APPENDIX A

Derivation of Simpson's rule for unequally spaced data values (centre points).

A Lagrange polynomial *p(x)* is obtained such that $p(x_i)=f(x_i)$ for $i=1,2,3$ and integrated over the interval $x_1 < x < x_3$.

$$p(x) = f(x_1)\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \tag{B.1}$$

So

$$\int_{x_1}^{x_3} f(x)dx \approx \int_{x_1}^{x_3} f(x_1)\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}\,dx \tag{B.2}$$

$$\begin{aligned}
&= \frac{f(x_1)}{(x_1-x_2)(x_1-x_3)}\int_{x_1}^{x_3}(x-x_2)(x-x_3)dx \\
&\quad + \frac{f(x_2)}{(x_2-x_1)(x_2-x_3)}\int_{x_1}^{x_3}(x-x_1)(x-x_3)dx \\
&\quad + \frac{f(x_3)}{(x_3-x_1)(x_3-x_2)}\int_{x_1}^{x_3}(x-x_1)(x-x_2)dx \\
&= \frac{f(x_1)}{(x_1-x_2)(x_1-x_3)}\int_{x_1}^{x_3}\left[x^2-(x_2+x_3)x+x_2x_3\right]dx \\
&\quad + \frac{f(x_2)}{(x_2-x_1)(x_2-x_3)}\int_{x_1}^{x_3}\left[x^2-(x_1+x_3)x+x_1x_3\right]dx \\
&\quad + \frac{f(x_3)}{(x_3-x_1)(x_3-x_2)}\int_{x_1}^{x_3}\left[x^2-(x_1+x_2)x+x_1x_2\right]dx \\
&= \frac{f(x_1)}{(x_1-x_2)(x_1-x_3)}\left[\tfrac{1}{3}\left(x_3^3-x_1^3\right)-(x_2+x_3)\tfrac{1}{2}\left(x_3^2-x_1^2\right)+x_2x_3(x_3-x_1)\right] \\
&\quad + \frac{f(x_2)}{(x_2-x_1)(x_2-x_3)}\left[\tfrac{1}{3}\left(x_3^3-x_1^3\right)-(x_1+x_3)\tfrac{1}{2}\left(x_3^2-x_1^2\right)+x_1x_3(x_3-x_1)\right] \\
&\quad + \frac{f(x_3)}{(x_3-x_1)(x_3-x_2)}\left[\tfrac{1}{3}\left(x_3^3-x_1^3\right)-(x_1+x_2)\tfrac{1}{2}\left(x_3^2-x_1^2\right)+x_1x_2(x_3-x_1)\right]
\end{aligned}$$

$$= f(x_1)\left[\frac{\frac{1}{3}\left(x_3^3 - x_1^3\right) - (x_2 + x_3)\frac{1}{2}\left(x_3^2 - x_1^2\right) + x_2 x_3 (x_3 - x_1)}{(x_1 - x_2)(x_1 - x_3)}\right]$$

$$+ f(x_2)\left[\frac{\frac{1}{3}\left(x_3^3 - x_1^3\right) - (x_1 + x_3)\frac{1}{2}\left(x_3^2 - x_1^2\right) + x_1 x_3 (x_3 - x_1)}{(x_2 - x_1)(x_2 - x_3)}\right]$$

$$+ f(x_3)\left[\frac{\frac{1}{3}\left(x_3^3 - x_1^3\right) - (x_1 + x_2)\frac{1}{2}\left(x_3^2 - x_1^2\right) + x_1 x_2 (x_3 - x_1)}{(x_3 - x_1)(x_3 - x_2)}\right]$$

$$= f(x_1)W_1 + f(x_2)W_2 + f(x_3)W_3$$

Derivation of Simpson's rule for unequally spaced data values (end points).

A Lagrange polynomial $p(x)$ **is obtained such that** $p(x_i)=f(x_i)$ **for** $i=1,2,3$ **and integrated over the intervals** $x_1 < x < x_2$ and $x_2 < x < x_3$ **respectively**.

Now,

$$\int_{x_1}^{x_2} f(x)dx \approx \int_{x_1}^{x_2} f(x_1)\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + f(x_2)\frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + f(x_3)\frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}\,dx \quad \text{(B.3)}$$

$$= f(x_1)\left[\frac{\frac{1}{3}\left(x_2^3 - x_1^3\right) - (x_2 + x_3)\frac{1}{2}\left(x_2^2 - x_1^2\right) + x_2 x_3 (x_2 - x_1)}{(x_1 - x_2)(x_1 - x_3)}\right]$$

$$+ f(x_2)\left[\frac{\frac{1}{3}\left(x_2^3 - x_1^3\right) - (x_1 + x_3)\frac{1}{2}\left(x_2^2 - x_1^2\right) + x_1 x_3 (x_2 - x_1)}{(x_2 - x_1)(x_2 - x_3)}\right]$$

$$+ f(x_3)\left[\frac{\frac{1}{3}\left(x_2^3 - x_1^3\right) - (x_1 + x_2)\frac{1}{2}\left(x_2^2 - x_1^2\right) + x_1 x_2 (x_2 - x_1)}{(x_3 - x_1)(x_3 - x_2)}\right]$$

$$= f(x_1)W_1 + f(x_2)W_2 + f(x_3)W_3$$

and

$$\int_{x_2}^{x_3} f(x)dx \approx \int_{x_2}^{x_3} f(x_1)\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}\,dx \quad \text{(B.4)}$$

$$\begin{aligned}
&= f(x_1)\left[\frac{\frac{1}{3}\left(x_3^3-x_2^3\right)-(x_2+x_3)\frac{1}{2}\left(x_3^2-x_2^2\right)+x_2x_3\left(x_3-x_2\right)}{(x_1-x_2)(x_1-x_3)}\right] \\
&\quad + f(x_2)\left[\frac{\frac{1}{3}\left(x_3^3-x_2^3\right)-(x_1+x_3)\frac{1}{2}\left(x_3^2-x_2^2\right)+x_1x_3\left(x_3-x_2\right)}{(x_2-x_1)(x_2-x_3)}\right] \\
&\quad + f(x_3)\left[\frac{\frac{1}{3}\left(x_3^3-x_2^3\right)-(x_1+x_2)\frac{1}{2}\left(x_3^2-x_2^2\right)+x_1x_2\left(x_3-x_2\right)}{(x_3-x_1)(x_3-x_2)}\right] \\
&= f(x_1)W_1 + f(x_2)W_2 + f(x_3)W_3
\end{aligned}$$