)|^2_{\mathbb{V}}+\frac{1}{L_\varepsilon}\bar{\mathbb{E}}\sup_{t\leq T}|u^N(t)|^2_{\mathbb{W}}\\ &+\sum_M\frac{1}{R_\varepsilon\nu_M}\bar{\mathbb{E}}\sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}|u^N(t+\theta)-u^N(t)|^2_{\mathbb{W}^*}.\end{aligned}$$

Owing to Lemma 4.4, Lemma 4.8 and by choosing $m=2$, we have

$$\begin{aligned}\mathfrak{A}_\varepsilon &\leq \frac{C_2T^2}{L_\varepsilon^4}\sum_{M=1}^{\infty}2^{-\frac{1}{2}M}+C\Big(\frac{1}{K_\varepsilon}+\frac{1}{L_\varepsilon}+\frac{1}{R_\varepsilon}\sum_M\frac{\sqrt{\mu_M}}{\nu_M}\Big)\\ &\leq\frac{C_2T^2}{J_\varepsilon^4}(2+\sqrt{2})+C\Big(\frac{1}{K_\varepsilon}+\frac{1}{L_\varepsilon}+\frac{1}{R_\varepsilon}\sum_M\frac{\sqrt{\mu_M}}{\nu_M}\Big).\end{aligned}$$

A convenient choice of J_ε, K_ε, L_ε, R_ε completes the proof of the claim, and hence the proof of the lemma. □

It follows by Prokhorov's Theorem (Theorem 2.17) that the family $(\mathfrak{P}^N)_{N\geq1}$ is relatively compact in the set of probability measures (equipped with the weak convergence topology) on $\mathfrak{S}$. Then, we can extract a subsequence $\mathfrak{P}^{N_\mu}$ that weakly converges to a probability measure $\mathfrak{P}$. By Skorokhod's Theorem (Theorem 2.18), there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables (W^{N_μ}, u^{N_μ}) and (W, u) on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathfrak{S}$ such that

$$W^{N_\mu} \to W \text{ in } C(0,T;\mathbb{R}^m) \ \mathbb{P} - a.s, \tag{4.8}$$

$$u^{N_\mu} \to u \text{ in } L^2(0,T;\mathbb{V}) \ \mathbb{P} - a.s. \tag{4.9}$$

Moreover,

$$\text{the probability law of } (W^{N_\mu}, u^{N_\mu}) \text{ is } \mathfrak{P}^{N_\mu} \text{ and that of } (W,u) \text{ is } \mathfrak{P}. \tag{4.10}$$

For the filtration $\mathbb{F}^t$, it is enough to choose $\mathbb{F}^t = \sigma(W(s), u(s) : 0 \leq s \leq t), t \in (0,T]$.
It remains to prove that the limit process W is a Wiener process. To fix this, it is sufficient to show that for any $0 < t_1 < t_2 < \ldots < t_m = T$, the increments process $W(t_j) - W(t_{j-1}))$ are independent with respect to $\mathbb{F}^{t_{j-1}}$, distributed normally with mean 0 and variance $t_j - t_{j-1}$. That is, to show that for any $\lambda_j \in \mathbb{R}^m$ and $i^2 = -1$

$$\mathbb{E}\exp\left(i\sum_{j=1}^{m}\lambda_j(W(t_j) - W(t_{j-1}))\right) = \prod_{j=1}^{m}\exp\left(-\frac{1}{2}\lambda_j^2(t_j - t_{j-1})\right). \tag{4.11}$$

The equation (4.11) will follow if we have

$$\mathbb{E}\left[\exp\left(i\lambda(W(t+\theta) - W(t))\right)/\,\mathcal{F}^t\right] = \exp\left(-\frac{\lambda\theta^2}{2}\right). \tag{4.12}$$

We rely on the fact that for any random variables X and Y on any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that X is $\mathcal{F}$-measurable and $\mathbb{E}|Y| < \infty$, $\mathbb{E}|XY| < \infty$, we have

$$\mathbb{E}(XY/\mathcal{F}) = X\mathbb{E}(Y/\mathcal{F}), \quad \mathbb{E}\mathbb{E}(Y/\mathcal{F}) = \mathbb{E}(Y),$$

that is,

$$\mathbb{E}(XY) = \mathbb{E}(X\mathbb{E}(Y/\mathcal{F})). \tag{4.13}$$

Now, let us consider an arbitrary bounded continuous functional $\vartheta_t(W, v)$ on $\mathfrak{S}$ depending only on the values of W and v on $(0,T)$. Owing to the independence of $\bar{W}(t)$ to $\vartheta_t(\bar{W}, v)$ and the fact that $\bar{W}$ is a Wiener process, we have

$$\begin{aligned}
&\bar{\mathbb{E}}\left[\exp\left(i\lambda(\bar{W}(t+\theta) - \bar{W}(t))\right)\vartheta_t(\bar{W}, v)\right] \\
&= \bar{\mathbb{E}}\left[\exp\left(i\lambda(\bar{W}(t+\theta) - \bar{W}(t))\right)\right]\bar{\mathbb{E}}\left[\vartheta_t(\bar{W}, v)\right] \\
&= \exp\left(-\frac{\lambda\theta^2}{2}\right)\bar{\mathbb{E}}\left[\vartheta_t(\bar{W}, v)\right].
\end{aligned}$$

In view of (4.10), this implies that

$$\begin{aligned}&\mathbb{E}\left[\exp\left(i\lambda(W^{N_\mu}(t+\theta)-W^{N_\mu}(t))\right)\vartheta_t(W^{N_\mu},v)\right]\\&=\mathbb{E}\left[\exp\left(i\lambda(W^{N_\mu}(t+\theta)-W^{N_\mu}(t))\right)\right]\mathbb{E}\left[\vartheta_t(W^{N_\mu},v)\right]\\&=\exp\left(-\frac{\lambda\theta^2}{2}\right)\mathbb{E}\left[\vartheta_t(W^{N_\mu},v)\right].\end{aligned}$$

Now, the convergences (4.8) and (4.9) and the continuity of ϑ allow us to pass to the limit in this latter equation and obtain

$$\mathbb{E}\left[\exp\left(i\lambda(W(t+\theta)-W(t))\right)\vartheta_t(W,v)\right]=\exp\left(-\frac{\lambda\theta^2}{2}\right)\mathbb{E}\left[\vartheta_t(W,v)\right],$$

which, in view of (4.13), implies (4.12). The choice of the above filtration implies then that W is a $\mathbb{F}^t$-standard m-dimensional Wiener process.

Theorem 4.11. *The pair (u^{N_μ},W^{N_μ}) satisfies the equation*

$$\begin{aligned}(u^{N_\mu}(s),e_i)_{\mathbb{V}}+\nu\int_0^t((u^{N_\mu}(s),e_i))ds+\int_0^t(\text{curl}(u^{N_\mu}(s)-\alpha\Delta u^{N_\mu}(s))\times u^{N_\mu}(s),e_i)ds\\=(u_0^{N_\mu},e_i)_{\mathbb{V}}+\int_0^t(F(u^{N_\mu}(s),s),e_i)ds+\int_0^t(G(u^{N_\mu}(s),s),e_i)dW^{N_\mu}.\end{aligned}\tag{4.14}$$

for any $i\geq 1$.

Proof. Let $i\geq 1$ be an arbitrary fixed integer. Following [8], we set

$$\begin{aligned}\mathfrak{X}^N=\int_0^T\Big|(u^N(s),e_i)_{\mathbb{V}}-(u_0^N,e_i)_{\mathbb{V}}+\nu\int_0^t((u^N(s),e_i))ds-\int_0^t(F(u^N(s),s),e_i)ds\\+\int_0^t(\text{curl}(u^N(s)-\alpha\Delta u^N(s))\times u^N(s),e_i)ds-\int_0^t(G(u^N(s),s),e_i)d\bar{W}\Big|^2dt.\end{aligned}$$

Obviously

$$\mathfrak{X}^N=0\ \bar{\mathbb{P}}-a.s,$$

which implies in particular that

$$\bar{\mathbb{E}}\frac{\mathfrak{X}^N}{1+\mathfrak{X}^N}=0.$$

Now we let

$$\begin{aligned}\mathfrak{Y}^{N_\mu}=\int_0^T\Big|(u^{N_\mu}(s),e_i)_{\mathbb{V}}+\nu\int_0^t((u^{N_\mu}(s),e_i))ds-\int_0^t(F(u^{N_\mu}(s),s),e_i)ds\\-(u_0^{N_\mu},e_i)_{\mathbb{V}}+\int_0^t(\text{curl}(u^{N_\mu}(s)-\alpha\Delta u^{N_\mu}(s))\times u^N(s),e_i)ds\\+\int_0^t(G(u^{N_\mu}(s),s),e_i)dW^{N_\mu}\Big|^2dt.\end{aligned}$$

We shall prove that

$$\mathbb{E}\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} = 0. \tag{4.15}$$

The difficulty we encounter is that $\mathfrak{X}^N$ is not a deterministic functional of u^N and $\bar{W}$ because of the stochastic term. To overcome this obstacle we introduce

$$G^\varepsilon(u(t),t) = \frac{1}{\varepsilon}\int_0^T \phi\left(-\frac{t-s}{\varepsilon}\right) G(u(s),s)ds, \tag{4.16}$$

where ϕ is a mollifier. It is clear that

$$\bar{\mathbb{E}}\int_0^T |G^\varepsilon(u(t),t)|^2 dt \leq \bar{\mathbb{E}}\int_0^T |G(u(t),t)|^2 dt. \tag{4.17}$$

Moreover,

$$G^\varepsilon(u(.),.) \to G(u(.),.) \text{ in } L^2(\bar{\Omega},\bar{\mathbb{P}};L^2(0,T;\mathbb{V})),$$

which implies in particular that

$$(G^\varepsilon(u(.),.),e_i) \to (G(u(.),.),e_i) \text{ in } L^2(\bar{\Omega},\bar{\mathbb{P}};L^2(0,T)) \text{ for any } i\geq 1.$$

Let us denote by $\mathfrak{X}^{N,\varepsilon}$ and $\mathfrak{Y}^{N_\mu,\varepsilon}$ the analog of $\mathfrak{X}^N$ and $\mathfrak{Y}^{N_\mu}$ with G replaced by G^ε. Introduce the mapping

$$\begin{aligned}\varphi_{N,\varepsilon} :& C(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{V}) \to (\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})\\ \varphi_{N,\varepsilon}(\bar{W},v^N) &= \frac{\mathfrak{X}^{N,\varepsilon}}{1+\mathfrak{X}^{N,\varepsilon}}.\end{aligned}$$

Now, it is seen that $\varphi_{N,\varepsilon}$ is a bounded continuous functional on $\mathfrak{S}$. Next, let us define

$$\varphi_{N_\mu,\varepsilon}(\bar{W},u^{N_\mu}) = \frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}}.$$

We have

$$\mathbb{E}\frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}} = \mathbb{E}\varphi_{N_\mu,\varepsilon}(W^{N_\mu},u^{N_\mu}) \tag{4.18}$$

Since $\varphi_{N_\mu,\varepsilon}(W^{N_\mu},u^{N_\mu})$ is a bounded functional on $\mathfrak{S}$ and since the law of W^{N_μ},u^{N_μ} is $\mathfrak{P}^{N_\mu}$ (see (4.10)), then

$$\mathbb{E}\frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}} = \int_{\mathfrak{S}}\varphi(w,v)d\mathfrak{P}^{N_\mu}.$$

We note that $\text{law}(\bar{W}, u^{N_\mu}) = \mathfrak{P}^{N_\mu}$, so

$$\begin{aligned}\int_{\mathfrak{S}} \varphi(w,v) d\mathfrak{P}^{N_\mu} &= \bar{\mathbb{E}}\varphi(\bar{W}, u^{N_\mu}) \\ &= \bar{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}}.\end{aligned}$$

That is,

$$\mathbb{E}\frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}} = \bar{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}}.$$

Note that

$$\begin{aligned}\mathbb{E}\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} - \bar{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu}}{1+\mathfrak{X}^{N_\mu}} =& \mathbb{E}\left(\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} - \frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}}\right) + \mathbb{E}\frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}} - \bar{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}} \\ &+ \bar{\mathbb{E}}\left(\frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}} - \frac{\mathfrak{X}^{N_\mu}}{1+\mathfrak{X}^{N_\mu}}\right)\end{aligned}$$

We can check that

$$\mathbb{E}\left|\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} - \frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}}\right| = \mathbb{E}\left|\frac{\mathfrak{Y}^{N_\mu} - \mathfrak{Y}^{N_\mu,\varepsilon}}{(1+\mathfrak{Y}^{N_\mu})(1+\mathfrak{Y}^{N_\mu,\varepsilon})}\right|,$$

and it implies that

$$\mathbb{E}\left|\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}} - \frac{\mathfrak{Y}^{N_\mu,\varepsilon}}{1+\mathfrak{Y}^{N_\mu,\varepsilon}}\right| \leq C\left(\mathbb{E}\int_0^T |(G^\varepsilon(u^{N_\mu}(t),t) - G(u^{N_\mu}(t),t), e_i)|^2 dt\right)^{\frac{1}{2}}.$$

We also have

$$\mathbb{E}\left|\frac{\mathfrak{X}^{N_\mu}}{1+\mathfrak{X}^{N_\mu}} - \frac{\mathfrak{X}^{N_\mu,\varepsilon}}{1+\mathfrak{X}^{N_\mu,\varepsilon}}\right| \leq C\left(\bar{\mathbb{E}}\int_0^T |(G^\varepsilon(u^{N_\mu}(t),t) - G(u^{N_\mu}(t),t), e_i)|^2 dt\right)^{\frac{1}{2}}.$$

The above estimates and (4.18) yield

$$\left|\left|\mathbb{E}\frac{\mathfrak{Y}^{N_\mu}}{1+\mathfrak{Y}^{N_\mu}}\right| - \left|\bar{\mathbb{E}}\frac{\mathfrak{X}^{N_\mu}}{1+\mathfrak{X}^{N_\mu}}\right|\right| \leq C\left(\bar{\mathbb{E}}\int_0^T |(G^\varepsilon(u^{N_\mu}(t),t) - G(u^{N_\mu}(t),t), e_i)|^2 dt\right)^{\frac{1}{2}}.$$

Passing to limit to the above relation imply (4.15) and hence (4.14). □

4.4 Proof of the main result

4.4.1 Passage to the limit

From the tightness property we have

$$u^{N_\mu} \to u \text{ in } L^2(0,T;\mathbb{V}) \ \mathbb{P}\text{-a.s.} \tag{4.19}$$

Since u^{N_μ} satisfies the two equivalent equations (4.14), then it verifies the same estimates as u^N.

Let us consider the positive nondecreasing function $\varphi(x) = x^p$, $p \geq 4$ defined on $\mathbb{R}_+$. We have

$$\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty. \tag{4.20}$$

Thanks to the estimate $\mathbb{E}\sup_{t\in[0,T]} |u^{N_\mu}|_{\mathbb{V}} \leq C$, we have

$$\mathbb{E}(\phi(|u^{N_\mu}|_{L^2(0,T;\mathbb{V})})) < \infty. \tag{4.21}$$

Thanks to the uniform integrability criterion in Theorem 2.20 we see that $|u^{N_\mu}|_{L^2(0,T;\mathbb{V})}$ is uniform integrable with respect to the probability measure.

We can deduce from Vitali's Convergence Theorem 2.21 that

$$u^{N_\mu} \to u \text{ in } L^2(\Omega, \mathbb{P}, L^2(0,T;\mathbb{V})). \tag{4.22}$$

This implies in particular that

$$u^{N_\mu} \to u \text{ in } L^2(\Omega, \mathbb{P}, L^2(0,T;L^2(D))), \tag{4.23}$$

$$\frac{\partial u^{N_\mu}}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ in } L^2(\Omega, \mathbb{P}, L^2(0,T;L^2(D))),\ i = 1,2. \tag{4.24}$$

Thanks to (4.22), we can still extract a new subsequence from u^{N_μ} denoted again by u^{N_μ} so that

$$u^{N_\mu} \to u \text{ in } \mathbb{V}\ dt \times d\mathbb{P} - \text{almost everywhere}. \tag{4.25}$$

It is readily seen that

$$((u^{N_\mu}, e_i)) \to ((u, e_i)) \text{ strongly in } L^2(\Omega, \mathbb{P}; L^2(0,T)).$$

Let χ be an element of $L^\infty(\Omega \times [0,T], d\mathbb{P} \otimes dt)$.

Since $e_i \in \mathbb{H}^3(D) \subset \mathbb{L}^\infty(D)$, then $\chi e_i \in L^\infty(\Omega \times [0,T] \times D, d\mathbb{P} \otimes dt \otimes dx)$. Thanks to (4.23), (4.24) we have that

$$u_j^{N_\mu} \frac{\partial u_k^{N_\mu}}{\partial x_j} (\chi e_i)_k \to u_j \frac{\partial u_k}{\partial x_j} (\chi e_i)_k \in L^1(\Omega \times (0,T] \times D), \tag{4.26}$$

which implies that

$$\mathbb{E}\int_0^T b(u^{N_\mu}, u^{N_\mu}, \chi e_i) dt \to \mathbb{E}\int_0^T b(u, u, \chi e_i) dt \text{ for any } i. \tag{4.27}$$

Since in view of Lemma 3.6 $e_i \in \mathbb{H}^4(D)$, then

$$\frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \in L^\infty(\Omega, \mathbb{P}, L^\infty(0,T;\mathbb{H}^2(D))).$$

Since $\mathbb{H}^2(D) \subset \mathbb{L}^\infty(D)$, then

$$\frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \in L^\infty(\Omega, \mathbb{P}, L^\infty(0,T;\mathbb{L}^\infty(D))).$$

With the help of (4.23) we obtain

$$(u^{N_\mu})_k \frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \to u_k \frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \text{ in } L^2(\Omega, \mathbb{P}; L^2(0,T;\mathbb{L}^2(D))).$$

We derive from this and (4.24) that

$$\mathbb{E}\int_0^T (u^{N_\mu})_k \frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \left(\frac{\partial u^{N_\mu}}{\partial x_j}\right)_l dt \to \mathbb{E}\int_0^T u_k \frac{\partial}{\partial x_k}\left(\chi \frac{\partial e_i}{\partial x_j}\right)_l \left(\frac{\partial u}{\partial x_j}\right)_l dt, \ \forall i,j,k,l. \tag{4.28}$$

In the above equations $(f)_k$ denotes the k-th component of the vector function f.

We can use the same argument to show that

$$\mathbb{E}\int_0^T \left(\frac{\partial u^{N_\mu}}{\partial x_j}\right)_k \chi \frac{\partial (e_i)_l}{\partial x_j} \left(\frac{\partial u^{N_\mu}}{\partial x_j}\right)_l dt \to \mathbb{E}\int_0^T \left(\frac{\partial u}{\partial x_j}\right)_k \chi \frac{\partial (e_i)_l}{\partial x_j} \left(\frac{\partial u}{\partial x_j}\right)_l dt, \ \forall i,j,k,l. \tag{4.29}$$

The density of $L^\infty(\Omega \times [0,T], d\mathbb{P} \otimes dt)$ in $L^2(\Omega \times [0,T], d\mathbb{P} \otimes dt)$, together with the relation (2.12), and the two equations (4.28), (4.29) imply that

$$b(u^{N_\mu}, \Delta u^{N_\mu}, e_i) \rightharpoonup b(u, \Delta u, e_i) \text{ weakly in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T)), \tag{4.30}$$

for any i.

Using the equation (2.13), we can imitate the argument used above to show that

$$b(e_i, \Delta u^{N_\mu}, u^{N_\mu}) \rightharpoonup b(e_i, \Delta u, u) \text{ weakly in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T)), \tag{4.31}$$

for any i.

We conclude with (2.9), (4.27), (4.30) and (4.31) that

$$(\mathrm{curl}(u^{N_\mu} - \alpha \Delta u^{N_\mu}) \times u^{N_\mu}, e_i) \rightharpoonup (\mathrm{curl}(u - \alpha \Delta u) \times u, e_i),$$

weakly in $L^2(\Omega, \mathbb{P}; L^2(0,T))$ for any i.

It follows from (4.25), the Lemma (4.5), the assumption on F and Vitali's Convergence Theorem 2.21 that

$$F(u^{N_\mu}(,),.) \to F(u(.),.) \text{ strongly in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T;\mathbb{V})). \tag{4.32}$$

This implies in particular that

$$(F(u^{N_\mu}(,),.), e_i) \to (F(u(.),.), e_i) \text{ strongly in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T)),$$

for any i.

It remains to prove that

$$\int_0^t (G(u^{N_\mu}, s), e_i) dW^{N_\mu} \rightharpoonup \int_0^t (G(u,s), e_i) dW \text{ weakly in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T)), \tag{4.33}$$

for any $t \in [0,T]$ and i as $\mu \to \infty$. Using a similar argument as in [114] we will just show that

$$\int_0^T (G(u^{N_\mu}, s), e_i) dW^{N_\mu} \rightharpoonup \int_0^T (G(u,s), e_i) dW \text{ weakly } L^2(\Omega, \mathcal{F}, \mathbb{P}), \tag{4.34}$$

from (4.33) follows. From now on we fix $i \geq 1$. First, Lemma 3.8, the convergence (4.25), the assumption on G and Vitali's Convergence Theorem 2.21 imply that

$$(G(u^{N_\mu}, .), e_i) \to (G(u,.), e_i) \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T)) \tag{4.35}$$

as $\mu \to \infty$. We consider the already introduced regularized function $G^\varepsilon(u(.),.)$ in (4.16). We readily check that

$$(G^\varepsilon(u(.),.), e_i) \to (G(u(.),.), e_i) \text{ in } L^2(\Omega, \mathbb{P}; L^2(0,T)), \tag{4.36}$$

as $\varepsilon \to 0$. We also have

$$\mathbb{E} \int_0^T |(G^\varepsilon(u^{N_\mu}, t) - G^\varepsilon(u,t), e_i)|^2 dt \leq \mathbb{E} \int_0^T |(G(u^{N_\mu}, t) - G(u,t), e_i)|^2 dt. \tag{4.37}$$

The crucial point is to show that

$$\int_0^T (G^\varepsilon(u^{N_\mu}, t), e_i) dW^{N_\mu} \rightharpoonup \int_0^T (G^\varepsilon(u,t), e_i) dW \text{ weakly in } L^2(\Omega, \mathbb{P}). \tag{4.38}$$

Since

$$\mathbb{E} \left| \int_0^T (G^\varepsilon(u^{N_\mu}, t), e_i) dW^{N_\mu} \right|^2 = \mathbb{E} \int_0^T (G^\varepsilon(u^{N_\mu}, t), e_i)^2 dt < \infty,$$

then $\int_0^T (G^\varepsilon(u^{N_\mu}, t), e_i) dW^{N_\mu}$ weakly converges to a certain β in $L^2(\Omega, \mathbb{P})$. An integration-by-parts yields

$$\int_0^T (G^\varepsilon(u^{N_\mu}, t), e_i) dW^{N_\mu} = (G^\varepsilon(u^{N_\mu}(T), T), e_i) - \int_0^T W^{N_\mu}(t) \frac{d}{dt}(G^\varepsilon(u^{N_\mu}(t), t), e_i) dt,$$

where

$$\frac{d}{dt}(G^\varepsilon(u^{N_\mu}(t), t), e_i) = \frac{1}{\varepsilon} \int_0^T \frac{d}{dt} \phi(-\frac{t-s}{\varepsilon})(G(v^{N_\mu}(s), s), e_i) ds.$$

By virtue of the convergence

$$(u^{N_\mu}, W^{N_\mu}) \to (u, W) \text{ in } C(0, T; \mathbb{R}^m) \times L^2(0, T; \mathbb{V})$$

$\mathbb{P}$-almost surely, we have

$$\int_0^T (G^\varepsilon(u^{N_\mu}, t), e_i) dW^{N_\mu} \to (G^\varepsilon(u(T), T), e_i) - \int_0^T W(t) \frac{d}{dt}(G^\varepsilon(u(t), t), e_i) dt \tag{4.39}$$

for almost all $\omega \in \Omega$. The term in the left hand side of the equation (4.39) is equal to

$$\int_0^T (G^\varepsilon(u(t), t), e_i) dW.$$

Now let us pick an element $\zeta \in L^\infty(\Omega, \mathbb{P})$. We have

$$\mathbb{E} \int_0^T (G^\varepsilon(u^{N_\mu}(t), t), \zeta e_i) dW^{N_\mu} \to \mathbb{E} \int_0^T (G^\varepsilon(u(t), t), \zeta e_i) dW, \tag{4.40}$$

that is

$$\beta = \int_0^T (G^\varepsilon(u(t), t), e_i) dW.$$

Indeed, thanks to the estimate (4.17), the Lemma 4.4 the sequence of random variables $\int_0^T (G^\varepsilon(u^{N_\mu}(t), t), \zeta e_i) dW^{N_\mu}$ is uniformly integrable. Owing to the convergence (4.39) and the applicability of Vitali's Convergence Theorem (Theorem 2.21), we get (4.40). We also have (4.38) since $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is dense in $L^2(\Omega, \mathbb{P})$.

Let $\zeta \in L^\infty(\Omega, \mathbb{P})$. We write

$$\left| \mathbb{E} \int_0^T (G(u^{N_\mu}(t), t), \zeta e_i) dW^{N_\mu} - \mathbb{E} \int_0^T (G(u(t), t), \zeta e_i) dW \right| \leq J_1 + J_2 + J_3, \tag{4.41}$$

where

$$J_1 = \left| \mathbb{E} \int_0^T (G^\varepsilon(u^{N_\mu}(t), t), \zeta e_i) dW^{N_\mu} - \mathbb{E} \int_0^T (G(u^{N_\mu}(t), t), \zeta e_i) dW^{N_\mu} \right|$$

$$J_2 = \left| \mathbb{E} \int_0^T (G^\varepsilon(u^{N_\mu}(t), t), \zeta e_i) dW^{N_\mu} - \mathbb{E} \int_0^T (G^\varepsilon(u(t), t), \zeta e_i) dW \right|$$

$$J_3 = \left| \mathbb{E} \int_0^T (G^\varepsilon(u(t), t), \zeta e_i) dW - \mathbb{E} \int_0^T (G(u(t), t), \zeta e_i) dW \right|.$$

By Cauchy-Schwarz's inequality and owing to (4.36), the term J_3 of the right hand side of (4.41) converges to zero as $\varepsilon \to 0$.
By (4.40), the term J_2 in the right hand side of (4.41) converges to zero as $\mu \to \infty$.
By Cauchy-Schwarz's inequality again, some simple calculations, and making use of the estimate (4.37) and the convergence (4.35) and (4.36) we see that J_1 converges to zero as $\varepsilon \to 0$ and $\mu \to \infty$.

In view of these convergences passing to the limit as $\varepsilon \to 0$ and $\mu \to \infty$ in (4.41) we get (4.33).
Combining all these results and passing to the limit in (4.14), we see that u satisfies the equation (4.3). This proves the first part of Theorem (4.3). The next subsection addresses the continuity in time of u.

4.4.2 Pathwise continuity in time of the weak probabilistic solution

We have already shown that for any $i \geq 1$ the equation

$$(u(t), e_i)_{\mathbb{V}} = (u_0, e_i)_{\mathbb{V}} + \int_0^t ((F(u(s), s) - \operatorname{curl}(u - \alpha\Delta u) \times u, e_i) - \nu((u(s), e_i)))\, ds + \int_0^t (G(u(s), s), e_i) dW$$

holds almost surely for any $t \in [0, T]$.
For any $i \geq 1$ let φ be the mapping

$$[0, T] \to \mathbb{R}$$
$$t \mapsto \varphi(t) = (u(t), e_i)_{\mathbb{V}}.$$

Let $\theta > 0$. We have

$$|\varphi(t) - \varphi(t+\theta)| \leq \left| \int_t^{t+\theta} ((F(u(s), s) - \operatorname{curl}(u - \alpha\Delta u) \times u, e_i) - \nu((u(s), e_i)))\, ds \right| + \left| \int_t^{t+\theta} (G(u(s), s), e_i) dW \right|$$

Let $p > 4$, we obtain by raising both sides of the last inequality to the power $\frac{p}{2}$

$$\begin{aligned}|\varphi(t)-\varphi(t+\theta)|^{\frac{p}{2}} \leq C\left|\int_t^{t+\theta}(G(u(s),s),e_i)dW\right|^{\frac{p}{2}} + C\left(\int_t^{t+\theta}|(F(u(s),s),e_i)|ds\right)^{\frac{p}{2}} \\ + C\left(\int_t^{t+\theta}|(\mathrm{curl}(u-\alpha\Delta u)\times u,e_i)|ds\right)^{\frac{p}{2}} \\ + C\left(\nu\int_t^{t+\theta}|((u(s),e_i))|ds\right)^{\frac{p}{2}}\end{aligned}$$

We infer from this that

$$\begin{aligned}\mathbb{E}|\varphi(t)-\varphi(t+\theta)|^{\frac{p}{2}} \leq \mathbb{E}\left(\int_t^{t+\delta}|(\mathrm{curl}(u-\alpha\Delta u)\times u,e_i)|ds\right)^{\frac{p}{2}} \\ + C\mathbb{E}\sup_{0\leq\delta\theta}\left|\int_t^{t+\delta}(G(u(s),s),e_i)dW\right|^{\frac{p}{2}} \\ + C\mathbb{E}\left(\int_t^{t+\delta}|(F(u(s),s),e_i)|ds\right)^{\frac{p}{2}} \\ + C\mathbb{E}\left(\int_t^{t+\delta}\nu|((u(s),e_i))|ds\right)^{\frac{p}{2}},\end{aligned}$$

which implies by the help of the martingale inequality that

$$\begin{aligned}\mathbb{E}|\varphi(t)-\varphi(t+\theta)|^{\frac{p}{2}} \leq C\theta^{\frac{p-2}{2}}\int_t^{t+\delta}|(\mathrm{curl}(u-\alpha\Delta u)\times u,e_i)|^{\frac{p}{2}}ds \\ + C\theta^{\frac{p-2}{2}}\int_t^{t+\delta}|(F(u(s),s),e_i)|^{\frac{p}{2}}ds \\ + C\theta^{\frac{p-2}{2}}\nu\int_t^{t+\delta}|((u(s),e_i))|^{\frac{p}{2}}ds \\ + C\mathbb{E}\left(\int_t^{t+\theta}|(G(u(s),s),e_i)|^2\right)^{\frac{p}{2}}.\end{aligned}$$

Using previous estimates and some elementary inequalities the following holds :

$$\mathbb{E}|\varphi(t)-\varphi(t+\theta)|^{\frac{p}{2}} \leq C(\theta^{1+\frac{p-2}{2}}+\theta^{1+\frac{p-4}{4}}),$$

for any $\theta > 0$. We conclude from Kolmogorov-Čentsov Theorem (Theorem 2.10) that the stochastic process $\varphi(.) = (u(.), e_i)_{\mathbb{V}}$ has almost surely a continuous modification with respect to the time variable t. Identifying u with this modification, we see that u has almost surely continuous paths taking values in $\mathbb{V}$-weak. Since u is also in the class $L^p(\Omega, \mathbb{P}; L^\infty(0, T; \mathbb{W}))$, then $u(.)$ also has almost surely continuous paths with respect to

t taking values in $\mathbb{W}$-weak (see [115] for justification). It follows that the initial condition $u(x,0) = u_0 \in \mathbb{W}$ in (1.5) makes sense.

To close this chapter we notice that the results about the stability of the solutions we gave in Chapter 3 still hold for the weak probabilistic solution. We also mention that the existence we have here and in the preceding chapter hold for the periodic boundary conditions.

Chapter 5

Asymptotic behavior of solutions of stochastic evolution equations for second grade fluids

5.1 Introduction

This chapter is devoted to the analysis of the behavior of the solutions of the stochastic equations for the motion of turbulent flows of a second grade fluids when the stress modulus α tends to zero. More precisely for a periodic square D, $D = [0,L]^2 \subset \mathbb{R}^2$, $L > 0$, we aim to study the convergence of the periodic-in-space velocity solution with period L of the following:

$$\begin{cases} d(u^\alpha - \alpha\Delta u^\alpha) + (-\nu\Delta u^\alpha + \operatorname{curl}(u^\alpha - \alpha\Delta u^\alpha) \times u^\alpha + \nabla P)dt = F(t,x)\,dt + G(t,x)dW \\ \text{in } \Omega \times (0,T] \times D, \\ \operatorname{div} u^\alpha = 0 \text{ in } \Omega \times (0,T] \times D, \\ \int_D u^\alpha dx = 0 \text{ in } \Omega \times (0,T], \\ u^\alpha(0) = u_0 \text{ in } \Omega \times D, \end{cases} \tag{5.1}$$

when the parameter α tends to 0. Here P is a scalar function representing a modified pressure, $(\Omega, \mathcal{F}, \mathbb{P})$, $0 \leq t \leq T$, is a given complete probability space on which a $\mathbb{R}^m$-valued standard Wiener process W is defined and $\mathbb{F}^t$ is an increasing filtration generated by W. Our aim is to show that we can construct a sequence u^{α_j} of strong probabilistic solutions of (5.1) that converges in appropriate sense the strong probabilistic solution of the stochastic Navier-Stokes equations as $\alpha_j \to 0$. That is, there exists another complete filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}^t, \bar{\mathbb{P}})$, a $\mathbb{R}^m$-valued Wiener process $\bar{\mathcal{W}}$ and a stochastic process v such that the following holds in the distribution sense

$$\begin{cases} dv + (\nu A v + \mathbf{P}(v \cdot \nabla v))dt = F(t,x)dt + G(t,x)d\bar{\mathcal{W}} \\ \text{in } \bar{\Omega} \times (0,T] \times D, \\ \operatorname{div} v = 0 \text{ in } \bar{\Omega} \times (0,T] \times D, \\ \int_D v dx = 0 \text{ in } \bar{\Omega} \times (0,T], \\ v(0) = u_0 \text{ in } \bar{\Omega} \times D; \end{cases} \tag{5.2}$$

here the operators A and $\mathbf{P}$ denote the Stokes operator and the Leray's projector, respectively. In the deterministic case, that is, when $G(t,x)dW = 0$ it is known from [61] that under general assumptions on the data (u_0^α, F) the weak solution (in the partial differential equations sense) of the second grade fluids equations converges weakly to the weak solution (in the partial differential equations sense) of the Navier-Stokes equations.

In addition to the current introduction, this chapter consist of three other sections. In Section 2, we give the hypotheses on our problem and state the main result whose proof is given in the last section. Section 3 is devoted for the derivation of crucial uniform estimates that are needed for the proof of the result.

5.2 Hypotheses and a convergence theorem

Throughout this section we assume that

(I) $F = F(t,x)$ is a $\mathbb{V}_{per}$-valued function defined on $[0,T] \times D$ such that the following holds

$$\int_0^T |F(t,x)|^p_{\mathbb{V}_{per}} < \infty,$$

for any $2 \le p < \infty$.

(II) $G = G(t,x)$ is a $\mathbb{V}_{per}^{\otimes m}$-valued function defined on $[0,T] \times D$ such that the following holds

$$\int_0^T |G(t,x)|^p_{\mathbb{V}_{per}^{\otimes m}} < \infty,$$

for any $2 \le p < \infty$.

(III) We further assume that $u_0 \subset \mathbb{V}_{per} \cap \mathbb{H}^3_{per}$ is nonrandom and there exists a positive constant C independent of α such that $|u_0|_{\mathbb{V}_{per}} < C$. Suppose also that the viscosity $\nu > 0$.

Throughout $|.|$ and $||.||$ denote the norm in $\mathbb{H}_{per}$ and the gradient norm on $\mathbb{V}_{per}$, respectively. For our convenience, let us recall the definition of the concept of the strong probabilistic solution for problem (5.1).

Definition 5.1. By a strong probabilistic solution of the system (5.1), we mean a stochastic process u^α such that

1. $u^\alpha \in L^p(\Omega, \mathbb{P}; L^\infty(0,T;\mathbb{W})) \cap L^p(\Omega, \mathbb{P}; L^\infty(0,T;\mathbb{V}_{per}))$ with $2 \le p < \infty$,
2. For all t, $u^\alpha(t)$ is $\mathbb{F}^t$-measurable,
3. $\mathbb{P}$-almost surely the following integral equation holds

$$(u^\alpha(t) - u^\alpha(0), \phi)_{\mathbb{V}_{per}} + \int_0^t [\nu((u^\alpha, \phi)) + (\text{curl}(u^\alpha(s) - \alpha\Delta u^\alpha(s)) \times u^\alpha(s), \phi)]\, ds$$
$$= \int_0^t (F,\phi)ds + \int_0^t (G,\phi)dW(s)$$

   for any $t \in [0,T]$ and $\phi \in \mathcal{V}_{per}$.

We know from Theorem 3.4 and Theorem 3.12 of Chapter 3 that (5.1) has a unique strong probabilistic solution.

Before we proceed to the statement of our main result we give the definition of weak probabilistic solution of the SNSE, this is taken from [8].

Definition 5.2. By a weak probabilistic solution of the SNSE, we mean a system

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}}^t, \bar{\mathcal{W}}, v),$$

where

1. $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is a complete probability space, $\bar{\mathbb{F}}^t$ is a filtration on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$,

2. $\bar{\mathcal{W}}(t)$ is an m-dimensional $\bar{\mathbb{F}}^t$-standard Wiener process,

3. $v \in L^p(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T; \mathbb{V}_{per})) \cap L^p(\bar{\Omega}, \bar{\mathbb{P}}; L^\infty(0,T; \mathbb{H}_{per})), \forall \quad 2 \leq p < \infty$,

4. For all t, $v(t)$ is $\bar{\mathbb{F}}^t$-measurable,

5. $\bar{\mathbb{P}}$-almost surely the following integral equation holds

$$\begin{aligned} &(v(t) - v(0), \phi) + \int_0^t [\nu((v,\phi)) + < v \cdot \nabla v, \phi >] \, ds \\ &= \int_0^t (F,\phi) ds + \int_0^t (G,\phi) d\bar{\mathcal{W}}(s) \end{aligned} \tag{5.3}$$

   for any $t \in [0,T]$ and $\phi \in \mathcal{V}_{per}$.

Our main result is the following convergence theorem.

Theorem 5.3. *Under the hypotheses (I)-(III) there exist a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a family of probability measures $(\mathfrak{Q}^{\alpha_j})$, a probability measure $\mathfrak{Q}$, and stochastic processes $(W^{\alpha_j}, u^{\alpha_j})$, $(\bar{\mathcal{W}}, v)$ such that the law of $(W^{\alpha_j}, u^{\alpha_j})$ (resp. $(\bar{W}, v)$) is $\mathfrak{Q}^{\alpha_j}$ (resp. $\mathfrak{Q}$) and*

$$W^{\alpha_j} \to \bar{\mathcal{W}} \text{ in } C(0,T;\mathbb{R}^m) \; \bar{\mathbb{P}} - \text{almost surely},$$
$$u^{\alpha_j} \to v \text{ in } L^2(0,T;\mathbb{H}_{per}) \; \bar{\mathbb{P}} - \text{almost surely},$$

as $j \to \infty$ $(\alpha_j \to 0)$. The pair $(W^{\alpha_j}, u^{\alpha_j})$ satisfies $\bar{\mathbb{P}}-$almost surely (5.1) in the sense of distribution and as $j \to \infty$ $(\alpha_j \to 0)$

$$u^{\alpha_j} \rightharpoonup v, \text{ weakly in } L^p(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T;\mathbb{V}_{per})), \tag{5.4}$$
$$u^{\alpha_j} \rightharpoonup v \text{ weakly-* in } L^p(\bar{\Omega}, \bar{\mathbb{P}}; L^\infty(0,T;\mathbb{H}_{per})), \tag{5.5}$$

for all $2 \leq p < \infty$. Furthermore, $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, v, \bar{\mathcal{W}})$ is a weak probabilistic solution of the SNSE:

$$\begin{cases} dv + (\nu A v + \mathbf{P}(v \cdot \nabla v))\, dt = F dt + G d\bar{\mathcal{W}} \\ \text{in } \bar{\Omega} \times (0,T] \times D, \\ \operatorname{div} v = 0 \text{ in } \bar{\Omega} \times (0,T] \times D, \\ \int_D v dx = 0 \text{ in } \bar{\Omega} \times (0,T], \\ v(0) = u_0 \text{ in } \bar{\Omega} \times D. \end{cases} \tag{5.6}$$

Remark 5.4. Since we are in 2-D then it is known that under the conditions (I)-(III) the problem (5.6) has a strong probabilistic solution which is unique, see for example [81]. This implies that the process v of the above theorem is a strong probabilistic solution of the stochastic Navier-Stokes equations (5.6).

5.3 Uniform a priori estimates

In this section we derive some estimates uniform in α. These inequalities do not follow from previous ones (see Chapter 3) which explode when $\alpha \to 0$. As usual C denotes some unessential positive constants independent of α, and which may change from one line to the next. Since we will let $\alpha \to 0$ then we may assume that $\alpha \in (0,1)$.

Lemma 5.5. *For $\alpha \in (0,1)$ we have*

$$\mathbb{E} \sup_{0\le s\le T} (|u^\alpha(s)|^2 + \alpha||u^\alpha(s)||^2) + \mathbb{E}\int_0^T ||u^\alpha(s)||^2 ds < C, \tag{5.7}$$

$$\mathbb{E} \sup_{0\le s\le T} (|u^\alpha(s)|^2 + \alpha||u^\alpha(s)||^2)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T ||u^\alpha(s)||^2 ds\right)^{\frac{p}{2}} < C, \tag{5.8}$$

for any $2 \le p < \infty$.

Before we prove this result it is important to make the following remark.

Remark 5.6. We recall that the continuous linear operator $(I+\alpha A)^{-1}$, where A is the usual Stokes operator, establishes a bijective correspondence between the spaces $\mathbb{H}^l_{per}(D) \cap \mathbb{V}_{per}$ (resp. $\mathbb{H}_{per}$) and $\mathbb{H}^{l+2}_{per}(D) \cap \mathbb{V}_{per}$, $l>1$ (resp. $l=0$) (see Theorem 2.7). Furthermore for any $w \in \mathbb{V}_{per}$, and $f \in \mathbb{H}^l_{per}(D)$, $l \ge 0$,

$$((I+\alpha A)^{-1} f, w)_{\mathbb{V}_{per}} = (f, w), \tag{5.9}$$

$$|(I+\alpha A)^{-1} f|_{\mathbb{V}_{per}} \le C|f| \tag{5.10}$$

Proof of Lemma 5.5. We start by establishing (5.7). To get rid of the gradient of the modified pressure we project the first equation of (5.1) onto $\mathbb{H}_{per}$ using Leray's projector $\mathbf{P}$. By doing so we obtain the equation

$$d(u + \alpha A u^\alpha) + A u^\alpha dt + \widehat{B}(u^\alpha, u^\alpha) dt = F dt + G dW, \tag{5.11}$$

which holds $\mathbb{P}$-almost surely since u^α is a solution of (1.5). Here we have set

$$\widehat{B}(u^\alpha, u^\alpha) = \mathbf{P}(\text{curl}(u^\alpha - \alpha\Delta u^\alpha) \times u^\alpha).$$

Applying the linear mapping $(I + \alpha A)^{-1}$ to both sides of (5.11) we have the following equation

$$du^\alpha + (I + \alpha A)^{-1} A u^\alpha dt + (I + \alpha A)^{-1}\widehat{B}(u^\alpha, u^\alpha)dt = (I + \alpha A)^{-1} F dt + (I + \alpha A)^{-1} G dW,$$

which holds $\mathbb{P}$-almost surely for any $t \in [0, T]$. For the rest of this chapter we write

$$(I + \alpha A)^{-1} F = \widehat{F},$$
$$(I + \alpha A)^{-1} G = \widehat{G}.$$

Itô's formula for the square of the norm of u^α in $\mathbb{V}$ (see [94] or [73]) implies that

$$\begin{aligned} d|u^\alpha|^2_{\mathbb{V}_{per}} + 2((I + \alpha A)^{-1} A u^\alpha, u^\alpha)_{\mathbb{V}_{per}} dt + 2((I + \alpha A)^{-1}\widehat{B}(u^\alpha, u^\alpha), u^\alpha)_{\mathbb{V}_{per}} dt \\ = (\widehat{F}, u^\alpha)_{\mathbb{V}_{per}} dt + |\widehat{G}|^2_{\mathbb{V}_{per}^{\otimes m}} dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW. \end{aligned}$$

Using the relation (5.9) of Remark 5.6 and the equation

$$(\widehat{B}(u^\alpha, u^\alpha), u^\alpha) = 0,$$

we obtain

$$d|u^\alpha|^2_{\mathbb{V}_{per}} + 2||u^\alpha||^2 dt = 2(F, u^\alpha)dt + |\widehat{G}|^2_{\mathbb{V}_{per}^{\otimes m}} dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW.$$

This implies with the help of Cauchy-Schwarz's inequality and (5.10) that

$$d|u^\alpha|^2_{\mathbb{V}_{per}} + 2||u^\alpha||^2 dt \leq 2|F||u^\alpha|dt + |\widehat{G}|^2_{\mathbb{V}_{per}^{\otimes m}} dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW.$$

By Cauchy's inequality we see from this last estimate that

$$d|u^\alpha|^2_{\mathbb{V}_{per}} + 2||u^\alpha||^2 dt \leq (|F|^2 + |u^\alpha|^2)dt + |\widehat{G}|^2_{\mathbb{V}_{per}^{\otimes m}} dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW,$$

and by the definition of $|.|^2_{\mathbb{V}_{per}}$ we have that

$$d|u^\alpha|^2_{\mathbb{V}_{per}} + 2||u^\alpha||^2 dt \leq |F|^2_{\mathbb{V}_{per}} + |u^\alpha|^2_{\mathbb{V}_{per}} dt + |\widehat{G}|^2_{\mathbb{V}_{per}^{\otimes m}} dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW. \tag{5.12}$$

Taking the sup over $0 \leq s \leq t$, $t \in [0, T]$ and passing to the mathematical expectation yield

$$\mathbb{E} \sup_{0\leq s\leq t} |u^\alpha|^2_{\mathbb{V}_{per}} + 2\mathbb{E}\int_0^t ||u^\alpha||^2 ds \leq C + \mathbb{E}\int_0^t |u^\alpha|^2_{\mathbb{V}_{per}} ds + 2\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s (\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW\right|,$$

where the assumptions on F and G were used. Burkhölder-Davis-Gundy's inequality (see Corollary 2.15) implies

$$\mathbb{E} \sup_{0\leq s\leq t} |u^\alpha|^2_{\mathbb{V}_{per}} + 2\mathbb{E}\int_0^t ||u^\alpha||^2 ds \leq C + \mathbb{E}\int_0^t |u^\alpha|^2_{\mathbb{V}_{per}} ds + 6\mathbb{E}\left(\int_0^t (\widehat{G}, u^\alpha)^2_{\mathbb{V}_{per}} ds\right)^{\frac{1}{2}}.$$

Cauchy's inequality implies

$$\begin{aligned}\mathbb{E} \sup_{0\leq s\leq t} |u^\alpha|^2_{\mathbb{V}_{per}} + 2\mathbb{E}\int_0^t ||u^\alpha||^2 ds \leq C + \mathbb{E}\int_0^t |u^\alpha|^2_{\mathbb{V}_{per}} ds + \frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t} |u^\alpha(s)|^2_{\mathbb{V}_{per}} \\ +C\mathbb{E}\int_0^t |\widehat{G}|^2_{\mathbb{V}_{per}^{\otimes m}} ds,\end{aligned}$$

or

$$\mathbb{E} \sup_{0\leq s\leq t} |u^\alpha|^2_{\mathbb{V}_{per}} + 4\mathbb{E}\int_0^t ||u^\alpha||^2 ds \leq C + C\mathbb{E}\int_0^t |u^\alpha|^2_{\mathbb{V}_{per}} ds.$$

Here we have used (5.10) and the assumption on G. It follows from Gronwall's inequality that

$$\mathbb{E} \sup_{0\leq s\leq t} |u^\alpha|^2_{\mathbb{V}_{per}} + 2\mathbb{E}\int_0^t ||u^\alpha||^2 ds < C,$$

for any $t \in [0, T]$. This completes the proof of (5.7).

We continue with the proof of (5.8). For $2 \leq p < \infty$ and $t \in [0, T]$ the following holds:

$$\begin{aligned}|u^\alpha|^p_{\mathbb{V}_{per}} + p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}_{per}} ||u^\alpha||^2 ds = p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}_{per}} (\widehat{F}, u^\alpha) ds + \frac{p}{2}\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}_{per}} |\widehat{G}|^2_{\mathbb{V}_{per}} ds \\ +|u_0|^p_{\mathbb{V}_{per}} + \frac{(p-2)p}{2}\int_0^t |u^\alpha|^{p-4}_{\mathbb{V}_{per}} (\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} ds + p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}_{per}} (\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW,\end{aligned} \tag{5.13}$$

here we proceeded as in the proof of Lemma 3.8 of Chapter 3. Owing to (5.10) and the above estimate we see that

$$\begin{aligned}|u^\alpha|^p_{\mathbb{V}_{per}} \leq p\int_0^t |u^\alpha|^{p-1}_{\mathbb{V}_{per}} |F| ds + \frac{p}{2}C\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}_{per}} |G|^2 ds + \frac{(p-2)p}{2}C\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}_{per}} |G|^2 ds \\ +|u_0|^p_{\mathbb{V}_{per}} + p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}_{per}} (\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW.\end{aligned} \tag{5.14}$$

This last estimate implies together with Young's inequality that

$$|u^\alpha|^p_{\mathbb{V}_{per}} \leq |u_0|^p_{\mathbb{V}_{per}} + C\int_0^t |u^\alpha|^p_{\mathbb{V}} ds + C\int_0^t |F|^p ds + C\int_0^t |G|^p ds + p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}} (\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW.$$

Taking the sup over $0 \leq s \leq t$, passing to the mathematical expectation and using the assumptions on F and G we get

$$\mathbb{E} \sup_{0\leq s\leq t} |u^\alpha|^p_{\mathbb{V}_{per}} \leq |u_0|^p_{\mathbb{V}_{per}} + C\mathbb{E}\int_0^t |u^\alpha|^p_{\mathbb{V}_{per}} ds + p\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s |u^\alpha|^{p-2}_{\mathbb{V}_{per}}(\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW\right|.$$

Invoking Burkhölder-Davis-Gundy's inequality (Corollary 2.15) yields

$$\mathbb{E} \sup_{0\leq s\leq t} |u^\alpha|^p_{\mathbb{V}_{per}} \leq C\mathbb{E}\int_0^t |u^\alpha|^p_{\mathbb{V}_{per}} ds + p\mathbb{E}\left(\int_0^t |u^\alpha|^{2p-2}_{\mathbb{V}_{per}}|\widehat{G}|^2_{\mathbb{V}_{per}} ds\right)^{\frac{1}{2}}.$$

We infer from this estimate, Young's inequality, (5.10) along with the assumption on G and Gronwall's inequality that

$$\mathbb{E} \sup_{0\leq s\leq t} |u^\alpha(s)|^p_{\mathbb{V}_{per}} < C, \tag{5.15}$$

for any $t \in [0, T]$ and $2 \leq p < \infty$. We deduce from (5.12) with the help of this last estimate that

$$\mathbb{E}\left(\int_0^t ||u^\alpha||^2 ds\right)^{\frac{p}{2}} \leq C + C\mathbb{E}\left|\int_0^t (\widehat{G}, u^\alpha)_{\mathbb{V}_{per}} dW\right|^{\frac{p}{2}}. \tag{5.16}$$

Applying Burkhölder-Davis-Gundy's inequality (Corollary 2.15) and using (5.15) we deduce that

$$\mathbb{E}\left(\int_0^t ||u^\alpha||^2 ds\right)^{\frac{p}{2}} \leq C.$$

And this completes the proof of (5.8), hence the lemma. □

The application of Prokhorov's Theorem relies on the following key lemma. From now on we set $\mathbb{H}^\beta_{per}(D) = \mathbb{H}^\beta_{per}$ and $\mathbb{H}^\beta_{mp}(D) = \mathbb{H}^\beta_{mp}$ for any $\beta \in \mathbb{R}$; we recall that the subscript "mp" stands for "zero mean periodic".

Lemma 5.7. *For any $\delta \in (0, 1)$ we have*

$$\mathbb{E}\sup_{|\theta|\leq\delta}\int_0^{T-\delta} |u^\alpha(t+\theta) - u^\alpha(t)|^2_{\mathbb{H}^{-4}_{mp}} \leq C\delta. \tag{5.17}$$

Proof. In what follows we set

$$\frac{\partial}{\partial x_i} = \partial_i, \text{ for any } i,$$

and we rewrite the first equation in (5.1) as follows (see [61] for the details)

$$\begin{aligned}\frac{\partial}{\partial t}(u^\alpha - \alpha\Delta u^\alpha) - \nu\Delta u^\alpha + u^\alpha\cdot\nabla u^\alpha - \alpha\sum_{j,k}\partial_j\partial_k(u^\alpha_j\partial_k u^\alpha) + \alpha\sum_{j,k}\partial_j(\partial_k u^\alpha_j\partial_k u^\alpha)\\ = \alpha\sum_{j,k}\partial_k(\partial_k u^\alpha_j\nabla u^\alpha_j) - \nabla P^\sharp + F + G\frac{dW}{dt},\end{aligned} \tag{5.18}$$

where

$$\nabla P^{\sharp} = \frac{1}{2}\nabla(|u^\alpha|^2 + \alpha|\nabla u^\alpha|^2) + \nabla P.$$

The projection of (5.18) onto the space of divergence free fields eliminates the term $\nabla P^{\sharp}$, and we obtain

$$\begin{aligned} d\Phi + \{\nu A u^\alpha + \mathbf{P}(u^\alpha \cdot \nabla u^\alpha) - \alpha \sum_{j,k} \mathbf{P}(\partial_j \partial_k (u_j^\alpha \partial_k u^\alpha)) + \alpha \sum_{j,k} \mathbf{P}(\partial_j (\partial_k u_j^\alpha \partial_k u^\alpha))\} dt \\ = \alpha \sum_{j,k} \mathbf{P}(\partial_k (\partial_k u_j^\alpha \nabla u_j^\alpha)) dt + F dt + G dW, \end{aligned} \tag{5.19}$$

where $\Phi = \mathbf{P}(u^\alpha - \alpha \Delta u^\alpha)$. We derive from (5.19) that

$$\begin{aligned} \Phi(t+\theta) - \Phi(t) = \int_t^{t+\theta} \{-\nu A u^\alpha + \alpha \sum_{j,k} \big(\mathbf{P}(\partial_j \partial_k (u_j^\alpha \partial_k u^\alpha)) - \mathbf{P}(\partial_j (\partial_k u_j^\alpha \partial_k u^\alpha))\big)\} ds \\ - \int_t^{t+\theta} \mathbf{P}(u^\alpha \cdot \nabla u^\alpha) ds + \int_t^{t+\theta} \{\alpha \sum_{j,k} \mathbf{P}(\partial_k (\partial_k u_j^\alpha \nabla u_j^\alpha)) + F\} ds + \int_t^{t+\theta} G dW, \end{aligned} \tag{5.20}$$

for any $\theta > 0$. We infer from (5.20) that

$$\begin{aligned} |\Phi(t+\theta) - \Phi(t)|^2_{\mathbb{H}^{-4}_{mp}} \le 2 \left(\int_t^{t+\theta} \left\{ +|u^\alpha \cdot \nabla u^\alpha|_{\mathbb{H}^{-4}_{per}} + \alpha \sum_{j,k} |\partial_j \partial_k (u_j^\alpha \partial_k u^\alpha)|_{\mathbb{H}^{-4}_{per}} \right\} ds \right)^2 \\ +4 \left(\int_t^{t+\theta} \left\{ \alpha \sum_{j,k} [|\partial_j (\partial_k u_j^\alpha \partial_k u^\alpha)|_{\mathbb{H}^{-4}_{per}} + |\partial_k (\partial_k u_j^\alpha \nabla u_j^\alpha)|_{\mathbb{H}^{-4}_{per}}] + |F|_{\mathbb{H}^{-4}_{per}} \right\} dst \right)^2 \\ +2 \int_t^{t+\theta} \nu |A u^\alpha|_{\mathbb{H}^{-4}_{per}} ds + 2 \left| \int_t^{t+\theta} G dW \right|^2_{\mathbb{H}^{-4}_{per}}, \end{aligned}$$

which implies

$$\begin{aligned} |\Phi(t+\theta) - \Phi(t)|^2_{\mathbb{H}^{-4}_{per}} \le C\theta \int_t^{t+\theta} \left\{ +|u^\alpha \cdot \nabla u^\alpha|^2_{\mathbb{H}^{-4}_{mp}} + \alpha^2 \sum_{j,k} |\partial_j \partial_k (u_j^\alpha \partial_k u^\alpha)|^2_{\mathbb{H}^{-4}_{per}} \right\} dt \\ +C\theta \int_t^{t+\theta} \left\{ \alpha^2 \sum_{j,k} [|\partial_j (\partial_k u_j^\alpha \partial_k u^\alpha)|^2_{\mathbb{H}^{-4}_{per}} + |\partial_k (\partial_k u_j^\alpha \nabla u_j^\alpha)|^2_{\mathbb{H}^{-4}_{per}}] + |F|^2 \right\} dt \\ +C\theta \int_t^{t+\theta} \nu |A u^\alpha|^2_{\mathbb{H}^{-4}_{per}} ds + 2 \left| \int_t^{t+\theta} G dW \right|^2_{\mathbb{H}^{-4}_{per}}. \end{aligned}$$

It is not hard to see that

$$|A u^\alpha|^2_{\mathbb{H}^{-4}_{per}} \le C |u^\alpha|^2. \tag{5.21}$$

For $n = 2$ Theorem 2.2 implies that

$$|u^\alpha \cdot \nabla u^\alpha|^2_{\mathbb{H}^{-4}_{per}} \leq C|u^\alpha|^2 |\nabla u^\alpha|^2, \tag{5.22}$$

$$\alpha^2 |\partial_j \partial_k (u^\alpha_j \partial_k u^\alpha)|^2_{\mathbb{H}^{-4}_{per}} \leq C|u^\alpha_j \partial_k u^\alpha|^2_{\mathbb{H}^{-2}_{per}} \leq C|u^\alpha|^2 |\nabla u^\alpha|^2. \tag{5.23}$$

From Theorem 2.2 again we have that

$$|\partial_k u^\alpha_j \partial_k u^\alpha|_{\mathbb{H}^{-2}_{per}} \leq C|\nabla u^\alpha|^2 \quad \forall k, j,$$

from which we derive that

$$\alpha^2 |\partial_j (\partial_k u^\alpha_j \partial_k u^\alpha)|^2_{\mathbb{H}^{-4}_{per}} \leq \alpha C \alpha |\nabla u^\alpha|^2 |\nabla u^\alpha|^2. \tag{5.24}$$

A similar argument can be used to show that

$$\alpha^2 |\partial_k (\partial_k u^\alpha_j \nabla u^\alpha_j)|^2_{\mathbb{H}^{-4}_{per}} \leq \alpha C \alpha |\nabla u^\alpha|^2 |\nabla u^\alpha|^2. \tag{5.25}$$

The estimates (5.21)-(5.25) along with (5.8) allow us to write

$$\mathbb{E} \int_0^{T-\delta} \sup_{0 \leq \theta \leq \delta} |\Phi(t+\theta) - \Phi(t)|^2_{\mathbb{H}^{-4}_{mp}} dt \leq C\delta^2 + C\delta + C\delta \mathbb{E} \int_0^{T-\delta} \int_t^{t+\theta} \alpha |\nabla u^\alpha|^2 |\nabla u^\alpha|^2 ds dt$$
$$+ C\mathbb{E} \int_0^{T-\theta} \sup_{0 \leq \theta \leq \delta} \left| \int_t^{t+\theta} G dW \right|^2_{\mathbb{H}^{-4}_{per}} dt.$$

But (5.8) implies that

$$\mathbb{E} \sup_{0 \leq t \leq T} \alpha^{\frac{p}{2}} |\nabla u^\alpha(t)|^p + \nu \mathbb{E} \left(\int_0^T |\nabla u^\alpha(t)|^2 dt \right)^{\frac{p}{2}} \leq C, \ 2 \leq p < \infty.$$

From which we deduce that

$$\mathbb{E} \int_0^{T-\delta} \sup_{0 \leq \theta \leq \delta} |\Phi(t+\theta) - \Phi(t)|^2_{\mathbb{H}^{-4}_{mp}} dt \leq C\delta^2 + C\delta + C\delta + C\mathbb{E} \int_0^{T-\delta} \sup_{0 \leq \theta \leq \delta} \left| \int_t^{t+\theta} G dW \right|^2_{\mathbb{H}^{-4}_{per}} dt.$$

By making use of Burkhölder-Davis-Gundy's inequality (Corollary 2.15), the assumption on G we obtain that

$$\mathbb{E} \int_0^{T-\delta} \sup_{0 \leq \theta \leq \delta} |\Phi(t+\theta) - \Phi(t)|^2_{\mathbb{H}^{-4}_{mp}} dt \leq C\delta.$$

For almost all $(\omega, t) \in \Omega \times [0, T]$ we have

$$u^\alpha(t+\theta) - u^\alpha(t) = (I + \alpha A)^{-1} (\Phi(t+\theta) - \Phi(t)),$$

which implies that

$$|u^\alpha(t+\theta) - u^\alpha(t)|^2_{\mathbb{H}^\beta_{mp}} < |\Phi(t+\theta) - \Phi(t)|^2_{\mathbb{H}^\beta_{mp}}, \forall \beta \in \mathbb{R}. \tag{5.26}$$

Indeed for any $\phi \in \mathbb{H}^\beta_{per}(D)$ such that $\operatorname{div} \phi = 0$ and $\int_D \phi(x)dx = 0$ we have (see for example [35] [48] and [117])

$$|\phi|^2_{\mathbb{H}^\beta_{mp}} = \sum_{j=1}^{\infty} |\phi_j|^2 \lambda_j^{2\beta} < \sum_{j=1}^{\infty} (1+\alpha\lambda_j)|\phi_j|^2 \lambda_j^{2\beta},$$

that is,

$$|\phi|^2_{\mathbb{H}^\beta_{mp}} < |\phi + \alpha A \phi|^2_{\mathbb{H}^\beta_{mp}},$$

where $\phi = \sum_{j=1}^\infty \phi_j e_j$, and $Ae_j = \lambda_j e_j$, $j = 1, 2...$; the e_j-s are the eigenfunctions of the operator A and the λ_j-s are the corresponding eigenvalues. It then follows from (5.26) that

$$\mathbb{E} \int_0^{T-\delta} \sup_{0 \le \theta \le \delta} |u^\alpha(t+\theta) - u^\alpha(t)|^2_{\mathbb{H}^{-4}_{mp}} dt \le C\delta.$$

A similar argument can be carried out to prove the same estimate for the case $\theta < 0$. □

5.4 Proof of Theorem 5.3

Let us introduce the mapping

$$\Psi : \omega \mapsto (W(\omega), u^\alpha(\omega, .)) .$$

The family of probability measures $\mathfrak{Q}^\alpha$ is defined on $\mathfrak{S} = C(0, T; \mathbb{R}^m) \times L^2(0, T; \mathbb{H}_{per})$ by

$$\mathfrak{Q}^\alpha(S) = \mathbb{P}(\Psi^{-1}(S)),$$

for any $S \in \mathcal{B}(\mathfrak{S})$, where $\mathcal{B}(\mathfrak{S})$ is the Borel σ-algebra of $\mathfrak{S}$. We have the lemma

Lemma 5.8. *The family $\{\mathfrak{Q}^\alpha : 0 < \alpha < 1\}$ is tight.*

Proof. With the help of the Lemmas 2.8, 5.5 and 5.7 the proof follows the same lines as in the proof of Lemma 4.10, so we omit it. □

Prokhorov's relative compactness theorem enables us to extract from $\mathfrak{Q}^\alpha$ a subsequence $\mathfrak{Q}^{\alpha_j}$ such that $\mathfrak{Q}^{\alpha_j}$ weakly converges to a probability measure $\mathfrak{Q}$ on $\mathfrak{S}$.

And finally Skorokhod's Theorem ensures the existence of a complete probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and random variables $(W^{\alpha_j}, u^{\alpha_j})$ and $(\bar{\mathcal{W}}, v)$ defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with values in $\mathfrak{S}$ such that

$$\text{The probability law of } (W^{\alpha_j}, u^{\alpha_j}) \text{ is } \mathfrak{Q}^{\alpha_j},$$

$$\text{The probability law of } (\bar{\mathcal{W}}, v) \text{ is } \mathfrak{Q},$$

$$W^{\alpha_j} \to \bar{\mathcal{W}} \text{ in } C(0,T;\mathbb{R}^m)\ \bar{\mathbb{P}} - \text{a.s.}, \tag{5.27}$$

$$u^{\alpha_j} \to v \text{ in } L^2(0,T;\mathbb{H}_{per})\ \bar{\mathbb{P}} - \text{a.s.}. \tag{5.28}$$

Moreover, letting $\bar{\mathbb{F}}^t$ be the σ-algebra generated by $(\bar{\mathcal{W}}(s), v(s)), 0 \le s \le t$ and the null sets of $\bar{\mathcal{F}}$, we can show as in Subsection 3.3.3 of Chapter 3 that $\bar{\mathcal{W}}$ is a $\bar{\mathbb{F}}^t$-adapted standard $\mathbb{R}^m$-valued Wiener process. Furthermore, we can prove the following result (see the proof of Theorem 4.11).

Theorem 5.9. *For any $j \ge 1$, $\phi \in \mathcal{V}$, for all $t \in [0,T]$ the following holds almost surely*

$$(u^{\alpha_j}, \phi)_{\mathbb{V}_{per}} + \int_0^t \{(\nu A u^{\alpha_j} + B(u^{\alpha_j}, u^{\alpha_j}), \phi)\} dt = (u_0, \phi)_{\mathbb{V}_{per}} + \int_0^t (R(u^{\alpha_j}) + F(u^{\alpha_j}), \phi) dt + \int_0^t (G, \phi) dW^{\alpha_j}, \tag{5.29}$$

where

$$B(u^{\alpha_j}, u^{\alpha_j}) = \mathbf{P}(u^{\alpha_j} \cdot \nabla u^{\alpha_j}),$$

$$R(u^{\alpha_j}) = \alpha \sum_{i,k} \mathbf{P}\left(\partial_i \partial_k (u_i^{\alpha_j} \partial_k u^{\alpha_j}) + \partial_i (\partial_k u_i^{\alpha_j} \partial_k u^{\alpha_j}) - \partial_k (\partial_k u_i^{\alpha_j} \nabla u_i^{\alpha_j})\right).$$

We are now left with the proof of the last statement of Theorem 5.3. To achieve that we have to pass to the limit in equation (5.29). Since u^{α_j} satisfies (5.29) then u^{α_j} satisfies the estimates in Lemma 5.5. Consequently, we can extract from (u^{α_j}) a subsequence denoted by the same symbol such that

$$u^{\alpha_j} \rightharpoonup v \text{ weak-}* \text{ in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^\infty(0,T;\mathbb{H}_{per})),$$

$$u^{\alpha_j} \rightharpoonup v \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T;\mathbb{V}_{per})). \tag{5.30}$$

We derive from (5.23)-(5.25) that

$$R(u^{\alpha_j}) \to 0 \text{ in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T;\mathbb{H}^{-4}_{per})).$$

Thus

$$(R(u^{\alpha_j}), \phi) \to 0 \text{ in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T)), \forall \phi \in \mathcal{V}_{per}.$$

Since A is linear and strongly continuous then owing to (5.30) we have

$$Au^{\alpha_j} \rightharpoonup Av \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T; \mathbb{H}^{-1}_{per})).$$

Hence

$$< Au^{\alpha_j}, \phi > \rightharpoonup < Av, \phi > \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T)) \text{ for any } \phi \in \mathcal{V}_{per}. \tag{5.31}$$

We derive from (5.28), the estimate (5.8) of Lemma 5.5 and Vitali's Convergence Theorem 2.21 that

$$u^{\alpha_j} \to v \text{ strongly in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T; \mathbb{H}_{per})). \tag{5.32}$$

For any element $\zeta \in L^\infty(\bar{\Omega} \times [0,T], d\bar{\mathbb{P}} \otimes dt)$ and for any $\phi \in \mathcal{V}_{per}$ we have

$$\mathbb{E} \int_0^T < B(u^{\alpha_j}, u^{\alpha_j}), \zeta\phi > dt = -\sum_{i,k} \mathbb{E} \int_{D\times[0,T]} u_i^{\alpha_j} \partial_i \phi_k \zeta u_k^{\alpha_j} dx \otimes dt.$$

Owing to (5.32)

$$\zeta \partial_i \phi_k u_k^{\alpha_j} \to \zeta \partial_i \phi_k v_k \text{ strongly in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^2(0,T; \mathbb{H}_{per})).$$

This and (5.32) again imply that

$$\begin{aligned} -\sum_{i,k} \mathbb{E} \int_{D\times[0,T]} u_i^{\alpha_j} \partial_i \phi_k \zeta u_k^{\alpha_j} dx \otimes dt \to -\sum_{i,k} \mathbb{E} \int_{D\times[0,T]} v_i \partial_i \phi_k \zeta v_k dx \otimes dt \\ = \mathbb{E} \int_0^T < B(v,v), \zeta\phi > dt. \end{aligned}$$

That is

$$< u^{\alpha_j} \cdot \nabla u^{\alpha_j}, \phi > \rightharpoonup < v \cdot \nabla v, \phi > \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^\infty(0,T)) \text{ for any } \phi \in \mathcal{V}_{per}.$$

We readily have that

$$\int_0^t (G,\phi) dW^{\alpha_j} \rightharpoonup \int_0^t (G,\phi) d\bar{\mathcal{W}} \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^\infty(0,T)) \text{ for any } \phi \in \mathcal{V}_{per}.$$

We have that

$$\begin{aligned} (u^{\alpha_j}, \phi)_{\mathbb{V}_{per}} &= (u^{\alpha_j} - \alpha_j \Delta u^{\alpha_j}, \phi) \\ &= (u^{\alpha_j}, \phi) + \alpha_j((u^{\alpha_j}, \phi)). \end{aligned}$$

This implies that

$$(u^{\alpha_j} - \alpha_j \Delta u^{\alpha_j} - v, \phi) = (u^{\alpha_j} - v, \phi) + \alpha_j((u^{\alpha_j}, \phi)).$$

It follows from Lemma 5.5 and (5.32) that

$$(u^{\alpha_j} - \alpha_j \Delta u^{\alpha_j} - v, \phi) \to 0 \text{ strongly in } L^2(\bar{\Omega}, \bar{\mathbb{P}}; L^\infty(0,T)) \text{ for any } \phi \in \mathcal{V}_{per}.$$

Using all these convergences we can derive from (5.29) that the following holds almost surely

$$(v, \phi) + \nu \int_0^t \{((v, \phi)) + (\mathbf{P}(v \cdot \nabla v), \phi)\} ds = (u_0, \phi) + \int_0^t (F(v), \phi) ds + \int_0^t (G, \phi) d\bar{\mathcal{W}},$$

for any $\phi \in \mathcal{V}_{per}$ and $t \in [0,T]$. That is the system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}^t, \bar{\mathbb{P}}); (\bar{\mathcal{W}}, v)$ is a weak solution of the stochastic Navier-Stokes equations. This completes the proof of Theorem 5.3.

Conclusion

In this thesis we obtained original results on several mathematical problems arising in the dynamics of stochastic second grade fluids governed by the equations (1.5).

In the case of Lipschitz conditions on the forces we established the following new results:

- existence and uniqueness of the strong probabilistic solution.
- The stability of the strong probabilistic solution in the sense that as $t \to \infty$ it converges to a stationary solution of a deterministic time-independent second grade fluid.
- A long-time asymptotic behavior of the strong probabilistic solution characterized by an exponential decay.

In the case when the forces no longer satisfy the Lipschitz condition we established the existence of a weak probabilistic solution.

Lastly, we proved that as the stress modulus $\alpha \to 0$ a sequence of strong probabilistic solutions indexed by α converges in law to the strong probabilistic solution of the stochastic Navier-Stokes equations.

These results are pioneering in some sense since the present thesis is the first work dealing with stochastic second grade fluids. It is our opinion that our results contribute substantially to the mathematical study of turbulent flows governed by the stochastic second grade fluids.

Though we considered forces driven by finite dimensional Wiener processes the rsults can be extended to the case of cylindrical Wiener processes on a separable Hilbert space K. The Wiener process W can then be formally written as $W = \sum_{j=1}^{\infty} B_j k_j$ where

$\{k_j, j \in \mathbb{N}\}$ is an orthonormal basis of K and $\{B_j, j \in \mathbb{N}\}$ is an infinite sequence of independent standard real-valued Wiener processes (see for instance [36]). We may then assume that G takes its value on $L_2(K, \mathbb{V})$; $L_2(K, \mathbb{V})$ is the space of Hilbert-Schmidt operators defined on K and taking values in $\mathbb{V}$. By Considering the following finite dimensional approximation of (1.5):

$$\begin{aligned} &d(u^N, e_i)_{\mathbb{V}} + \nu((u^N, e_i))dt + b(u^N, u^N, e_i)dt - \alpha b(u^N, \Delta u^N, e_i)dt + \alpha b(e_i, \Delta u^N, u^N)dt \\ &= (F(u^N, t), e_i)dt + \sum_{j=1}^{N} (G(u^N, t)k_j, e_i)dB_j, i \in \{1, ..., N\}, \end{aligned} \tag{5.33}$$

we can prove that all the a priori estimates obtained in the thesis remain valid. We can also see that the same procedures for the passage to the limit still hold as well.

Issues arising from the thesis which might be the object for future research are for instance the existence and behavior of the flow generated by problem (1.5) and possible implications such as ergodicity, random attractors; we refer to [6, 18, 42, 54, 55, 56, 74, 75, 84, 85, 86, 87] for some examples of work treating such topics for some classes of stochastic evolution equations. Another issue is the extension of our model to the density dependent second grade fluids. As far as we know the last topic has not been addressed even in the deterministic case.