The Szemerédi property in noncommutative dynamical systems

by

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Submitted in partial fulfillment of the requirements for the degree

Philosophiae Doctor

in the Department of Mathematics and Applied Mathematics

in the Faculty of Natural and Agricultural Sciences

University of Pretoria

Pretoria

May 2008

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YUNIBESITHI YA PRETORIA

Dankbetuigings

Hiermee my opregte waardering aan Dr.Rocco Duvenhage en Prof.Anton Ströh vir hul ondersteuning en leiding gedurende hierdie studie.

Verder, my innige dank aan my familie, spesifiek my vrou Hanni en my ouers vir hul deurgaanse ondersteuning.

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Chapter 1

Background

1.1 Introduction

First we give a brief overview of classical dynamics (partly drawn from [6] and [10]). A fundamental question in statistical mechanics concerns the existence of certain types of time averages. The problem may be formulated as follows: The state of a physical system at a certain time is described by specifying a point in a "phase space" X. When a mechanical system is subject to a principle of scientific determinism, e.g. when it is assumed to follow the classical Hamiltonian equations, it is known that an initial state x will, after t seconds have elapsed, have passed into a unique new state y. Since y is uniquely determined by x and t, a function $T : X \to X$ is defined by the equation $y = T_t(x)$. The flow T_t in this case has the property that

$$T_t(T_s(x)) = T_{t+s}(x)$$

for all points x in phase space and for all times s and t.

If we obtain a numerical quantity from the state of the physical system at some given time by an observation, such a quantity can be viewed as a value of a complex valued function f defined on X. If the initial state of the system is specified by the point x in X, the value of the quantity f at a time t will be $f(T_t(x))$. In practice, however, we are in most cases unable to observe a state directly, but rather an average value of $f(T_t(x))$ i.e.

$$\frac{1}{N}\int_0^N f(T_t(x))dt$$

computed over a time interval $0 \leq t \leq N$.

If observations regarding "micro"-processes are made we find that time intervals on "macro" level are very large compared to the natural rate of evolution of the given micro-system. An example is observations on gas in a vessel. In each second, the molecules travel vast distances and recoil from the wall millions of times. Thus, the time N involved in the experiment is large enough to give a good approximation for the limit

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N f(T_t(x))dt.$$

Thus it is central in ergodic theory to determine whether or not, or under what circumstances the limit above exists.

Historically, a mechanical system is said to be ergodic if it has the property that the above limit (the time mean) is the constant space mean taken with respect to the Lebesque measure ν in the phase space X, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N f(T_t(x))dt = \frac{\int_X f d\nu}{\nu(X)} = \int_X f d\nu.$$

Hence ergodicity implies that the averages obtained over sufficiently large time intervals can be used to obtain global information about a state in X.

In many cases the flow T_t is taken over discrete time instead of continuous time. Then $T_{n+m} = T_n T_m$ and $T_n = T_1^n$, and hence, for a given measure preserving transformation, the map $n \mapsto T^n x$ defines an action of the group of integers on X. In this instance we consider averages of the form

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T_1^n(x)), \quad f \in L_p(X, \mathcal{B}, \nu)$$

where T_1 is a mapping of X into itself. Hence, in this case the problem is to determine whether the time mean

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

exists and is equal to the space mean for almost all $x \in X$ with respect to ν. In other words, the problem is to determine whether the space mean of an observable quantity can be derived almost surely from discrete measurements along the time evolution of a single state $x \in X$.

In this thesis we consider more general group actions on X, specifically actions more general than $\mathbb{Z}$ and $\mathbb{R}$, as well as stronger assumptions regarding the dynamical systems, such as weak mixing (which implies ergodicity). We will also extend certain results from classical dynamics to more general, non-commutative dynamical systems.

The abstract framework for classical dynamics is given by the following. We say that (X, Σ, ν, T) is a measure preserving dynamical system, if (X, Σ, ν) is a complete probability space and $T : X \to X$ is a *measure preserving transformation (m.p.t)* in the sense that T satisfies the following conditions:

(a) T is bijective
(b) $TA, T^{-1}A \in \Sigma$ for all $A \in \Sigma$ and
(c) $\nu(T^{-1}A) = \nu(A)$ for all $A \in \Sigma$.

In some instances the assumption that T is bijective is not included. It is, however, convenient to assume that T is bijective, but it should be noted that many ergodic results also hold under weaker assumptions.

Besides its application to physics, ergodic theorems also have applications in other fields such as number theory. Consider a measure preserving dynamical system (X, Σ, ν, T). Furstenberg [17], [18] gave an ergodic-theoretic proof of Szemerédi's theorem in combinatoric number theory by proving a multiple recurrence result, i.e. that for any measure preserving dynamical system

$$\liminf_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} \nu(A \cap T^{-n}(A) \cap T^{-2n}(A) \cap ... \cap T^{-kn}(A)) > 0$$

if $\nu(A) > 0$. We will refer to this as a *Szemerédi property*; also see [20]. From a statistical viewpoint the two most important types of dynamical systems are the weakly mixing systems and the compact systems when one studies recurrence properties. In fact, it was shown in [20] that if one would like to prove the aforementioned multiple recurrence result for general dynamical systems, one could reduce the problem by only considering the two mentioned cases. It should however be noted that this has not yet be proven for 'non-classical' dynamics, i.e the non-commutative case.

A major part of this thesis will involve weak mixing, and also weak mixing of "higher orders". Weak mixing is an important notion in ergodic theory, introduced by Koopman and Von Neumann [24] in 1932 for actions of the group $\mathbb{R}$. Iterates of T above can be viewed as an action of the group $\mathbb{Z}$, and in this case the system above is called weakly mixing if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\nu(A \cap T^{-n}(B)) - \nu(A)\nu(B)\right| = 0$$

for all $A, B \in \Sigma$. It is straightforward to show that weak mixing implies ergodicity. The system (X, Σ, ν, T) is said to be weakly mixing of order k if for all sets $A_0, A_1, \ldots, A_k$ in Σ, we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \nu(A_0 \cap T^{-n} A_1 \cap \ldots \cap T^{-kn} A_k) - \nu(A_0)\nu(A_1) \ldots \nu(A_k) \right| = 0. \tag{1.1}$$

In a more general setting, Furstenberg proved that weakly mixing systems (X, Σ, ν, T) are weakly mixing of all orders, namely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \nu(A_0 \cap T^{-m_1 n}(A_1) \cap ... \cap T^{-m_k n}(A_k)) - \nu(A_0)\nu(A_1)...\nu(A_k) \right| = 0 \tag{1.2}$$

for all $A_0, ..., A_k \in \Sigma$, all $m_1, ..., m_k \in \mathbb{N}$ with $m_1 < m_2 < ... < m_k$, and all $k \in \mathbb{N} = \{1, 2, 3, ...\}$, from which the Szemerédi property then follows easily for weakly mixing systems.

On the other hand, the system is called compact if the orbit $\{f \circ T^n : n \in \mathbb{Z}\}$ of every $f \in L^2(\nu)$ is relatively compact in $L^2(\nu)$. Such systems can also be shown to have the Szemerédi property. As one might expect, these and related ideas have been studied for actions of more general groups; see for example [12], [2] (Section 4), [3] and [4].

Ergodic theory has historically been studied with regard to classical dynamical systems, where the assumption of commutativity of the underlying phase space is plausible. However, many analogues of classical (commutative) ergodic theoretical results exist in a non-commutative setting. This thesis will mainly be concerned with the study of such results.

The results of this thesis form part of a programme to extend the structure theorems developed by Furstenberg and others to the operator algebraic setting. We first give an overview of necessary concepts like amenability and Følner sequences (Section 1.2), as well as basic results and tools (like the GNS construction) needed for the study. In Chapter 2 we supply a catalogue of definitions regarding the dynamical systems we work with.

We then turn to weak mixing of all orders in a non-commutative C*-algebraic setting where (X, Σ, ν, T) is replaced by a C*-dynamical system (A, ω, τ) where ω is a state on the unital C*-algebra A, and τ a group of $*$-automorphisms of A keeping ω invariant. This problem has also been studied by Niculescu, Ströh, and Zsidó [28] for actions of $\mathbb{Z}$. However we allow more general groups, namely abelian second countable locally compact groups which contain a

Følner sequence satisfying certain conditions. The role of a Følner sequence is to replace the sequence of sets $\{1, ..., n\}$ appearing in the averages in the expressions above.

One of the technical tools we use in this case is a so-called Van der Corput lemma which we discuss in Chapter 3.2. This type of lemma and related inequalities, inspired by the classical Van der Corput difference theorem and Van der Corput inequality, have been used by Bergelson et al [1], [3], Furstenberg [19], Niculescu, Ströh, and Zsidó [28], and others, to study polynomial ergodic theorems, nonconventional ergodic averages, and noncommutative recurrence, for example. We extend the Van der Corput lemma to more general groups, namely second countable amenable locally compact groups. The main result of Section 3.2 is given by Theorem 3.2.5. After some preliminaries on weak mixing in Section 3.1, we devote Section 3.3 to showing how weak mixing implies weak mixing of all orders. The form of weak mixing of all orders we prove, involves replacing the multiplication with $m_1, ..., m_k$ in (1.2), by homomorphisms of the group over which we work, and this motivates why we incorporate such homomorphisms in a generalized definition of weak mixing in Section 3.1. The main result of Chapter 3 is Theorem 3.3.4.

We then proceed to the Szemerédi property for compact C*-dynamical systems (Section 4.1). Finally, in Section 4.2, we use the results of the previous Sections to study ergodic W*-dynamical systems (where A above is a finite Von Neumann algebra), however we only show that an asymptotic abelian ergodic system either has the Szemerédi property, or has a subsystem (called a factor) that has this property . This final result (Theorem 4.2.6) is proved for a smaller class of groups which however still contains $\mathbb{Z}^q$ and $\mathbb{R}^q$. Many of the intermediate results hold for more general groups or semigroups, as we will indicate. The asymptotic abelianness we refer to here is of a relatively weak form, namely "in the average" or "in density" as defined in Section 3.3. Asymptotic abelianness is needed to handle the weakly mixing case, while the compact case works without it, however in the latter we assume ω to be tracial while in the former we do not. So a certain level of commutativity is always present.

Note that this thesis is based on [5], explaining some of the concepts therein more thoroughly.

1.2 Amenable groups and Følner sequences

The dynamical systems we use are almost invariably defined over amenable groups for which a Følner sequence exists. We now briefly expand on these and some other central concepts.

In a group G we will use the notations $Vg := \{vg : v \in V\}$, $VW := \{vw : v \in V, w \in W\}$, $V^{-1} := \{v^{-1} : v \in V\}$, etc. for any $V, W \subset G$ and $g \in G$, and we will use multiplicative notation even when working in an abelian group.

Definition 1.2.1. Let G be a locally compact group and $L^\infty(G)$ be the Banach space of all essentially bounded functions $G \to \mathbb{C}$ with respect to the Haar measure. Let V be a subspace of $L^\infty(G)$.

(i) A linear functional M on V is called a *mean* on V if for all $f \in V$

$$f \geq 0 \Rightarrow M(f) \geq 0 \text{ and if } M(1) = 1, \tag{1.3}$$

where 1 is the constant function (see [16] for a definition of a mean which is implied by the definition above).

(ii) Let ${}_gf$ (respectively f_g) denote the *left* (respectively *right*) *action* of $g \in G$ on V, i.e. ${}_gf(x) = f(gx)$, (respectively $f_g(x) = f(xg)$. Then, a mean M is said to be *left-invariant on* V (respectively *right-invariant on* V) if $M({}_gf) = M(f)$ (respectively $M(f_g) = M(f)$) for all $g \in G$ and $f \in V$. If M is both left- and right-invariant it is simply called *invariant*.

(iii) A locally compact group G is *amenable* if there is a left- (or right-) invariant mean on $L^\infty(G)$.

Since we will work with abelian groups, the following theorem must first be established.

Theorem 1.2.2. *Every abelian locally compact group is amenable.*

This theorem follows from the following three results, which is adapted here from [22], Section 17.5.

Proposition 1.2.3. *Let G be a locally compact group and let $L_r^\infty(G)$ denote the $\mathbb{R}$-subspace of $L^\infty(G)$. If M is a left-invariant mean on $L_r^\infty(G)$ then M' defined by*

$$M'(f) := M(\mathrm{re}(f)) + iM(\mathrm{im}(f))$$

is a left-invariant mean on $L^\infty(G)$.

Proof. Let $f \in L^\infty(G)$. Then $\mathrm{re}(f), \mathrm{im}(f) \in L_r^\infty(G)$, M' is clearly linear and $M'(1) = 1$. Also, if $f \geq 0$ then $f \in L_r^\infty(G)$ and hence

$$M'(f) = M(f) \geq 0.$$

Left-invariance of M follows from the fact that

$$\mathrm{re}({}_gf(x)) = \mathrm{re}(f(gx)) = \mathrm{re}(f)(gx) = ({}_g\mathrm{re}(f))(x)$$

for all $g, x \in G$ and similarly for $\mathrm{im}(f)$. $\square$

Lemma 1.2.4. *Let G be a locally compact group. Let F consist of all functions $h \in L_r^\infty(G)$ of the form*

$$h = \sum_{k=1}^{n} (f_k - ({}_{a_k}f_k))$$

where $f_1, \ldots, f_n \in L_r^\infty(G)$ and $a_1, \ldots, a_n \in G$. Then there exists a left invariant mean for $L_r^\infty(G)$ if and only if

$$\operatorname*{ess\,sup}_{g \in G}\{h(g)\} \geq 0 \tag{1.4}$$

for all $h \in F$.

Proof. If M is a left-invariant mean for $L_r^\infty(G)$ then $M(h) = 0$ a.e. for all $h \in F$. Note that since M is a real linear functional, (1.3) implies that

$$\operatorname*{ess\,inf}_{g \in G}\{f(g)\} \leq M(f) \leq \operatorname*{ess\,sup}_{g \in G}\{f(g)\} \tag{1.5}$$

a.e. for all $f \in L_r^\infty(G)$. To see this, let $\operatorname{ess\,sup}_{g \in G}\{f(g)\} = k$, where k is a constant real number. By linearity $M(k) = k$, and since $k - f \geq 0$ a.e. we have

$$0 \leq M(k - f) = k - M(f),$$

so $M(f) \leq k$ a.e. Similarly $M(f) \geq \operatorname{ess\,inf}_{g \in G}\{f(g)\}$ a.e. for all $f \in L_r^\infty(G)$. Hence, since $h \in L_r^\infty(G)$,

$$\operatorname*{ess\,sup}_{g \in G}\{h(g)\} \geq M(h) = 0.$$

Conversely suppose that (1.4) holds for all $h \in F$. F is clearly a linear subspace of $L_r^\infty(G)$. Let M_0 be the null linear functional on F, i.e. $M_0(h) = 0$ for all $h \in F$. By (1.4) we then have

$$M_0(h) = 0 \leq \operatorname*{ess\,sup}_{g \in G}\{h(g)\}$$

for all $h \in F$. If we let $p(f) := \operatorname{ess\,sup}_{g \in G}\{f(g)\}$ for $f \in L_r^\infty(G)$ it follows by the Hahn-Banach Theorem that M_0 can be extended to a linear functional M on $L_r^\infty(G)$ satisfying

$$M(f) \leq \operatorname*{ess\,sup}_{g \in G}\{f(g)\}$$

for all $f \in L_r^\infty(G)$. We also have

$$-M(f) = M(-f) \leq \operatorname*{ess\,sup}_{g \in G}\{-f(g)\} = -\operatorname*{ess\,inf}_{g \in G}\{f(g)\},$$

so that $\operatorname{ess\,inf}_{g\in G}\{f(g)\} \leq M(f)$ i.e. (1.5) holds. Hence clearly $M(1) = 1$. Also, if $f \geq 0$,

$$M(f) \geq \operatorname*{ess\,inf}_{g\in G}\{f(g)\} \geq 0.$$

Finally, since $f - ({}_g f) \in F$, we have $M(f) = M({}_g f)$ for all $g \in G$. □

Theorem 1.2.5. *Let G be an abelian locally compact group. Then there is an invariant mean M on $L_r^\infty(G)$.*

Proof. Let $f_1, \ldots, f_n \in L_r^\infty(G)$, let $a_1, \ldots, a_n \in G$ and let

$$h = \sum_{k=1}^{n} \left(f_k - ({}_{a_k} f_k)\right).$$

By Lemma 1.2.4 it is sufficient to show that $\operatorname{ess\,sup}_{g\in G}\{h(g)\} \geq 0$. Suppose then that for some $\varepsilon > 0$, we have

$$\operatorname*{ess\,sup}_{g\in G}\{h(g)\} < -\varepsilon. \tag{1.6}$$

Let p be any positive integer and let B consist of all functions λ with domain $\{1, \ldots, n\}$ and range in $\{1, \ldots, p\}$. Then B contains exactly p^n elements. Let ξ be the mapping of B into G defined by

$$\xi(\lambda) := a_1^{\lambda(1)} a_2^{\lambda(2)} \ldots a_n^{\lambda(n)}.$$

For a fixed $k \in \{1, \ldots, n\}$ we consider the sum

$$\sum_{\lambda\in B} \left[f_k(\xi(\lambda)) - f_k(a_k\xi(\lambda))\right]. \tag{1.7}$$

Since G is abelian it follows that all of the terms in (1.7) cancel each other except possibly those $f_k(\xi(\lambda))$ such that $\lambda(k) = 1$ and those $f_k(a_k\xi(\lambda))$ such that $\lambda(k) = p$. The number of these terms is $2p^{n-1}$. Hence

$$\sum_{\lambda\in B} \left[f_k(\xi(\lambda)) - f_k(a_k\xi(\lambda))\right] \geq -2p^{n-1}\|f_k\|_\infty.$$

Applying (1.6) we obtain

$$\begin{aligned}
-\varepsilon p^n &\geq \sum_{\lambda\in B} h(\xi(\lambda)) \\
&= \sum_{\lambda\in B}\sum_{k=1}^{n} [f_k(\xi(\lambda)) - f_k(a_k\xi(\lambda))] \\
&= \sum_{k=1}^{n}\sum_{\lambda\in B} [f_k(\xi(\lambda)) - f_k(a_k\xi(\lambda))] \\
&\geq -\sum_{k=1}^{n} 2p^{n-1}C = -2np^{n-1}C,
\end{aligned}$$

where $C = \max\{\|f_k\|_\infty : k = 1, \ldots, n\}$. Consequently we have $\varepsilon p \leq 2nC$. Since p can be chosen arbitrarily, this leads to a contradiction. Hence (1.6) cannot hold and there is an invariant mean on $L_r^\infty(G)$. □

We may remark that Theorem 1.2.2 also holds if we use a semigroup instead of a group.

In the remainder of this Chapter and up to Chapter 3, G denotes an abelian second countable locally compact group with identity e, and regular Haar measure μ. In this Chapter and the next the commutativity of G is in fact not crucial; the proofs are valid even if G is not abelian but still amenable, and μ is right invariant (but see the remarks just before Theorem 3.2.5). Unfortunately in Chapter 3 this is not the case.

Since G is second countable and locally compact, G can clearly be covered by countably many compact sets, i.e. it is σ-compact and hence its amenability (even for a nonabelian group) is equivalent to the existence of a Følner sequence (Λ_n) in G defined as follows:

Definition 1.2.6. A *Følner sequence* in G is a sequence (Λ_n) of compact subsets of G such that $0 < \mu(\Lambda_n)$ for all n, and

$$\lim_{n\to\infty} \frac{\mu\left(\Lambda_n \Delta(\Lambda_n g)\right)}{\mu(\Lambda_n)} = 0 \tag{1.8}$$

for all $g \in G$.

Refer to Theorem 4 in [13] and Theorems 1 and 2 in [14] for a very clear exposition of this. In fact, these papers show that we can choose a Følner sequence with stronger properties than those in Definition 1.2.6, but our definition will suffice for our purposes. Furthermore, Theorem 3 in [13] shows that Definition 1.2.6 implies uniform convergence of (1.8) on compact sets, i.e.

$$\lim_{n\to\infty} \sup_{g\in K} \frac{\mu\left(\Lambda_n \Delta(\Lambda_n g)\right)}{\mu(\Lambda_n)} = 0$$

for any non-empty compact $K \subset G$. We will have occasion to use this important fact later on. Throughout Chapters 1 to 3, (Λ_n) will denote a Følner sequence in G. At the end of Chapter 3, we briefly consider simple examples of such sequences in $\mathbb{Z}^q$ and $\mathbb{R}^q$.

Definition 1.2.7. A Følner sequence (Λ_n) in G is said to satisfy the *Tempelman condition* if there is a real number $c > 0$ such that

$$\mu(\Lambda_n^{-1}\Lambda_n) \leq c\mu(\Lambda_n)$$

for n large enough.

See [26] for some discussion and further references related to this condition.

Remark 1.2.8. It is interesting to note that if we have a Følner sequence (Λ_n) in G and a sequence (Λ'_n) satisfying the Tempelman condition and with $\Lambda_n \subset \Lambda'_n$ for all n, then (Λ'_n) is not necessarily a Følner sequence in G, as one might intuitively expect. A counterexample is given by the sequences $\Lambda_n = \{0, \pm 1, \ldots \pm n\}$ and $\Lambda'_n = \{0, \pm 1, \pm 2, \ldots \pm n\} \cup (2\mathbb{Z} \cap \{0, \pm 1, \pm 2, \ldots \pm 2n\})$ (see also [15]). To see this, note that

$$(\Lambda'_n)^{-1}\Lambda'_n = \{0, \pm 1, \pm 2, \ldots \pm 3n\} \cup (2\mathbb{Z} \cap \{0, \pm 1, \pm 2, \ldots \pm 4n\}).$$

Since

$$\mu(\Lambda'_n) = \begin{cases} 3n+1, & \text{n even} \\ 3n+2, & \text{n odd} \end{cases}$$

and $\mu\left((\Lambda'_n)^{-1}\Lambda'_n\right) \leq 8n$, we have that

$$\mu\left((\Lambda'_n)^{-1}\Lambda'_n\right) < 3\mu(\Lambda'_n)$$

satisfying the Tempelman condition. It can also readily be checked that $\mu\left(\Lambda'_n \Delta(\Lambda'_n + k)\right) \geq n$ for any positive odd integer k where $n \geq k$. Hence

$$\lim_{n\to\infty} \frac{\mu\left(\Lambda'_n \Delta(\Lambda'_n + k)\right)}{\mu(\Lambda'_n)} \geq \frac{1}{3}$$

proving that (Λ'_n) is not Følner.

Definition 1.2.9. Let K be a semigroup. We call a set $E \subset K$ *relatively dense* in K if there exist an $r \in \mathbb{N}$ and $g_1, \dots, g_r \in K$ such that

$$E \cap \{gg_1, \dots, gg_r\} \neq \varnothing$$

for all $g \in K$.

Strictly speaking one could call this *left* relative denseness, with the right hand case being defined similarly in terms of $g_j g$, but we will only work with Definition 1.2.9. The usual definition of relative denseness of a subset E in $\mathbb{N}$ is in terms of "bounded gaps" (see [31] for example), and it is easy to check that in this special case the two definitions are equivalent.

Lemma 1.2.10. *Let K be a semigroup. Take any $g_n \in K$ for each n. Then the sequence*

$$(\Lambda_n g_n)$$

is also a Følner sequence in K.

Proof. Since K has the right cancelation property, we have $(Ag)\Delta(Bg) = (A\Delta B)g$ for all $A, B \subset K$ and $g \in K$. Hence

$$\begin{aligned}\frac{\mu\left((\Lambda_n g_n)\Delta(g(\Lambda_n g_\alpha))\right)}{\mu(\Lambda_n g_n)} &= \frac{\mu\left((\Lambda_n \Delta(g\Lambda_n))g_n\right)}{\mu(\Lambda_n g_n)} \\ &= \frac{\mu\left(\Lambda_n \Delta(g\Lambda_n)\right)}{\mu(\Lambda_n)} \\ &\longrightarrow 0\end{aligned}$$

as $n \to \infty$. □

Definition 1.2.11. Let K be a semigroup. Let (Λ_n) be any Følner sequence in K. Consider any $V \in \Sigma$ and set

$$D_{(\Lambda_n)}(V) := \lim_{n\to\infty}\left[\inf_{m\geq n}\left(\frac{\mu(\Lambda_m \cap V)}{\mu(\Lambda_m)}\right)\right] \equiv \liminf_{n\to\infty}\frac{\mu(\Lambda_n \cap V)}{\mu(\Lambda_n)}.$$

If $D_{(\Lambda_n)}(V) > 0$, then we say that V has *positive lower density* relative to (Λ_n).

To see that $D_{(\Lambda_n)}(V)$ in this definition always exists, note that if

$$a_n := \inf_{m\geq n}\left(\frac{\mu(\Lambda_m \cap V)}{\mu(\Lambda_m)}\right),$$

then (a_n) is an increasing sequence with $a_n \leq 1$.

Lemma 1.2.12. *Let K be a semigroup. Let $E \in \Sigma$ be relatively dense in K. Then:*

(1) There exists an $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in K$ such that the following holds: for each $B \in \Sigma$ with $\mu(B) < \infty$ there exists a $j \in \{1, \ldots, r\}$ such that $\mu((Bg_j) \cap E) \geq \mu(B)/r$.

(2) E has positive lower density relative to some Følner net in K.

(3) Let $f : K \to \mathbb{R}$ a Σ-measurable function with $f \geq 0$. Assume that $f(g) \geq \alpha$ for some $\alpha > 0$ and all $g \in E \in \Sigma$. Then there exists a Følner sequence (Λ_n) in K such that

$$\liminf_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f d\mu > 0.$$

Proof. (1) Let $g_1, ..., g_r$ be given by Definition 1.2.9. Set

$$B_j := \{b \in B : bg_j \in E\}$$

for $j = 1, \ldots, r$, so $B_j g_j = (Bg_j) \cap E \in \Sigma$ and hence $B_j \in \Sigma$. Now, for any $b \in B$ we know from Definition 1.2.9 that $E \cap \{bg_1, \ldots, bg_r\} \neq \varnothing$. So $bg_j \in E$ for some $j \in \{1, \ldots, r\}$, i.e. $b \in B_j$. Hence $B = \bigcup_{j=1}^r B_j$ and therefore

$$\mu(B) = \mu(\bigcup_{j=1}^r B_j) \leq \sum_{j=1}^r \mu(B_j) = \sum_{j=1}^r \mu(B_j g_j) = \sum_{j=1}^r \mu((Bg_j) \cap E)$$

from which the conclusion follows.

(2) Consider any Følner sequence (Λ_n) in K. Let $g_1, \ldots, g_r \in K$ be as in Definition 1.2.9. For each n it follows from (1) that there exists a $j(n) \in \{1, \ldots, r\}$ such that

$$\frac{\mu((\Lambda_n g_{j(n)}) \cap E)}{\mu(\Lambda_n g_{j(n)})} \geq \frac{1}{r}$$

where we also made use of $\mu(\Lambda_n g_{j(n)}) = \mu(\Lambda_n)$. But it follows from Lemma 1.2.10 that (Λ'_n) given by $\Lambda'_n := \Lambda_n g_{j(n)}$ is a Følner sequence in K. Furthermore,

$$D_{(\Lambda'_n)}(E) = \liminf_{n\to\infty} \frac{\mu(\Lambda'_n \cap E)}{\mu(\Lambda'_n)} \geq \lim_{n\to\infty} \frac{1}{r} = \frac{1}{r}.$$

(3) By (2) there exists a Følner sequence (Λ_n) in K such that

$$\liminf_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f d\mu \geq \liminf_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap E} \alpha \, dg = \alpha D_{(\Lambda_n)}(E) > 0.$$

$\square$

1.3 Density limits

In this Section we define and develop some basic tools needed in later Sections.

Definition 1.3.1. (i) A set $R \subset G$ is said to have *density zero relative to* (Λ_n), and we write $D_{(\Lambda_n)}(R) = 0$, if and only if there exists a measurable set $S \subset G$, with $R \subset S$ such that

$$\lim_{n\to\infty} \frac{\mu(\Lambda_n \cap S)}{\mu(\Lambda_n)} = 0.$$

(ii) We say that $f : G \to L$, with L a real or complex normed space, has *density limit* $a \in L$ relative to (Λ_n), if and only if for each $\varepsilon > 0$, $D_{(\Lambda_n)}(S_\varepsilon) = 0$, where

$$S_\varepsilon := \{h \in G : \|f(h) - a\| \geq \varepsilon\},$$

and we write it as

$$D_{(\Lambda_n)}\text{-}\lim f = D_{(\Lambda_n)}\text{-}\lim_h f(h) = a.$$

Proposition 1.3.2. *If R and S have density zero relative to (Λ_n) and $V \subset S$, then $V, R \cap S$ and $R \cup S$ also have density zero relative to (Λ_n).*

Proof. V has zero density per definition. Also, if R and S have density zero relative to (Λ_n), then there are measurable sets $S_1 \supset R$ and $S_2 \supset S$ such that

$$\lim_{n\to\infty} \frac{\mu(\Lambda_n \cap S_1)}{\mu(\Lambda_n)} = 0 \text{ and } \lim_{n\to\infty} \frac{\mu(\Lambda_n \cap S_2)}{\mu(\Lambda_n)} = 0.$$

Since $R \cap S \subset S_1 \cap S_2 \subset S_1$, and from the fact that S_1 trivially has density zero, we have that $R \cap S$ also has density zero.

Finally since also $S_1 \cap S_2$ has density zero, we have that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mu\left(\Lambda_n \cap (S_1 \cup S_2)\right)}{\mu(\Lambda_n)} &= \lim_{n\to\infty} \frac{\mu(\Lambda_n \cap S_1)}{\mu(\Lambda_n)} + \lim_{n\to\infty} \frac{\mu(\Lambda_n \cap S_2)}{\mu(\Lambda_n)} \\
&\quad - \lim_{n\to\infty} \frac{\mu\left(\Lambda_n \cap (S_1 \cap S_2)\right)}{\mu(\Lambda_n)} \\
&= 0.
\end{aligned}$$

□

Proposition 1.3.3. *Let $f, g : G \to L$ with L as in Definition 1.3.1, and assume that*

$$D_{(\Lambda_n)}\text{-}\lim f = a \quad \textit{and} \quad D_{(\Lambda_n)}\text{-}\lim g = b.$$

Then

$$D_{(\Lambda_n)}\text{-}\lim(f+g) = a+b$$

and

$$D_{(\Lambda_n)}\text{-}\lim(\beta f) = \beta a$$

for any $\beta \in \mathbb{C}$. Furthermore, if f, g are real-valued functions and $f(h) \leq g(h)$ for all $h \in G$, then $a \leq b$.

Proof. For each $\varepsilon > 0$, let

$$R_\varepsilon := \{h \in G : \|f(h) - a\| \geq \varepsilon\} \quad \text{and} \quad S_\varepsilon := \{h \in G : \|g(h) - b\| \geq \varepsilon\}.$$

By definition, R_ε and S_ε have density zero relative to (Λ_n). Let

$$V_\varepsilon := \{h \in G : \|(f+g)(h) - (a+b)\| \geq \varepsilon\}$$

and

$$V'_\varepsilon := \{h \in G : \|f(h) - a\| + \|g(h) - b\| \geq \varepsilon\}.$$

Since $\|(f+g)(h) - (a+b)\| \leq \|f(h) - a\| + \|g(h) - b\|$, it is clear that $V_\varepsilon \subset V'_\varepsilon$. Also, clearly $V'_\varepsilon \subset R_{\frac{\varepsilon}{2}} \cup S_{\frac{\varepsilon}{2}}$. But $R_{\frac{\varepsilon}{2}} \cup S_{\frac{\varepsilon}{2}}$ has density zero relative to (Λ_n), and hence the same holds for V'_ε and then V_ε. Hence

$$D_{(\Lambda_n)}\text{-}\lim(f+g) = a+b.$$

Letting $W_\varepsilon := \{h \in G : \|(\beta f)(h) - \beta a\| \geq \varepsilon\}$, it is easily seen that W_ε has density zero relative to (Λ_n), hence

$$D_{(\Lambda_n)}\text{-}\lim(\beta f) = \beta a.$$

Finally, suppose that f, g are real-valued functions, i.e. $L = \mathbb{R}$, and $f(h) \leq g(h)$ for all $h \in G$. From the previous two results in this proposition, we have that

$$D_{(\Lambda_n)}\text{-}\lim(g - f) = b - a.$$

Hence for any $\varepsilon > 0$, the set

$$W'_\varepsilon := \{h \in G : |(g-f)(h) - (b-a)| \geq \varepsilon\}$$

has density zero relative to (Λ_n). Suppose now that $b - a =: \rho < 0$. Since $(g-f)(h) \geq 0$ for all $h \in G$, we must have that the set $W'_{|\rho|/2}$ consists of all of G. Hence

$$\frac{\mu(\Lambda_n \cap W'_{|\rho|/2})}{\mu(\Lambda_n)} = \frac{\mu(\Lambda_n)}{\mu(\Lambda_n)} = 1,$$

contradicting the stated fact that $W'_{|\rho|/2}$ has density zero relative to (Λ_n). Therefore $b - a \geq 0$. $\square$

We now give a Koopman-Von Neumann type lemma:

Lemma 1.3.4. *Let $f : G \to [0, \infty)$ be bounded and measurable. Then the following are equivalent:*
(1) $D_{(\Lambda_n)}$-$\lim f = 0$
(2) $\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f \, d\mu = 0.$

Proof. For every $\varepsilon > 0$, let $S_\varepsilon := \{h \in G : f(h) \geq \varepsilon\}$, which is a measurable set, since f is measurable.

(1) $\Rightarrow$ (2): From (1) we have that each S_ε has density zero relative to (Λ_n). Given any $\varepsilon > 0$ and index α, consider the term

$$\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f d\mu = \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap S_\varepsilon} f \, d\mu + \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap S_\varepsilon^c} f d\mu.$$

Since S_ε has density zero relative to (Λ_n)

$$0 \leq \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap S_\varepsilon} f \, dh\mu \leq \frac{\mu\left(\Lambda_n \cap S_\varepsilon\right)}{\mu(\Lambda_n)} \sup f(G) \to 0$$

as $n \to \infty$. Also,

$$0 \leq \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap S_\varepsilon^c} f \, d\mu \leq \frac{\mu\left(\Lambda_n \cap S_\varepsilon^c\right)}{\mu(\Lambda_n)} \varepsilon \leq \varepsilon$$

hence

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f d\mu = 0.$$

(2) $\Rightarrow$ (1): Clearly $\varepsilon \chi_{S_\varepsilon} \leq f$. Also note that $D_{(\Lambda_n)}\left(S_\varepsilon\right) = 0$, since S_ε is measurable and

$$\varepsilon \frac{\mu\left(\Lambda_n \cap S_\varepsilon\right)}{\mu\left(\Lambda_n\right)} \leq \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f d\mu$$

which tends to zero as $n \to \infty$. □

Corollary 1.3.5. *Let $f : G \to \mathbb{R}$ be bounded and measurable. Then*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} [f(h)]^2 \, dh = 0$$

if and only if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |f(h)| \, dh = 0.$$

Proof. Given any $\varepsilon > 0$. Let

$$S_\varepsilon := \{h \in G : [f(h)]^2 \geq \varepsilon^2\} = \{h \in G : |f(h)| \geq \varepsilon\}.$$

Suppose that $\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} [f(h)]^2 \, dh = 0$, i.e.

$$D_{(\Lambda_n)}\text{-}\lim_h [f(h)]^2 = 0$$

by Lemma 1.3.4. By the definition of the density limit we have $D_{(\Lambda_n)}(S_\varepsilon) = 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $D_{(\Lambda_n)}\text{-}\lim |f| = 0$, and hence

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |f(h)| \, dh = 0$$

by Lemma 1.3.4. The converse follows similarly. □

As a result, the $|\cdot|$ in Definition 2.1.7(i) of M-weak mixing, can be replaced by $|\cdot|^2$, which is useful below and in Chapter 3.3.

Lemma 1.3.6. *Let $f : G \to \mathbb{C}$ bounded and measurable. Let $\beta \in \mathbb{C}$. If*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(h) dh = \beta \text{ and } \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |f(h)|^2 dh = |\beta|^2,$$

then

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |f(h) - \beta|^2 \, dh = 0.$$

Proof. This follows immediately if we note that

$$\begin{aligned}
&\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |f(h) - \beta|^2 \, dh \\
&= \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} (f(h) - \beta)(\overline{f(h) - \beta}) dh \\
&= \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left(|f(h)|^2 - \beta\overline{f(h)} - \overline{\beta} f(h) + |\beta|^2\right) dh \\
&\to 0
\end{aligned}$$

in the n limit. □

1.4 The GNS construction

The Gelfand-Naimark-Segal (GNS) construction provides us with a powerful tool for the study of ergodic theory in non-commutative dynamical systems, as it enables us to approach some problems through Hilbert space theory. In the discussion below $L(X)$ refers to the algebra of all linear operators $X \to X$ while $\mathfrak{L}(X)$ refers to all bounded linear operators.

Definition 1.4.1. Let A be a unital $*$-algebra with ω a state on A (i.e. a linear functional on A such that $\omega(a^*a) \geq 0$ and $\omega(1) = 1$). Let Q be an inner product space and $\pi : A \to L(Q)$ a homomorphism. A vector Ω in Q is said to be *cyclic for* $\pi : A \to L(Q)$ if $\pi(A)\Omega$ is dense in Q. If $\Omega \in Q$ is cyclic for π and $\langle \pi(a)\Omega, \pi(b)\Omega \rangle = \omega(a^*b)$ for all $a, b \in A$, then the triple (Q, π, Ω) is called a *cyclic representation* of (A, ω).

Proposition 1.4.2. *Let A be a unital $*$-algebra with ω a positive linear functional on A. Define*

$$\|a\|_\omega := \sqrt{\omega(a^*a)}$$

for all $a \in A$. Then $\|\cdot\|_\omega$ defines a seminorm on A. For a cyclic representation (Q, π, Ω) of (A, Ω) and $\iota : A \to Q : a \mapsto \pi(a)\Omega$ we have

$$\|a\|_\omega = \|\iota(a)\|$$

for all $a \in A$.

Proof. Let $a, b \in A$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}
\|a+b\|_\omega^2 &= \omega\left((a+b)^*(a+b)\right) \\
&= \|a\|_\omega^2 + \omega(a^*b) + \omega(b^*a) + \|b\|_\omega^2 \\
&\leq \|a\|_\omega^2 + |\omega(a^*b)| + |\omega(b^*a)| + \|b\|_\omega^2 \\
&\leq \|a\|_\omega^2 + \|a\|_\omega\|b\|_\omega + \|b\|_\omega\|a\|_\omega + \|b\|_\omega^2 \\
&= \left(\|a\|_\omega + \|b\|_\omega\right)^2
\end{aligned}$$

by the Cauchy-Schwartz inequality ([7], Lemma 2.3.10). Also,

$$\|\alpha a\|_\omega = \sqrt{\omega((\alpha a)^*(\alpha a))} = \sqrt{|\alpha|^2\omega(a^*a)} = |\alpha|\|a\|_\omega.$$

Finally,

$$\|a\|_\omega = \sqrt{\omega(a^*a)} = \sqrt{\langle \pi(a)\Omega, \pi(a)\Omega \rangle} = \sqrt{\langle \iota(a), \iota(a) \rangle} = \sqrt{\|\iota(a)\|^2} = \|\iota(a)\|.$$

$\square$

Theorem 1.4.3. ***The GNS-construction.*** *Let $\omega : A \to \mathbb{C}$ be a state on a unital $*$-algebra A. Then there exists a cyclic representation (Q, π, Ω) of (A, ω) with $\omega(a) = \langle \Omega, \pi(a)\Omega \rangle$ for all $a \in A$.*

Proof. We follow the structure of the proof in [11]. We first show that there exists a representation (Q, π) of (A, ω). We construct the inner product space Q. Consider the set

$$\mathfrak{T} = \{a \in A : \|a\|_\omega = 0\}.$$

$\mathfrak{T}$ is clearly a linear subspace of A due to the fact that $\|\cdot\|_\omega$ is a seminorm. Indeed, if $a, b \in \mathfrak{T}$ and $\alpha \in \mathbb{C}$ then $\alpha a + b \in \mathfrak{T}$ since

$$0 \leq \|\alpha a + b\|_\omega \leq |\alpha| \|a\|_\omega + \|b\|_\omega = 0.$$

Then $Q := A/\mathfrak{T}$ is also a vector space. Define $\iota : A \to Q$ by

$$\iota(a) = a + \mathfrak{T}$$

for all $a \in A$. Note that ι is surjective by definition and linear since $\iota(\alpha a+b) = (\alpha a + b) + \mathfrak{T} = (\alpha a + \mathfrak{T}) + (b + \mathfrak{T}) = \alpha(a + \mathfrak{T}) + (b + \mathfrak{T}) = \alpha\iota(a) + \iota(b)$.

Using ι, we define an inner product on Q by

$$\langle \iota(a), \iota(b) \rangle := \omega(a^* b)$$

for all $a, b \in A$. Note that here we follow the convention that $\langle \cdot, \cdot \rangle$ is conjugate linear in the first slot. We must show that this inner product is well-defined and that it is indeed an inner product.

Let $a, b, c, d \in A$ such that $\iota(c) = \iota(a)$ and $\iota(d) = \iota(b)$. Set $p := c - a$ and $q := d - b$. Clearly $p, q \in \mathfrak{T}$ since, noting that $\mathfrak{T}$ is the zero element of Q,

$$0 = \iota(c - a) = \iota(p) \Leftrightarrow p + \mathfrak{T} = \mathfrak{T} \Leftrightarrow p \in \mathfrak{T}.$$

Similarly for $q \in \mathfrak{T}$. We have that

$$\omega(a^* b) = \omega((c - p)^*(d - q)) = \omega(c^* d) - \omega(c^* q) - \omega(p^* d) + \omega(p^* q).$$

From the Cauchy-Schwartz inequality, and the fact that $q \in \mathfrak{T}$ it follows that

$$|\omega(c^* q)| \leq \|c\|_\omega \|q\|_\omega = 0$$

and similarly that $\omega(p^* d) = \omega(p^* q) = 0$. So $\omega(c^* d) = \omega(a^* b)$ and hence the inner product is well-defined. The fact that $\langle \cdot, \cdot \rangle$ is an inner product on Q can be verified as follows: Let $a, b \in A$ and $\alpha \in \mathbb{C}$. $\langle \cdot, \cdot \rangle$ is conjugate linear in the first slot since

$$\langle \iota(\alpha a + b), \iota(c) \rangle = \omega((\alpha a + b)^* c) = \omega(\overline{\alpha} a^* c + b^* c) = \overline{\alpha} \langle \iota(a), \iota(c) \rangle + \langle \iota(b), \iota(c) \rangle.$$

Also,

$$\overline{\langle \iota(a), \iota(b) \rangle} = \overline{\omega(a^*b)} = \omega(b^*a) = \langle \iota(b), \iota(a) \rangle$$

(see Lemma 2.3.10 in [7]). We have

$$\langle \iota(a), \iota(a) \rangle = \omega(a^*a) \geq 0$$

since ω is a positive linear functional, and furthermore

$$\langle \iota(a), \iota(a) \rangle = 0 \Leftrightarrow \omega(a^*a) = \|a\|_\omega = 0 \Leftrightarrow a \in \mathfrak{T} \Leftrightarrow \iota(a) = 0.$$

Hence Q is an inner product space. Next we construct $\pi : A \to L(Q)$ and show that it is a homomorphism (i.e that π is linear, multiplicative and $\pi(1) = 1$). Define π by

$$\pi(a)\iota(b) = \iota(ab).$$

for all $a, b \in A$. For each $a \in A$, $\pi(a)$ is a well-defined element of $L(Q)$. To see this, first note that $\mathfrak{T}$ is a left ideal of A since, by the Cauchy-Schwartz inequality,

$$p \in \mathfrak{T} \Rightarrow \|ap\|_\omega^2 = \omega((ap)^*ap) = |\omega(a^*ap)^*p)| \leq \|a^*ap\|_\omega \|p\|_\omega = 0$$

implying that $ap \in \mathfrak{T}$ for all $a \in A$. Let $a, b \in A$ such that $\iota(b) = \iota(c)$. Then $p := c - b \in \mathfrak{T}$ and

$$\iota(ac) = \iota(a(b + p)) = \iota(ab) + \iota(ap) = \iota(ab) + \mathfrak{T} = \iota(ab)$$

proving, together with the fact that ι is surjective, that $\pi(a) \in A$ is well-defined. π is linear since for $a, b, c \in A$ and $\alpha \in \mathbb{C}$ we have

$$\pi(\alpha a + b)\iota(c) = \iota((\alpha a + b)c) = \alpha\iota(ac) + \iota(bc) = (\alpha\pi(a) + \pi(b))\iota(c).$$

π is multiplicative since for $a, b, c \in A$

$$\pi(ab)\iota(c) = \iota(abc) = \pi(a)\iota(bc) = \pi(a)\pi(b)\iota(c)$$

so $\pi(ab) = \pi(a)\pi(b)$. Hence π is a homomorphism. Also

$$\pi(1)\iota(a) = \iota(1.a) = \iota(a) \tag{1.9}$$

for all $a \in A$ and it then follows from the surjectivity of ι that $\pi(1)$ is the identity of $L(Q)$. In order to find a cyclic representation of (A, ω), define $\Omega := \iota(1)$. We have that

$$\pi(a)\Omega = \pi(a)\iota(1) = \iota(a)$$

for all $a \in A$ and hence

$$\pi(A)\Omega = \iota(A) = Q$$

since ι is surjective. Hence Ω is a cyclic vector for π and so (Q, π, Ω) is a cyclic representation of (A, ω).

Finally we see that

$$\begin{aligned}\langle \pi(a)\Omega, \pi(b)\Omega\rangle = \langle \iota(a), \iota(b)\rangle &= \omega(a^*b) \\ &= \omega(1^*(a^*b)) \\ &= \langle \iota(1), \iota(a^*b)\rangle \\ &= \langle \Omega, \pi(a^*b)\Omega\rangle.\end{aligned}$$

Setting $a = 1$, we have $\omega(b) = \langle \Omega, \pi(b)\Omega\rangle$ for all $b \in A$. $\square$

Remark 1.4.4. If A is a C^*-algebra, then Definition 1.4.1 can be modified so that Q is replaced by its completion, i.e. the Hilbert space H, and $L(Q)$ is replaced by $\mathfrak{L}(H)$. Also π obtained in the GNS construction is a $*$-homomorphism (as shown below) instead of simply being a homomorphism.

From Proposition 2.3.11(c) in [7] we have that

$$\begin{aligned}\|\pi(a)\iota(b)\|^2 &= \langle \iota(ab), \iota(ab)\rangle \\ &= \omega(b^*a^*ab) \\ &\leq \omega(b^*b)\|a^*a\| \\ &= \|a\|^2\|\iota(b)\|^2\end{aligned}$$

and from this boundedness it follows that each $\pi(a)$ can be uniquely extended to an element of $\mathfrak{L}(H)$. It can also be shown that this π is a homomorphism in the same way as in the proof above. To see that it is $*$-homomorphism, we note that

$$\begin{aligned}\langle \pi(a)\iota(b), \iota(c)\rangle &= \langle \iota(ab), \iota(c)\rangle \\ &= \omega((ab)^*c) = \omega(b^*a^*c) \\ &= \langle \iota(b), \pi(a^*)\iota(c)\rangle \\ &= \langle (\pi(a^*))^*\iota(b), \iota(c)\rangle\end{aligned}$$

for all $a, b \in A$. Hence $\pi(a^*) = \pi(a)^*$ since $\pi(a) \in \mathfrak{L}(H)$ for all $a \in A$. Since Q is dense in H and $\pi(A)\Omega \supseteq Q$, $\pi(A)\Omega$ is dense in H, hence (H, π, Ω) is a cyclic representation of (A, ω). Also see [7] Definitions 2.3.2 and 2.3.5 and the discussion in Section 2.3.3 in [7] for the case when A is a C^*-algebra.

1.5 The Mean Ergodic Theorem

One of the cornerstone theorems of ergodic theory is the mean ergodic theorem. It is well-known in classical dynamics and we give a brief overview below, drawn mainly from [21]. We then state and prove the theorem in the more general setting that is relevant to this thesis.

Poincaré's Recurrence Theorem states that almost every point of each measurable set B returns to B infinitely often under a transformation T, and under appropriate conditions. It may now be asked how long the recurring points remain in B. This problem can be formulated as follows: given a point x (in B or not), and given a positive integer n, form the ratio of the number of these points that belong to B to the total number (i.e. to n), and evaluate the limit of these ratios as n tends to infinity, if this limit exists in a meaningful sense. Hence we should consider the ratio

$$\frac{1}{n}\sum_{k=0}^{n-1}\chi_B(T^k x).$$

This average is called the mean sojourn of x and we are therefore concerned with the problem of its convergence.

We do not need to restrict ourselves to characteristic functions. If f is any arbitrary function on X, then a function g on X may be defined by $g(x) = f(Tx)$ and we can define a mapping U by $g = Uf$. The mapping U has some important properties.

1. The most obvious property of U is its linearity, i.e.

$$\begin{aligned} U(af + bg)(x) &= (af + bg)(Tx) = (af)(Tx) + (bg)(Tx) \\ &= af(Tx) + bg(Tx) = aUf(x) + bUg(x) \end{aligned}$$

   for any complex-valued functions f and g on X, complex scalars a and b and any $x \in X$.

2. If T is measure preserving, then U sends $L^1(X, \Sigma, \mu)$ into itself, and moreover, it is an isometry on $L^1(X, \Sigma, \mu)$. This implies that if $f \in L^1$, then

$$Uf \in L^1 \text{ and } \|f\|_1 = \|Uf\|_1.$$

   To show that U is an isometry, we follow a standard approximation tool. If χ_B is the characteristic function of the set B of finite measure, then $U\chi_B$ is the characteristic function of $T^{-1}B$. Also, $\|\chi_B\|_1 = \mu(B)$. From this and from the linearity of U it follows that U is norm-preserving on finite linear combinations of such characteristic functions, i.e. on simple functions. If f is a non-negative function, then f is the pointwise limit of an increasing sequence f_n of simple functions. Since Uf_n is also an increasing sequence of non-negative functions, it follows from the theorem on integration of monotone sequences that

$$\lim_{n\to\infty} \|Uf_n\|_1 = \|Uf\|_1$$

as well as

$$\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1.$$

This proves the result for non-negative functions. The general case follows from the fact that the norm of every f in L^1 is the same as the norm of $|f|$. (Note that it was not necessary to assume that $\mu(X) < \infty$).

3. The fact that U is an isometry on L^1 implies that U is an isometry on L^2. To see this, note that $\|f\|_2 = \sqrt{\|f^2\|_1}$. If T is a bijective measure preserving transformation, then U is a bijective isometry, with $U^{-1}f(x) = f(T^{-1}x)$. An invertible isometry on a Hilbert space is a unitary operator ([16] Theorem 3.10-6(f)). This $U : L^2 \to L^2$ is called *the unitary operator induced by* T.

4. Furthermore, if U is an isometry, then $Uf = f$ if and only if $U^*f = f$. To see this, note that $Uf = f \implies U^*Uf = U^*f \implies f = U^*f$, if we recall for an isometry U, $U^*U = 1$. Conversely, if $U^*f = f$, then

$$\|Uf - f\|^2 = \langle Uf - f, Uf - f\rangle = \|Uf\|^2 - \langle f, Uf\rangle - \langle Uf, f\rangle + \|f\|^2.$$

Since $\langle f, Uf\rangle = \langle U^*f, f\rangle = \|f\|^2$, and $\langle Uf, f\rangle = \langle f, U^*f\rangle = \|f\|^2$. Hence

$$\|Uf - f\|^2 = 0 \text{ and } Uf = f.$$

Considering the properties of the unitary operator U, we find that one of the basic problems of ergodic theory consists of studying the limiting behavior of averages

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k$$

where U is an isometry on a Hilbert space.

Example 1.5.1. If the Hilbert space under consideration is one-dimensional, the Mean Ergodic Theorem is quite simple, but still interesting. In this case, the isometry is determined by a complex number u such that $|u| = 1$. Now consider the average

$$\frac{1}{n}\sum_{k=0}^{n-1} u^k.$$

If $u = 1$, then each average is equal to 1. If $u \neq 1$, then

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} u^k\right| = \left|\frac{1-u^n}{n(1-u)}\right| \leq \frac{2}{n|1-u|} \longrightarrow 0,$$

as $n \to \infty$. Hence the averages converge to 0. We see that the averages converge to a function p, which can be seen to be a projection on the space of all elements f such that $uf = f$.

In the finite-dimensional case, every isometry is given by a unitary matrix, which, without loss of generality, may be assumed to be a diagonal matrix. Since the diagonal entries of such a matrix U are complex numbers with absolute value 1, it follows that the averages converge to a diagonal matrix with diagonal entries 0's and 1's. The limit matrix, say P, is also a projection in this case, i.e. the projection on the space of all vectors f such that $Uf = f$.

The Mean Ergodic Theorem in Hilbert spaces is given below (as stated in [6]).

Theorem 1.5.2. *Let H be a Hilbert space, $U : H \mapsto H$ a unitary operator and let $M = \{f \in H : Uf = f\}$. If $P : H \mapsto M$ is the projection of H onto M, then*

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f - Pf\right\| \longrightarrow 0.$$

The Mean Ergodic Theorem Theorem also holds in a more general setting (also see [9] for a review). First we need to define Hilbert space-valued integrals as follows. For (Y, μ) a measure space and $\mathfrak{H}$ a Hilbert space, consider a bounded $f : \Lambda \to \mathfrak{H}$ with $\Lambda \subset Y$ measurable and $\mu(\Lambda) < \infty$, and $\langle f(\cdot), x\rangle$ measurable for every $x \in \mathfrak{H}$. Using the Riesz Representation Theorem, define $\int_\Lambda f d\mu$ by requiring

$$\left\langle \int_\Lambda f d\mu, x \right\rangle := \int_\Lambda \langle f(y), x\rangle \, d\mu(y)$$

for all $x \in \mathfrak{H}$. We will often use the notation $\int_\Lambda f(y)dy = \int_\Lambda f d\mu$, since there will be no ambiguity in the measure being used.

Theorem 1.5.3. *Let G be an amenable group (and hence contains a Følner sequence) with right Haar measure μ. Let H be a Hilbert space and $U : G \to \mathfrak{L}(H) : g \mapsto U_g$ be such that $\|U_g\| \leq 1$ and $U_g U_h = U_{gh}$ for all $g, h \in G$. Let $G \ni g \mapsto \langle U_g x, y \rangle$ be Borel measurable for all $x, y \in H$. Take P to be the projection of H onto $V := \{x \in H : U_g x = x \text{ for all } g \in G\}$. For any Følner sequence (Λ_n) in G we then have*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} U_g x dg = Px$$

for all $x \in H$.

Proof. Set $N := \overline{\operatorname{span}\{x - U_g x : x \in H, g \in G\}}$. For any g, a fixed point of U_g^* is a fixed point of U_g, and vice versa, since $||U_g^*|| \leq 1$. From this it follows that $V = N^\perp$, and in particular that V is a closed subspace of H.

Set

$$I_n(x) := \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} U_g x dg.$$

We first prove that $\lim_{n\to\infty} I_n(x) = 0$ for $x \in N$. Let $x = y - U_h y$ for some $y \in H$ and $h \in G$. Then we have that

$$\begin{aligned}
I_n(x) &= \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} (U_g y - U_{gh} y) dg \\
&= \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} U_g y dg - \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n h} U_g y dg \\
&= \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \setminus (\Lambda_n \cap (\Lambda_n h))} U_g y dg - \frac{1}{\mu(\Lambda_n)} \int_{(\Lambda_n h) \setminus (\Lambda_n \cap (\Lambda_n h))} U_g y dg
\end{aligned}$$

hence

$$\begin{aligned}
||I_n(x)|| &\leq \frac{1}{\mu(\Lambda_n)} \left|\left| \int_{\Lambda_n \setminus (\Lambda_n \cap (\Lambda_n h))} U_g y dg \right|\right| + \frac{1}{\mu(\Lambda_n)} \left|\left| \int_{(\Lambda_n h) \setminus (\Lambda_n \cap (\Lambda_n h))} U_g y dg \right|\right| \\
&\leq ||y|| \frac{\mu(\Lambda_n \Delta (\Lambda_n h))}{\mu(\Lambda_n)}
\end{aligned}$$

since $||U_g|| \leq 1$. Since (Λ_n) is a Følner sequence in G, $\lim_{n\to\infty} I_n(x) = 0$.

We then have that $\lim_{n\to\infty} I_n(x) = 0$ for all $x \in N$. Let $x \in N$ and set $N_0 := \{y - U_g y : y \in H, g \in G\}$. Then for any $\varepsilon > 0$ there is a $y \in \operatorname{span} N_0$ such that $||x - y|| < \varepsilon$, say $y = \sum_{j=1}^m x_j$ where $x_j \in N_0$. Therefore

$$|\|I_n(x)\| - \|I_n(y)\|| \leq ||I_n(x) - I_n(y)|| \leq \frac{1}{\mu(\Lambda_n)} ||x - y|| \int_{\Lambda_n} d\mu < \varepsilon$$

while

$$||I_n(y)|| \leq \sum_{j=1}^{m} ||I_n(x_j)|| \to 0$$

as $n \to \infty$, as shown above. Hence $\lim_{n\to\infty} I_n(x) = 0$.

Let $z := x - Px$ for any $x \in H$. Then $x = Px + z$ and since $Px \in V$ and $H = V \oplus N$, we have $z \in N$. Also, note that

$$I_n(Px) = \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} U_g(Px) d\mu = \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} Px d\mu = \frac{\mu(\Lambda_n)Px}{\mu(\Lambda_n)} = Px.$$

Then

$$||I_n(x) - Px|| = ||I_n(z) + I_n(Px) - Px|| = ||I_n(z)|| \to 0$$

as $n \to \infty$. $\square$

See [29] for an even more general version in Banach space.

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Chapter 2

Dynamical systems

2.1 Definitions

In this Section we list some definitions, notations and some basic results that we will need throughout the thesis. As mentioned in Section 1.2, G denotes an abelian second countable locally compact group with identity e, and regular Haar measure μ. It is however important to note that at this stage G need not be abelian in the definitions below. In fact, even the properties of Følner sequences are not needed until Corollary 3.1.2 (this means that in Definition 2.1.7 below one could in principle work with an arbitrary sequence (Λ_n) of Borel sets in G with $0 < \mu(\Lambda_n) < \infty$). In Definition 2.1.1 and onwards, the following notational agreement is used: If A is a unital $*$-algebra and τ is a mapping $\tau : G \to Aut(A)$, the $\tau_g := \tau(g)$ for all $g \in G$.

Definition 2.1.1. Let ω be a state on a unital $*$-algebra A. Let τ_g be a $*$-automorphism of A for every $g \in G$ such that $\tau_g \circ \tau_h = \tau_{gh}$ for all $g, h \in G$, and such that τ_e is the identity on A and $G \to \mathbb{C} : g \mapsto \omega(a\tau_g(b))$ is Borel measurable for all $a, b \in A$. Then we'll call (A, ω, τ, G) a *$*$-dynamical system*. If in addition $\omega \circ \tau_g = \omega$ for all g in G, we say that (A, ω, τ, G) is a *state-preserving* dynamical system. If we further have that A is a C*-algebra, then the state-preserving dynamical system (A, ω, τ, G) is called a *C*-dynamical system*.

Remark 2.1.2. As noted in Section 1.5, we can use the GNS representation to represent τ on the Hilbert space H by

$$U_g\iota(a) := \iota(\tau_g(a))$$

for all $a \in A$ and $g \in G$, and then extending each U_g uniquely to H. We then have that $U_g U_h = U_{gh}$, since

$$\begin{aligned} U_{gh}\iota(a) = \iota(\tau_{gh}(a)) = \iota(\tau_g(\tau_h(a))) &= U_g(\iota(\tau_h(a))) \\ &= U_g(U_h(\iota(a))) = U_g U_h \iota(a) \end{aligned}$$

Furthermore, since τ_g is a $*$-homomorphism and $\omega \circ \tau_g = \omega$,

$$\begin{aligned} \|U_g(\iota(a))\|^2 &= \langle U_g(\iota(a)), U_g(\iota(a))\rangle \\ &= \langle \iota\tau_g(a), \iota\tau_g(a)\rangle \\ &= \omega(\tau_g(a)^*\tau_g(a)) \\ &= \omega(\tau_g(a^*a)) \\ &= \omega(a^*a) = \langle \iota(a), \iota(a)\rangle = \|\iota(a)\|^2. \end{aligned}$$

Hence U_g is an isometry, satisfying the requirements of Theorem 1.5.3.

Remark 2.1.3. We also note that the measure theoretic definition of a dynamical system is a special case of this definition. For a σ-algebra on a set X, $B_\infty(\Sigma)$ denotes the C^*-algebra of all bounded complex-valued Σ-measurable functions on X, with sup-norm, its operations defined pointwise and its involution given by complex conjugation.

Now, given a measure theoretic dynamical system (X, Σ, μ, T), we obtain a $*$-dynamical system $(B_\infty(\Sigma), \varphi, \tau)$ where we have that $\varphi(f) = \int f d\mu$ and $\tau(f) = f \circ T$ for all $f \in B_\infty(\Sigma)$. We will denote the equivalence class $[g]$ of all measurable complex-valued functions on the measure space that are almost everywhere equal to g, simply by g. A cyclic representation (Q, π, Ω) of $(B_\infty(\Sigma), \varphi, \tau)$ is given by $Q = \{g : g \in B_\infty(\Sigma)\}$, $\pi(f)g = fg$ for all $f, g \in B_\infty(\Sigma)$ and $\Omega = 1$. Note that ι becomes $\iota(f) = f$. The completion of Q is $L^2(\mu)$ by the following Proposition in [11]:

Proposition 2.1.4. *Let μ be a measure on a σ-algebra Σ on X. Then $Q = \{g : g \in B_\infty(\Sigma)\}$ is dense in $L^2(\mu)$.*

Proof. Clearly $Q \subset L^2(\mu)$. Let $g \in L^2(\mu)$ with $g \geq 0$. Then there exists a sequence $(s_n) \subset B_\infty(\Sigma)$ of simple functions such that $0 \leq s_1 \leq s_2 \leq \ldots \leq g$ and $s_n(x) \to g$ for all $x \in X$. Now since $g \in L^2(\mu)$ and clearly $|s_n - g|^2 \leq |g|^2$, we have that $|s_n - g|^2, |g|^2 \in L^1(\mu)$. Hence by Lebesque's Dominated Convergence Theorem it follows that

$$\|s_n - g\|_2^2 = \int |s_n - g|^2 d\mu \to 0$$

implying that $g \in \overline{Q}$ in $L^2(\mu)$. For an arbitrary $g \in L^2(\mu)$, we let $g = u^+ - u^- + i(v^+ - v^-)$. We then have $0 \leq u^+, u^-, v^+, v^- \in L^2(\mu)$. So, as shown above, $u^+, u^-, v^+, v^- \in \overline{Q}$ and hence also g. $\square$

Remark 2.1.5. Suppose that (A, ω, τ, G) is a $*$-dynamical system. Then so is $(\overline{A}, \overline{\omega}, \tau, G)$ where $\overline{\omega}(a) = \overline{\omega(a)}$, while $\overline{A}$ is the $*$-algebra A with the original scalar multiplication replaced by $\alpha \cdot a = \overline{\alpha} a$ for all $\alpha \in \mathbb{C}$ and $a \in A$. Let $A \otimes \overline{A}$ denote the algebraic tensor product of A with $\overline{A}$. When A is normed, we assign the same norm to $\bar{A}$, and on $A \otimes \overline{A}$ for our purposes any norm satisfying $\|a \otimes b\| \leq \|a\| \, \|b\|$ will do, for example the spatial C*-norm when A is a C*-algebra. However, even in the normed case, $A \otimes \bar{A}$ will denote the algebraic tensor product; we will not work with the completion in the norm.

The following will be useful in Chapter 3.

Proposition 2.1.6. *Suppose that (A, ω, τ, G) is a $*$-dynamical system. If $A \otimes \overline{A}$ denotes the algebraic tensor product of A with $\overline{A}$, then $(A \otimes \overline{A}, \omega \otimes \overline{\omega}, \tau \otimes \tau, G)$ is also a $*$-dynamical system, where $(\tau \otimes \tau)_g := \tau_g \otimes \tau_g$. Moreover, if (A, ω, τ, G) is state-preserving, then so is $(A \otimes \overline{A}, \omega \otimes \overline{\omega}, \tau \otimes \tau, G)$.*

Proof. $A \otimes \overline{A}$ is clearly a $*$-algebra with unit $1 \otimes 1$. Also, $\omega \otimes \overline{\omega}$ is a state on $A \otimes \overline{A}$, since $\omega \otimes \overline{\omega}(1 \otimes 1) = 1$, $\omega \otimes \overline{\omega}(c^* c) \geq 0$ for all $c \in A \otimes \overline{A}$ (see [27] Theorem 6.4.5) and $\omega \otimes \overline{\omega}$ is linear (see [27] Section 6.3). Also, $(\tau \otimes \tau)_g = \tau_g \otimes \tau_g$ is indeed a $*$-automorphism for all $g \in G$ ([27], Section 6.3). Furthermore,

$$(\tau \otimes \tau)_g \circ (\tau \otimes \tau)_h = \tau_{gh} \otimes \tau_{gh} = (\tau \otimes \tau)_{gh} \,,$$

for all $g, h \in G$ since for any $a \in A \otimes \overline{A}$ we can write

$$a = \sum_{i=1}^{n} a_i \otimes b_i$$

for some positive integer n and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ and

$$\begin{aligned}
(\tau \otimes \tau)_g \circ (\tau \otimes \tau)_h (a) &= (\tau \otimes \tau)_g \left((\tau \otimes \tau)_h (a) \right) \\
&= (\tau \otimes \tau)_g \left((\tau \otimes \tau)_h \left(\sum_{i=1}^{n} a_i \otimes b_i \right) \right) \\
&= \sum_{i=1}^{n} (\tau_g \otimes \tau_g \left(\tau_h(a_i) \otimes \tau_h(b_i) \right) \\
&= \sum_{i=1}^{n} \tau_{gh}(a_i) \otimes \tau_{gh}(b_i) \\
&= (\tau_{gh} \otimes \tau_{gh}) \left(\sum_{i=1}^{n} a_i \otimes b_i \right) \\
&= (\tau \otimes \tau)_{gh}(a)
\end{aligned}$$

for all $g, h \in G$. From linearity and the fact that τ_e is the identity on A, $(\tau\otimes\tau)_e$ is clearly the identity on $A\otimes\overline{A}$. Since $G \to \mathbb{C} : g \mapsto \omega(a\tau_g(b))$ is Borel measurable for all $a, b \in A$ it follows that the maps $g \mapsto \omega \otimes \overline{\omega}(a(\tau \otimes \tau)_g(b))$ are Borel measurable for all $a, b \in A \otimes \overline{A}$. To see this, use $a, b \in A \otimes \overline{A}$ with $a = \sum_{i=1}^{n} a_i \otimes c_i$ and $b = \sum_{i=1}^{n} b_i \otimes d_i$ and the fact that

$$\omega \otimes \overline{\omega}\left(\left(\sum_{i=1}^{n} a_i \otimes c_i\right)(\tau \otimes \tau)_g\left(\sum_{j=1}^{n} b_j \otimes d_j\right)\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \omega\left(a_i\tau_g(b_j)\right)\overline{\omega}\left(c_i\tau_g(d_j)\right),$$

which is Borel measurable. Finally, if (A, ω, τ, G) is state-preserving, then clearly also $(\overline{A}, \overline{\omega}, \tau, G)$ is state-preserving and hence for each $a = \sum_{i=1}^{n} a_i \otimes b_i \in A \otimes \overline{A}$ we have that

$$\omega \otimes \overline{\omega}(\tau \otimes \tau)_g(a) = \omega \otimes \overline{\omega}(a)$$

for all $g \in G$. □

For a group G, let $\mathrm{Hom}(G)$ denote the set of *all* group homomorphisms $G \to G$.

Definition 2.1.7. Let (A, ω, τ, G) be a $*$-dynamical system and consider an $M \subset \mathrm{Hom}(G)$ such that $G \to \mathbb{C} : g \mapsto \omega(a\tau_{\varphi(g)}(b))$ is Borel measurable for all $\varphi \in M$.

(i) (A, ω, τ, G) is said to be *M-weakly mixing relative to* (Λ_n), if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left|\omega(a\tau_{\varphi(g)}(b)) - \omega(a)\omega(b)\right| dg = 0$$

for all $a, b \in A$, and for all $\varphi \in M$.

(ii) (A, ω, τ, G) is said to be *M-ergodic relative to* (Λ_n), if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(a\tau_{\varphi(g)}(b))dg = \omega(a)\omega(b)$$

for all $a, b \in A$, and for all $\varphi \in M$.

Remark 2.1.8. If (A, ω, τ, G) is a $*$-dynamical system, then so is $\left(A, \omega, \tau_{\varphi(\cdot)}, G\right)$ for any $\varphi \in$ M. So essentially we're looking at a set of systems indexed by M, and one can therefore expect that the known properties of weakly mixing and ergodic systems will extend to the situation in Definition 2.1.7, as we'll review in the rest of the Section. In the case of $G = \mathbb{Z}$, $\Lambda_n = \{1, ..., n\}$ and with $M = \{id_{\mathbb{Z}}\}$, Definition 2.1.7(i) corresponds to the usual definition of weak mixing for an action of the group $\mathbb{Z}$. Since all homomorphisms of $\mathbb{Z}$ are of the form $n \mapsto kn$ for some $k \in \mathbb{Z}$, one can then easily show that $\{id_{\mathbb{Z}}\}$-weak mixing implies $\mathrm{Hom}(\mathbb{Z})$-weak mixing. Note that if a homomorphism given by $\varphi_0(g) = e$ for all $g \in G$ is in M then the system is not M-weakly mixing, hence we wouldn't want φ_0 to be in M. We mention this simply because φ_0 does appear in the theory to follow, but not as an element of M.

We will work with asymptotically abelian systems, e.g. in Proposition 3.3.3, Theorem 3.3.4 and our final result, Theorem 4.2.6.

Definition 2.1.9. Let (A, ω, τ, G) be a $*$-dynamical system where A has a submultiplicative norm. Such a $*$-dynamical system is said to be *M-asymptotically abelian relative to* (Λ_n), where $M \subset \mathrm{Hom}(G)$, if $G \to A : g \mapsto \tau_{\varphi(g)}(b)$ is continuous, and

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \|[a, \tau_{\varphi(g)}(b)]\| dg = 0$$

for all $a, b \in A$, and for all $\varphi \in M$, where $[a, b] := ab - ba$.

We will need the following in our discussion of compact systems in Section 4.1 and beyond.

Definition 2.1.10. A set V in a pseudo metric space (X, d) is said to be *ε-separated*, where $\varepsilon > 0$, if $d(x, y) \geq \varepsilon$ for all $x, y \in V$ with $x \neq y$. A set $B \subset X$ is said to be *totally bounded in* (X, d) if for every $\varepsilon > 0$ there exists a finite set $M_\varepsilon \subset X$, called a finite *ε-net*, such that for every $x \in B$ there is a $y \in M_\varepsilon$ with $d(x, y) < \varepsilon$.

Definition 2.1.11. Let ω be a positive linear functional on a $*$-algebra A, K a semigroup, and $\tau_g : A \to A$ a linear map for each $g \in K$ such that

$$\tau_g \circ \tau_h = \tau_{gh}$$

and

$$\|\tau_g(a)\|_\omega = \|a\|_\omega$$

for all $g, h \in K$ and $a \in A$. Assume that the *orbit*

$$B_a := \{\tau_g(a) : g \in K\}$$

is totally bounded in $(A, \|\cdot\|_\omega)$ for each $a \in A$. Then we call (A, ω, τ, K) a *compact system*. If furthermore A is a C*-algebra and $\|\tau_g(a)\| \leq \|a\|$ in A's norm for all $a \in A$ and $g \in K$, then we refer to (A, ω, τ, K) as a *compact C*-system*.

In particular, if the orbits of the $*$-dynamical systems and C*-dynamical systems in Definition 2.1.1 are totally bounded in $(A, \|\cdot\|_\omega)$, then those systems will be called *compact*.

Remark 2.1.12. In Section 1.1 we noted that a measure-theoretic dynamical system (X, Σ, ν, T) is said to be compact if the orbit $\{f \circ T^n : n \in \mathbb{Z}\}$ of every $f \in L^2(\nu)$ is relatively compact in $L^2(\nu)$. Using the GNS construction, we will now briefly show that the L^2 definition of compactness is a special case of Definition 2.1.11. Given a $*$-dynamical system (A, ω, τ, G), the GNS construction provides us with a representation of (A, ω), namely an inner product space Q, a linear surjection $\iota : A \to Q$, and a linear mapping $\pi : A \to L(Q)$, with $L(Q)$ the space of all linear maps $Q \to Q$ (not necessarily bounded), such that $\langle \iota(a), \iota(b) \rangle = \omega(a^*b)$, $\pi(a)\iota(b) = \iota(ab)$ and $\pi(ab) = \pi(a)\pi(b)$ for all $a, b \in A$. Then

$$U_g : Q \to Q : \iota(a) \mapsto \iota(\tau_g(a))$$

is a well-defined linear operator with $||U_g x|| = ||x||$ for all $x \in Q$ and $g \in G$. It is then straightforward to show that (A, ω, τ, G) is compact if and only if all the orbits

$$B_x := \{U_g x : g \in G\} \tag{2.1}$$

with $x \in Q$, are totally bounded in Q. However, U_g has a unique continuous extension to the completion H of Q, and one can show that all the orbits B_x, $x \in H$, again defined as in 2.1, are totally bounded if and only if they are totally bounded for all $x \in Q$. Hence (A, ω, τ, G) is compact if and only if all the orbits B_x, $x \in H$, in the Hilbert space H are totally bounded. The measure theoretic definition is a special case of this simply because $L^2(\nu)$ is a Hilbert space obtained exactly as H above through the GNS-construction applied to the state $\omega = \int (\cdot) d\nu$ on the C^*-algebra $B_\infty(\Sigma)$ of all bounded complex-valued Σ-measurable functions on X, or on $L^\infty(\nu)$ (also see Remark 2.1.3).

Recall that $B(H)$ denotes the algebra of all bounded linear operators in the Hilbert space H, and let S' denote the commutant of a set $S \subset B(H)$. We say that a state ω on a Von Neumann algebra A is *normal* if $\omega(\sup_\alpha(a_\alpha)) = \sup(\omega(a_\alpha))$ for all increasing nets $a_\alpha \subset A_+$ with an upper bound. Also, a state is called *faithful* if $\omega(a) > 0$ for all nonzero $a \in A_+$.

The definitions below are needed for Section 4.2.

Definition 2.1.13. A *W*-dynamical system* (A, ω, τ, G) consists of a Von Neumann algebra A on which we have a faithful normal state ω, and where

$\tau : G \to \text{Aut}(A) : g \mapsto \tau_g$ is a representation of any abelian group G as *-automorphisms of A (i.e. $\tau_e = \text{id}_A$ and $\tau_g \circ \tau_h = \tau_{gh}$), such that $\omega \circ \tau_g = \omega$ for all g in G.

Note that the existence of a faithful normal state on A in this definition implies that A is a σ-finite Von Neumann algebra (see for example Proposition 2.5.6 in [7]). Also recall that a vector x is called separating for a Von Neumann algebra R if $rx = 0$ for $r \in R$ implies $r = 0$. It is convenient to work in the GNS representation of such a system and for certain intermediate results the group G need not be abelian, therefore we will mostly work with the following:

Definition 2.1.14. A *represented system* $(R, \omega_\Omega, \alpha)$ consists of the following: Firstly a Von Neumann algebra R on a Hilbert space H, a unit vector $\Omega \in H$ which is cyclic and separating for R, in terms of which we define a state ω_Ω on R by $\omega_\Omega(a) = \langle \Omega, a\Omega \rangle$. Furthermore we have a unitary representation $U : G \to B(H) : g \mapsto U_g$ of an arbitrary group G (i.e. U_g is a unitary operator, $U_e = 1$ and $U_g U_h = U_{gh}$), such that $U_g \Omega = \Omega$ and $U_g M U_g^* \subset M$ for all $g \in G$, and in terms of which $\alpha : G \to \text{Aut}(R) : g \mapsto \alpha_g$ is defined by $\alpha_g(a) = U_g a U_g^*$.

The notation in these two definitions will be used consistently, for example reference to a represented system will imply the notation $(R, \omega_\Omega, \alpha)$, H, G and U, and throughout the rest of this Section $(R, \omega_\Omega, \alpha)$ is a represented system. Note that the GNS representation (H, π, Ω) of a W*-dynamical system (A, ω, τ, G) gives us a *corresponding* represented system $(R, \omega_\Omega, \alpha)$ where $R = \pi(A)$ (where $\pi(A)$ is a Von Neumann algebra by Theorem 2.4.24 in [7]) and $\alpha_g(\pi(a)) = \pi(\tau_g(a))$ in terms of which U is uniquely defined.

Also keep in mind that π is faithful in this situation. To see this, let $\pi(a)\Omega = 0$ for an $a \in A$. Then

$$\omega(a^*a) = \|\pi(a)\Omega\|^2 = 0$$

and since ω is faithful we have $a^*a = 0$, so $a = 0$, implying that π is injective.

For a represented system an *eigenoperator* of α is an $a \in A\backslash\{0\}$ such that there exists a function $\lambda_a : G \to \mathbb{C}$ with $\alpha_g(a) = \lambda_a(g)a$ for all $g \in G$. Note that in this case $|\lambda_a(g)| = 1$ for all g, i.e. λ_a is *unimodular*. Similarly an *eigenvector* of U is an $x \in H\backslash\{0\}$ such that there exists a function $\lambda_x : G \to \mathbb{C}$ with $U_g x = \lambda_x(g)x$ for all g. Again note that λ_x is unimodular.

For a represented system we will denote the Hilbert subspace of H spanned by the eigenvectors of U by H_0. The Hilbert subspace of H spanned by the eigenvectors x with $\lambda_x = 1$ will be denoted by H_1. Note that $\mathbb{C}\Omega \subset H_1 \subset H_0$, with equality allowed.

Definition 2.1.15. A represented system is called *ergodic* (respectively *weakly mixing*) when $\dim H_1 = 1$ (respectively $\dim H_0 = 1$). A W*-dynamical system is called *ergodic* (respectively *weakly mixing*) when its corresponding represented system is ergodic (respectively weakly mixing).

When G is as in Section 1.2, then ergodicity and weak mixing of the dynamical system (A, ω, α) as given in Definition 2.1.15 are equivalent to $\{\mathrm{id}_G\}$-ergodicity and $\{\mathrm{id}_G\}$-weak mixing as given in Definition 2.1.7. For ergodicity this follows from the mean ergodic theorem, and for weak mixing it can be shown to follow from the general theory in Section 2.4 of [25].

For a represented system we define a norm $\|\cdot\|_\Omega$ on R by $\|a\|_\Omega = \omega_\Omega(a^*a)^{1/2} = \|a\Omega\|$ where $\|\cdot\|$ denotes the norm of H.

Definition 2.1.16. A *factor* (N, ω, τ) of a W*-dynamical system (A, ω, τ, G) consists of a $*$-algebra $N \subset A$ and the restrictions of ω and τ_g to N, such that $\tau_g(N) \subset N$ for all g . Similarly a *factor* $(N, \omega_\Omega, \alpha)$ of a represented system $(R, \omega_\Omega, \alpha)$ consists of a $*$-algebra $N \subset R$ and the restrictions of ω_Ω and α_g to N, such that $\alpha_g(N) \subset N$ for all g. Such factors are called *compact* if respectively every orbit $\tau_G(a) = \{\tau_g(a) : g \in G\}$ is totally bounded in $(N, \|\cdot\|_\omega)$ or every orbit $\alpha_G(a) = \{\alpha_g(a) : g \in G\}$ is totally bounded in $(N, \|\cdot\|_\Omega)$. A factor will be called *nontrivial* if R strictly contains $\mathbb{C}1$.

To avoid confusion we stress that the term factor here refers to a subsystem of a dynamical system as defined below, and not to a von Neumann algebra which is a factor (i.e. has trivial center).

For a represented system the orbit of $x \in H$ will be denoted by $U_G x = \{U_g x : g \in G\}$.

Definition 2.1.17. A W*-dynamical system (A, ω, τ, G), with G as in Section 1.2, is said to have the *Szemerédi property* if there exists a Følner sequence (Λ_n) in G such that for any $k \in \mathbb{N}$ and $m_1, \ldots, m_k \in \mathbb{N}$ with $m_1 < \ldots < m_k$ and for all $a \in A^+$ with $\omega(a) > 0$,

$$\liminf_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left| \omega\left(a \prod_{j=1}^{k} \tau_{g^{m_j}}(a) \right) \right| dg > 0.$$

2.2 Examples

Although the focus in this thesis is mainly on structure, a few basic examples are included for completeness and as an illustration of some of the definitions above.

2.2.1 A noncommutative compact system

In this thesis, weakly mixing systems and compact systems appear as part of the structure of ergodic systems, so concrete examples of weakly mixing systems and compact systems are not essential for our ultimate goal. Nevertheless, it is interesting to look at an example of a compact C*-dynamical system in which the C*-algebra is noncommutative. To do this we need a few simple tools, which we now discuss.

First note that if a set in a C*-algebra A is totally bounded in A (i.e. in terms of A's norm), then it is also totally bounded in $(A, ||\cdot||_\omega)$ for any positive linear functional ω on A, since $||\cdot||_\omega \leq ||\omega||^{1/2} ||\cdot||$ (keep in mind that ω is bounded, since it is positive and A is a C*-algebra). Hence, if we can prove that the orbits of a given C*-dynamical system (A, ω, τ, K) are totally bounded in A, then it follows that the system is compact. Of course, this is then a stronger form of compactness, but Example 2.2.2 happens to possess this stronger property, and it turns out to be easier to prove than to prove compactness directly in terms of $||\cdot||_\omega$, since A's norm is submultiplicative, which makes it easier to work with than $||\cdot||_\omega$.

In Lemma 2.2.1 and Proposition 2.2.2 below, we work with a C*-algebra A, an arbitrary *set* K, and a $*$-homomorphism $\tau_g : A \to A$ for each $g \in K$. When we say that an "orbit" $(a_g) \equiv (a_n)_{g\in K}$ is *totally bounded* in a space, we mean that the set $\{a_g : g \in K\}$ is totally bounded in that space. For any subset $\mathfrak{V} \subset A$ we will denote the set of all polynomials over $\mathbb{C}$ generated by the elements of $\mathfrak{V}$ and their adjoints, by $p(\mathfrak{V})$, i.e. $p(\mathfrak{A})$ consists of all finite linear combinations of all finite products of elements of $\mathfrak{V} \cup \mathfrak{V}^*$ with $\mathfrak{V}^* := \{a^* : a \in \mathfrak{V}\}$. We will use the notation $XY := \{xy : x \in X, y \in Y\}$ whenever X and Y are sets for which this multiplication of their elements is defined.

Lemma 2.2.1. *If $(\tau_g(a))$ is totally bounded in A for every a in some subset $\mathfrak{V}$ of A, then $(\tau_g(a))$ is totally bounded in A for every $a \in p(\mathfrak{V})$.*

Proof. The following easily verifiable fact will be useful in this proof: Let $\|\cdot\|$ be a seminorm on a vector space X. A set $B \subset X$ is totally bounded in X if for every $\varepsilon > 0$ there exists a finite set $M_\varepsilon \subset X$ such that for all $y \in B$ there is a $z \in M_\varepsilon$ with $\|y - z\| < \varepsilon$.

Consider any $a, b \in A$ for which $(\tau_g(a))$ and $(\tau_g(b))$ are totally bounded in A, and any $\varepsilon > 0$. By the hypothesis there are finite sets $M, N \subset A$ such that for each $g \in K$ there is an $a_g \in M$ and a $b_g \in N$ such that $\|\tau_g(a) - a_g\| < \varepsilon$

and $\|\tau_g(b) - b_g\| < \varepsilon$. Clearly

$$\begin{aligned}\|\tau_g(a)\tau_g(b) - a_g b_g\| &\leq \|\tau_g(a)\|\|\tau_g(b) - b_g\| + \|\tau_g(a) - a_g\|\|b_g\| \\ &\leq \varepsilon\left(\|\tau_g(a)\| + \|b_g\|\right)\end{aligned}$$

but note that $\|\tau_g(a)\| \leq \|a\|$, since τ_g is a $*$-homomorphism and A is a C*-algebra, while $||b_g|| < \|\tau_g(b)\| + \varepsilon \leq ||b|| + \varepsilon$. Since MN is a finite subset of A, and $a_g b_g \in MN$, it follows that $(\tau_g(ab))$ is totally bounded in A. Similarly $(\tau_g(a^*))$ and $(\tau_g(\alpha a + \beta b))$ are totally bounded in A for any $\alpha, \beta \in \mathbb{C}$, and this is enough to prove the lemma. □

Proposition 2.2.2. *Now assume that A is generated by a subset $\mathfrak{V} \subset A$ for which $\tau_g(\mathfrak{V}) \subset p(\mathfrak{V})$ for every $g \in K$. Also assume that $(\tau_g(a))$ is totally bounded in A for every $a \in \mathfrak{V}$. Then $(\tau_g(a))$ is totally bounded in A for every $a \in A$.*

Proof. Firstly it is easily shown that if Y is a dense subspace of a normed space X, $U_g : Y \to Y$ is linear with $\|U_g\| \leq 1$ for all $g \in K$, and $(U_g y)$ is totally bounded in X for every $y \in Y$ (or in Y for every $y \in Y$), then for the unique bounded linear extension $U_g : X \to X$ the "orbit" $(U_g x)$ is totally bounded in X for every $x \in X$. (We also used this fact when we discussed the GNS-construction above.)

Now simply set $X = A$, $Y = p(\mathfrak{V})$ and $U_g = \tau_g$, then by our assumptions and Lemma 2.2.1 all the requirements in the remark above are met. □

Example 2.2.3.

We consider a so-called rotation C*-algebra, and use Proposition 2.2.2 to show that we obtain a compact C*-dynamical system. As described in Chapter VI of [8], let $\mathfrak{H} := L^2(\mathbb{R}/\mathbb{Z})$ and define two unitary operators U and V on $\mathfrak{H}$ by

$$(Uf)(t) = f(t + \theta)$$

and

$$(Vf)(t) = e^{2\pi i t} f(t)$$

for $f \in \mathfrak{H}$, where $\theta \in \mathbb{R}$ (though the interesting case is $\theta \in \mathbb{Q}$). These operators satisfy

$$UV = e^{2\pi i \theta} VU. \tag{2.2}$$

Let $\mathfrak{A}$ be the C*-algebra generated by U and V. Note that $\mathfrak{A}$ is noncommutative because of (2.2). Then, as shown in Chapter VI of [8], there is a unique trace ω on $\mathfrak{A}$, i.e. a state with $\omega(ab) = \omega(ba)$. Define $\tau : \mathfrak{A} \to \mathfrak{A}$ by $\tau(a) = U^* a U$ for all $a \in \mathfrak{A}$, then τ is a $*$-isomorphism and therefore

$||\tau(a)|| = ||a||$, since $\mathfrak{A}$ is a C*-algebra. Also, since ω is a trace and U is unitary, $\|\tau(a)\|_\omega = \|a\|_\omega$ for all $a \in \mathfrak{A}$. Hence $(\mathfrak{A}, \omega, \tau, \mathbb{N})$ is a C*-dynamical system, where by slight abuse of notation τ here denotes the function $n \mapsto \tau^n$ as well, to fit it into Definitions 2.1.1 and 2.1.11's notation.

We now show that $(\mathfrak{A}, \omega, \tau, \mathbb{N})$ is compact: It is trivial that $(\tau^n(U)) = (U)$ is totally bounded in $\mathfrak{A}$. Furthermore, $\tau^n(V) = (U^*)^n V U^n = e^{-2\pi i n\theta} V$ by (2.2). Since the unit circle is compact, it follows that $(\tau^n(V))$ is totally bounded in $\mathfrak{A}$. From Proposition 2.2.2 with $\mathfrak{V} = \{U, V\}$ we conclude that $(\tau^n(a))$ is totally bounded in $\mathfrak{A}$ for all $a \in \mathfrak{A}$. In particular $(\mathfrak{A}, \omega, \tau, \mathbb{N})$ is a compact C*-system. Similarly $(\mathfrak{A}, \omega, \tau, \mathbb{Z})$ is a compact C*-dynamical system.

2.2.2 A counterexample

The following example is drawn from [28]. Let H be a Hilbert space with orthonormal basis $\{\xi_0\} \cup \{\eta_j : j \in \mathbb{Z}\}$. Let U be the unitary operator on H defined by

$$U\xi_0 = \xi_0 \text{ and } U\eta_j = \eta_{j+1}$$

for all $j \in \mathbb{Z}$. Define

$$\tau : \mathfrak{L}(H) \to L(H) : x \mapsto UxU^*.$$

Then τ is a $*$-automorphism of the C^*-algebra $\mathfrak{L}(H)$. Indeed it is easily checked that τ is linear and injective. Surjectivity of τ follows from the fact that any $y \in \mathfrak{L}(H)$ can be written as $U(U^*yU)U^*$, with $U^*yU \in \mathfrak{L}(H)$. It is a $*$-homomorphism since $\tau(xy) = UxyU^* = UxU^*UyU^* = \tau(x)\tau(y)$ and $\tau(x^*) = Ux^*U^* = (UxU^*)^* = \tau(x)^*$. We have that

$$\omega : \mathfrak{L}(H) \to \mathbb{C} : x \mapsto \langle x\xi_0, \xi_0\rangle$$

is a state since it is clearly linear, $\omega(x^*x) = \langle x^*x\xi_0, \xi_0\rangle = \langle x\xi_0, x\xi_0\rangle \geq 0$ for all $x \in \mathfrak{L}(H)$ and $\omega(1) = \langle \xi_0, \xi_0\rangle = 1$. Also, τ leaves ω invariant since $\omega(\tau(x)) = \omega(UxU^*) = \langle UxU^*\xi_0, \xi_0\rangle = \langle xU^*\xi_0, U^*\xi_0\rangle = \langle x\xi_0, \xi_0\rangle$ since $U^*\xi_0 = \xi_0$. Also, $\mathbb{Z} \ni n \mapsto \omega(x\tau^n y)$ is clearly Borel measurable. Hence $(\mathfrak{L}(H), \omega, \tau, \mathbb{Z})$ is a state preserving C^*-dynamical system. We next show that $(\mathfrak{L}(H), \omega, \tau, \mathbb{Z})$ is weakly mixing. For any linear combinations

$$\xi = \alpha_0\xi_0 + \sum_{j=-k}^{k} \alpha_j\eta_j, \quad \eta = \beta_0\xi_0 + \sum_{j=-k}^{k} \beta_j\eta_j,$$

we have, by orthonormality and for $n > k$ that,

$$\langle U^n\xi, \eta\rangle = \alpha_0\overline{\beta_0} = \langle \xi, \xi_0\rangle\langle \xi_0, \eta\rangle = 0$$

and hence

$$\lim_{n\to\infty} \langle U^n \xi, \eta \rangle = \langle \xi, \xi_0 \rangle \langle \xi_0, \eta \rangle \tag{2.3}$$

for all $\xi, \eta \in H$. Consequently, from 2.3

$$\begin{aligned}
\lim_{n\to\infty} \omega(x\tau^n y) &= \lim_{n\to\infty} \langle xU^n y (U^*)^n \xi_0, \xi_0 \rangle \\
&= \lim_{n\to\infty} \langle U^n y \xi_0, x^* \xi_0 \rangle \\
&= \langle y\xi_0, \xi_0 \rangle \langle \xi_0, x^* \xi_0 \rangle \\
&= \omega(y)\omega(x),
\end{aligned}$$

i.e. $(\mathfrak{L}(H), \omega, \tau, \mathbb{Z})$ is strongly mixing, and hence also weakly mixing. The system is however not weakly mixing of order 2 (see equation 1.1). To see this, let x_2 denote the partial isometry which carries ξ_0 to η_0 and vanishes on the orthogonal complement of ξ_0. Let $x_0 = x_2^*$ and let x_1 be the unitary on H defined by

$$x_1 \xi_0 = \xi_0 \text{ and } x_1 \eta_{-j} = \eta_j$$

for all $j \in \mathbb{Z}$. It then follows that

$$\begin{aligned}
\omega\left(x_0 \tau^k(x_1) \tau^{2k}(x_2)\right) &= \langle x_0 \tau^k(x_1) \tau^{2k}(x_2) \xi_0, \xi_0 \rangle \\
&= \langle U^k x_1 (U^*)^k U^{2k} x_2 (U^*)^{2k} \xi_0, x_2 \xi_0 \rangle \\
&= \langle U^k x_1 U^k x_2 \xi_0, x_2 \xi_0 \rangle = \langle U^k x_1 U^k \eta_0, \eta_0 \rangle \\
&= \langle U^k x_1 \eta_k, \eta_0 \rangle = \langle U^k \eta_{-k}, \eta_0 \rangle = \langle \eta_0, \eta_0 \rangle = 1
\end{aligned}$$

and so

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \omega\left(x_0 \tau^k(x_1) \tau^{2k}(x_2)\right) = 1$$

while

$$\omega(x_0)\omega(x_1)\omega(x_2) = 0$$

as $\omega(x_2) = \langle x_2 \xi_0, \xi_0 \rangle = \langle \eta_0, \xi_0 \rangle = 0$.

In the next Chapter we will show that under the assumption that certain dynamical systems are asymptotic abelian we can prove weakly mixing of all orders.

2.2.3 An asymptotic abelian and weakly mixing system

Consider the C^*-algebra $A = \bigotimes_{n\in\mathbb{Z}} A_n$, where $A_n := M_2(\mathbb{C})$, i.e. A_n is the C^*-algebra of complex 2×2 matrices, for each $n \in \mathbb{Z}$. Set $B_n := A_{n+1}$ and let $\theta_n : A_n \to B_n$ be a $*$-isomorphism for each n. The tensor product $\theta = \bigotimes_{n\in\mathbb{Z}} \theta_n$ is then a $*$-automorphism of A (see [23] Section 11.4, and particularly Theorem 11.4.5). Since there exists a unique tracial state ω_n on each A_n (see [30] Theorem 6.4.3) we have that the tensor product $\omega = \bigotimes_{n\in\mathbb{Z}} \omega_n$ is also the unique tracial state ([23] Theorems 11.4.6 and 11.4.7). Since the composition of any $*$-isomorphism with a trace is again a trace, it follows from the uniqueness of the tracial state that ω is invariant with respect to θ (and also that each ω_n is invariant with respect to θ_n). Since also $\mathbb{Z} \to \mathbb{C} : n \mapsto \omega(a\theta^n(b))$ is Borel measurable for all $a, b \in A$, we then obtain a state preserving C^*-dynamical system $(A, \omega, \theta, \mathbb{Z})$. We next show that this system is "strongly asymptotically abelian" in the sense that

$$\lim_{|n|\to\infty} \|[a, \theta^n(b)]\| = 0$$

for all $a, b \in A$ and also "strongly mixing" namely

$$\lim_{|n|\to\infty} \omega(a\theta^n(b)) = \omega(a)\omega(b)$$

for all $a, b \in A$ and hence weakly mixing. Since A is generated by elementary tensor products of the form

$$\ldots \otimes a_{-2} \otimes a_{-1} \otimes a_0 \otimes a_1 \otimes \ldots$$

where only a finite number of the a_i's are not 1, we will first confirm that the strong asymptotically abelian and strong mixing criteria are satisfied for these type of elements. Let $a = \ldots \otimes a_{-1} \otimes a_0 \otimes a_1 \otimes \ldots$ and $b = \ldots \otimes b_{-1} \otimes b_0 \otimes b_1 \otimes \ldots$ be such elements of A. Then clearly for $|n|$ large enough $\theta^m(b) = \ldots \otimes c_{-1} \otimes c_0 \otimes c_1 \otimes \ldots$ will be such that for each i, either $a_i = 1$ or $c_i = 1$ or both, for all $|m| \geq |n|$, hence clearly a and $\theta^m(b)$ commute for all $|m| \geq |n|$, so $\lim_{|n|\to\infty} \|[a, \theta^n(b)]\| = 0$ for these type of elements in A. It can be shown in a similar way that all finite linear combinations of elements of the form $\ldots \otimes a_{-1} \otimes a_0 \otimes a_1 \otimes \ldots$ where only a finite number of the a_i's are not 1 are also strongly asymptotically abelian. Hence we have that $\lim_{|n|\to\infty} \|[a, \theta^n(b)]\| = 0$ for all a, b in a dense subset, say B, of A (see [23] Theorem 11.4.3). To show that it holds for all of A, let a and b be arbitrary elements of A. For any $\varepsilon > 0$ there are elements $c, d \in B$ such that

$$\|a - c\| < \varepsilon \text{ and } \|b - d\| < \varepsilon.$$

Noting that

$$\|[x,y]\| = \|xy - yx\| \le 2\|x\|\|y\|$$

for all $x, y \in A$ and $\|c\| \le \varepsilon + \|a\|$, we then have that

$$\begin{aligned}
\|[a,\theta^n(b)]\| &= \|[a-c,\theta^n(b)] + [c,\theta^n(b)]\| \\
&= \|[a-c,\theta^n(b)] + [c,\theta^n(b-d)] + [c,\theta^n(b)]\| \\
&\le \|[a-c,\theta^n(b)]\| + \|[c,\theta^n(b-d)]\| + \|[c,\theta^n(b)]\| \\
&\le 2\left(\|a-c\|\|b\| + \|c\|\|b-d\|\right) + \|[c,\theta^n(b)]\| \\
&\le 2\left(\|b\|\varepsilon + (\varepsilon + \|a\|)\varepsilon\right) + \|[c,\theta^n(b)]\|.
\end{aligned}$$

Since $\|[c,\theta^n(b)]\|$ can be made arbitrarily small (in fact 0) for $|n|$ large enough, we have $\lim_{|n|\to\infty} \|[a,\theta^n(b)]\| = 0$ as required.

In a similar way we can show that $(A, \omega, \theta, \mathbb{Z})$ is strongly mixing (and hence also weakly mixing). Again, for elements $a = \ldots \otimes a_{-1} \otimes a_0 \otimes a_1 \otimes \ldots$ and $b = \ldots \otimes b_{-1} \otimes b_0 \otimes b_1 \otimes \ldots$ with only a finite number of tensor components not equal to 1, consider $\omega(a\theta^n(b))$. For a and b, suppose that

$$a_{i_1}, a_{i_2}, \ldots, a_{i_k} \ne 1 \text{ and } b_{j_1}, b_{j_2}, \ldots, b_{j_l} \ne 1$$

for some positive integers k and l, and that all other tensor components of a and b are equal to 1. Then, as before, there is an $|n|$ such that for all $|m| \ge |n|$, $\theta^m(b) = \ldots \otimes c_{-1} \otimes c_0 \otimes c_1 \otimes \ldots$ will shift b such that for each i, either $a_i = 1$ or $c_i = 1$ or both. Since θ_i leaves ω_i invariant for all i we have

$$\begin{aligned}
&\omega(a\theta^m(b)) \\
&= \omega_{i_1}(a_{i_1})\omega_{i_2}(a_{i_2})\cdots\omega_{i_k}(a_{i_k})\omega_{b_{j_1}+m}(\theta^m_{j_1}(b_{j_1}))\omega_{b_{j_2}+m}(\theta^m_{j_2}(b_{j_2}))\cdots\omega_{b_{j_l}+m}(\theta^m_{j_l}(b_{j_l})) \\
&= \omega_{i_1}(a_{i_1})\omega_{i_2}(a_{i_2})\cdots\omega_{i_k}(a_{i_k})\omega_{b_{j_1}+m}(b_{j_1})\omega_{b_{j_2}+m}(b_{j_2})\cdots\omega_{b_{j_l}+m}(b_{j_l}) \\
&= \omega(a)\omega(b).
\end{aligned}$$

Similar to the asymptotically abelian case, we can then show that this property extends to all elements a, b in A. We may note that since ω is a factor trace ([23] Section 11.4) strong asymptotically abelianness implies strong mixing.

Chapter 3

Weakly mixing systems

3.1 Characterizations of weak mixing

The characterizations of weak mixing given below will set the stage for our study of weak mixing of all orders in Section 3.3. In this section G need not be abelian and the properties of Følner sequences are only needed in Corollary 3.1.2. The material in this Section is fairly standard, except that we work with the notion "M-weak mixing" (and "M-ergodicity"), which is important in Section 3.3. Also note that throughout this Chapter, the operator ι and the Hilbert space H are those obtained from the GNS construction as discussed in Section 1.4.

Recall that for a group G, $\mathrm{Hom}(G)$ denotes the set of all group homomorphisms $G \to G$.

Proposition 3.1.1. *Let (A, ω, τ, G) be a $*$-dynamical system and $M \subset \mathrm{Hom}(G)$ such that $g \mapsto \omega(a\tau_{\varphi(g)}(b))$ is measurable for all $a, b \in A$ and all $\varphi \in M$. Then the following are equivalent:*

(1) (A, ω, τ, G) is M-weakly mixing relative to (Λ_n).

(2) $(A \otimes \overline{A}, \omega \otimes \overline{\omega}, \tau \otimes \tau, G)$ is M-weakly mixing relative to (Λ_n).

(3) $(A \otimes \overline{A}, \omega \otimes \overline{\omega}, \tau \otimes \tau, G)$ is M-ergodic relative to (Λ_n).

Proof. (2) $\Rightarrow$ (3): Follows immediately from Definition 2.1.7.

(3) $\Rightarrow$ (1): Let $a, b \in A$ and $\varphi \in M$. We have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(a\tau_{\varphi(g)}(b))dg \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega \otimes \overline{\omega}\left((a\otimes 1)(\tau\otimes\tau)_{\varphi(g)}(b\otimes 1)\right) dg \\
&= \omega\otimes\overline{\omega}(a\otimes 1)\omega\otimes\overline{\omega}(b\otimes 1) \\
&= \omega(a)\omega(b).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left|\omega(a\tau_{\varphi(g)}(b))\right|^2 dg \\
&\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(a\tau_{\varphi(g)}(b))\overline{\omega(a\tau_{\varphi(g)}(b))}dg \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\otimes\overline{\omega}\left((a\tau_{\varphi(g)}(b)\otimes(a\tau_{\varphi(g)}(b)\right) dg \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\otimes\overline{\omega}\Big((a\otimes a)\,(\tau\otimes\tau)_{\varphi(g)}\,(b\otimes b)\Big) dg \\
&= \omega\otimes\overline{\omega}(a\otimes a)\omega\otimes\overline{\omega}(b\otimes b) \\
&= |\omega(a)\omega(b)|^2.
\end{aligned}$$

Therefore by Lemma 1.3.6 we have that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left|\omega(a\tau_{\varphi(g)}(b)) - \omega(a)\omega(b)\right|^2 dg = 0,$$

and it follows from Corollary 1.3.5 that (A, ω, τ, G) is M-weakly mixing relative to (Λ_n).

(1) $\Rightarrow$ (2): Given any $\varphi \in M$ and $a, b \in A\otimes\overline{A}$, with $a = \sum_{j=1}^{n} a_j \otimes c_j$ and $b = \sum_{k=1}^{m} b_k \otimes d_k$ where $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ and $c_1, \ldots, c_n, d_1, \ldots, d_m \in \overline{A}$, then

$$
\begin{aligned}
&\left|\omega\otimes\overline{\omega}\left(a(\tau\otimes\tau)_{\varphi(g)}(b)-\omega\otimes\overline{\omega}(a)\omega\otimes\overline{\omega}(b)\right)\right| \\
&=\left|\omega\otimes\overline{\omega}\left(\sum_{j=1}^{n}(a_j\otimes c_j)\sum_{k=1}^{m}\left(\tau_{\varphi(g)}(b_k)\otimes\tau_{\varphi(g)}(d_k)\right)-\omega\otimes\overline{\omega}(\sum_{j=1}^{n}a_j\otimes c_j)\omega\otimes\overline{\omega}(\sum_{k=1}^{m}b_k\otimes d_k)\right)\right| \\
&=\left|\sum_{j=1}^{n}\sum_{k=1}^{m}\omega\otimes\overline{\omega}\left(\left(a_j\tau_{\varphi(g)}(b_k)\right)\otimes\left(c_j\tau_{\varphi(g)}(d_k)\right)\right)-\omega(a_j)\overline{\omega}(c_j)\omega(b_k)\overline{\omega}(d_k)\right| \\
&\leq\sum_{j=1}^{n}\sum_{k=1}^{m}\left(\left|\omega\left(a_j\tau_{\varphi(g)}(b_k)\right)\overline{\omega}\left(c_j\tau_{\varphi(g)}(d_k)\right)-\omega\left(a_j\tau_{\varphi(g)}(b_k)\right)\overline{\omega}(c_j)\overline{\omega}(d_k)\right|\right. \\
&\quad\left.+\left|\omega\left(a_j\tau_{\varphi(g)}(b_k)\right)\overline{\omega}(c_j)\overline{\omega}(d_k)-\omega(a_j)\overline{\omega}(c_j)\omega(b_k)\overline{\omega}(d_k)\right|\right) \\
&\leq\sum_{j=1}^{n}\sum_{k=1}^{m}(\left|\omega\left(a_j\tau_{\varphi(g)}(b_k)\right)\right|\left|\overline{\omega}\left(c_j\tau_{\varphi(g)}(d_k)\right)-\overline{\omega}(c_j)\overline{\omega}(d_k)\right| \\
&\quad+|\overline{\omega}(c_j)\overline{\omega}(d_k)|\left|\omega\left(a_j\tau_{\varphi(g)}(b_k)\right)-\omega(a_j)\omega(b_k)\right|) \\
&\leq\sum_{j=1}^{n}\sum_{k=1}^{m}\left(\|a_j\|_\omega\|b_k\|_\omega\left|\omega\left(c_j\tau_{\varphi(g)}(d_k)\right)-\omega(c_j)\omega(d_k)\right|\right. \\
&\leq\sum_{j=1}^{n}\sum_{k=1}^{m}\left(\|a_j\|_\omega\|b_k\|_\omega\left|\omega\left(c_j\tau_{\varphi(g)}(d_k)\right)-\omega(c_j)\omega(d_k)\right|\right. \\
&\quad\left.+|\overline{\omega}(c_j)\overline{\omega}(d_k)|\left|\omega\left(a_j\tau_{\varphi(g)}(b_k)\right)-\omega(a_j)\omega(b_k)\right|\right)
\end{aligned}
$$

hence (2) follows from Definition 2.1.7(i), as $\left|\omega\left(c_j\tau_{\varphi(g)}(d_k)\right)-\omega(c_j)\omega(d_k)\right|\to 0$ and $\left|\omega\left(a_j\tau_{\varphi(g)}(b_k)\right)-\omega(a_j)\omega(b_k)\right|\to 0$ as $n\to\infty$. □

We can now show that the definition of M-weak mixing relative to a Følner sequence, is independent of the Følner sequence being used:

Corollary 3.1.2. *If a state preserving $*$-dynamical system (A,ω,τ,G) is M-weakly mixing relative to some Følner sequence in G, then it is M-weakly mixing relative to every Følner sequence in G*

Proof. By the Mean Ergodic Theorem the M-ergodicity of a $*$-dynamical system is independent of the Følner sequence being used (see [9]). Hence M-weak mixing is also independent of the Følner sequence by Proposition 3.1.1(1 and 3). □

3.2 A Van der Corput lemma

This Section is devoted to proving a Van der Corput lemma, stated in Theorem 3.2.5. Our proof of this lemma will roughly follow that of [19] over the group $\mathbb{Z}$.

As noted in Section 1.5, we define Hilbert space-valued integrals as follows. For (Y, μ) a measure space and $\mathfrak{H}$ a Hilbert space, consider a bounded $f : \Lambda \to \mathfrak{H}$ with $\Lambda \subset Y$ measurable and $\mu(\Lambda) < \infty$, and $\langle f(\cdot), x\rangle$ measurable for every $x \in \mathfrak{H}$. Due to the fact that f is bounded and Λ has finite measure, $\int_\Lambda \langle f(y), x\rangle\, d\mu(y)$ is a bounded linear functional in x, and we can use the Riesz Representation Theorem to define $\int_\Lambda f d\mu$ by requiring

$$\left\langle \int_\Lambda f d\mu, x \right\rangle := \int_\Lambda \langle f(y), x\rangle\, d\mu(y)$$

for all $x \in \mathfrak{H}$. If there is no ambiguity in the measure being used (as is often the case) we will use the notation $\int_\Lambda f(y)dy = \int_\Lambda f d\mu$. Iterated integrals (when they exist) will be written as $\int_B \int_A f(y, z)dydz$, which of course simply means $\int_B \left[\int_A f(y, z)dy\right] dz$, and similarly for triple integrals.

For second countable topological spaces X, Y, and their Borel σ-algebras S, T, the product σ-algebra obtained from S, T is the same as the Borel σ-algebra of the topological space $X \times Y$. This is needed in order to apply Fubini's theorem, which requires measurability in the product σ-algebra.

Proposition 3.2.1. *Consider a bounded $f : G \to \mathfrak{H}$ with $\mathfrak{H}$ a Hilbert space, such that $\langle f(\cdot), x\rangle$ is Borel measurable for every $x \in \mathfrak{H}$. Then*

$$\lim_{m\to\infty} \left\| \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} f d\mu - \frac{1}{\mu(\Lambda_m)} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_m} \int_{\Lambda_n} f(gh)dhdg \right\| = 0$$

for every n.

Proof. By (2.1) and Fubini's theorem

$$\begin{aligned}\int_{\Lambda_m} \int_{\Lambda_n} \langle f(gh), x\rangle\, dhdg &= \int_{\Lambda_m \times \Lambda_n} \langle f(gh), x\rangle\, d(g, h)\\ &= \int_{\Lambda_n} \int_{\Lambda_m} \langle f(gh), x\rangle\, dgdh\end{aligned}$$

which by definition means that

$$\int_{\Lambda_m} \int_{\Lambda_n} f(gh)dhdg = \int_{\Lambda_n} \int_{\Lambda_m} f(gh)dgdh$$

and in particular these iterated integrals exists. From this and the fact that μ is a right invariant measure, we have

$$\begin{aligned}
&\left\|\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_m} f d\mu - \frac{1}{\mu(\Lambda_m)}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_m}\int_{\Lambda_n} f(gh)dhdg\right\| \\
&= \left\|\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\left[\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_m} f(g)dg\right]dh - \frac{1}{\mu(\Lambda_n)}\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_n}\left[\int_{\Lambda_m} f(gh)dg\right]dh\right\| \\
&= \left\|\frac{1}{\mu(\Lambda_n)}\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_n}\left[\int_{\Lambda_m} f(g)dg - \int_{\Lambda_m} f(gh)dg\right]dh\right\| \\
&= \left\|\frac{1}{\mu(\Lambda_n)}\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_n}\left[\int_{\Lambda_m} f(g)dg - \int_{\Lambda_m h} f(g)dg\right]dh\right\| \\
&= \left\|\frac{1}{\mu(\Lambda_n)}\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_n}\left[\int_{\Lambda_m\setminus(\Lambda_m\cap(\Lambda_m h))} f(g)dg - \int_{(\Lambda_m h)\setminus(\Lambda_m\cap(\Lambda_m h))} f(g)dg\right]dh\right\|.
\end{aligned}$$

But if $b \in \mathbb{R}$ is an upper bound for $\|f(G)\|$, we have

$$\begin{aligned}
&\left\|\int_{\Lambda_m\setminus(\Lambda_m\cap(\Lambda_m h))} f(g)dg - \int_{(\Lambda_m h)\setminus(\Lambda_m\cap(\Lambda_m h))} f(g)dg\right\| \\
&\leq b\mu\left(\Lambda_m\setminus\left(\Lambda_m\cap(\Lambda_m h)\right)\right) + b\mu\left((\Lambda_m h)\setminus\left(\Lambda_m\cap(\Lambda_m h)\right)\right) \\
&= b\mu\left(\Lambda_m\Delta(\Lambda_m h)\right) \\
&\leq b\sup_{h\in\Lambda_n}\mu\left(\Lambda_m\Delta(\Lambda_m h)\right)
\end{aligned}$$

therefore

$$\begin{aligned}
&\left\|\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_m} f d\mu - \frac{1}{\mu(\Lambda_m)}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_m}\int_{\Lambda_n} f(gh)dhdg\right\| \\
&\leq \frac{1}{\mu(\Lambda_m)} b\sup_{h\in\Lambda_n}\mu\left(\Lambda_m\Delta(\Lambda_m h)\right) \\
&\to 0
\end{aligned}$$

as $m \to \infty$. □

Lemma 3.2.2. *Let $\mathfrak{H}$ be a Hilbert space, (Y, μ) a measure space, and $\Lambda \subset Y$ a measurable set with $\mu(\Lambda) < \infty$. Consider an $f : \Lambda \to \mathfrak{H}$ with $\|f(\cdot)\|$ measurable, and $\langle f(\cdot), x\rangle$ measurable for every $x \in \mathfrak{H}$, and with $\int_\Lambda \|f(y)\| dy < \infty$ (which means $\int_\Lambda f d\mu$ exists). Then*

$$\left\|\int_\Lambda f d\mu\right\|^2 \leq \mu(\Lambda)\int_\Lambda \|f(y)\|^2 dy$$

Proof. By definition of $\int_\Lambda f d\mu$,

$$\left|\left|\int_\Lambda f d\mu\right|\right|^2 = \left\langle \int_\Lambda f d\mu, \int_\Lambda f d\mu \right\rangle = \int_\Lambda \left\langle f(y), \int_\Lambda f d\mu \right\rangle dy$$
$$= \int_\Lambda \left[\int_\Lambda \langle f(y), f(z) \rangle \, dz \right] dy.$$

For any $a, b \in \mathfrak{H}$ we have $2 \operatorname{Re} \langle a, b \rangle \leq ||a||^2 + ||b||^2$, and since the object above is real, we have

$$\begin{aligned} \left|\left|\int_\Lambda f d\mu\right|\right|^2 &= \int_\Lambda \left[\int_\Lambda \operatorname{Re} \langle f(y), f(z) \rangle \, dz \right] dy \\ &\leq \frac{1}{2} \int_\Lambda \left[\int_\Lambda \left(||f(y)||^2 + ||f(z)||^2 \right) dz \right] dy \\ &= \mu(\Lambda) \int_\Lambda ||f(y)||^2 \, dy \quad . \end{aligned}$$

□

Proposition 3.2.3. *Consider the situation in Proposition 3.2.1. Assume furthermore that $F : G \times G \to \mathbb{C} : (g, h) \mapsto \langle f(g), f(h) \rangle$ is Borel measurable, and that $\Lambda_1, \Lambda_2 \subset G$ are Borel sets with $\mu(\Lambda_j) < \infty$. Then*

$$\begin{aligned} &\left|\left|\int_{\Lambda_2} \int_{\Lambda_1} f(gh) dh dg\right|\right|^2 \\ &\leq \mu(\Lambda_2) \int_{\Lambda_1} \int_{\Lambda_1} \int_{\Lambda_2} \langle f(gh_1), f(gh_2) \rangle \, dg dh_1 dh_2 \end{aligned}$$

and in particular these integrals exist.

Proof. The double integral exists as in Proposition 3.2.1's proof. Let's now consider the triple integral. Since F is Borel measurable and G 's product is continuous, $(g, h_1) \mapsto \langle f(gh_1), f(gh_2) \rangle$ is Borel measurable on $G \times G = G^2$ and hence measurable in the product σ-algebra on G^2. By Fubini's theorem we have

$$\int_{\Lambda_1} \int_{\Lambda_2} \langle f(gh_1), f(gh_2) \rangle \, dg dh_1 = \int_{\Lambda_1 \times \Lambda_2} \langle f(gh_1), f(gh_2) \rangle \, d(h_1, g)$$

and in particular the iterated integral exists. Furthermore, $G \times G^2 \to G^2 : (h_2, h_1, g) \mapsto (gh_1, gh_2)$ is continuous, so $G \times G^2 \to \mathbb{C} : (h_2, h_1, g) \mapsto$

$\langle f(gh_1), f(gh_2)\rangle$ is measurable in the product σ-algebra of G and G^2. Hence by Fubini's theorem

$$\int_{\Lambda_1}\int_{\Lambda_1\times\Lambda_2} \langle f(gh_1), f(gh_2)\rangle\, d(h_1,g)dh_2 = \int_{\Lambda_1\times\Lambda_1\times\Lambda_2} \langle f(gh_1), f(gh_2)\rangle\, d(h_2,h_1,g)$$

and in particular, the triple integral exists, and we can do the three integrals in any order. By Lemma 3.2.2 it follows that

$$\begin{aligned}
&\left|\left|\int_{\Lambda_2}\int_{\Lambda_1} f(gh)dhdg\right|\right|^2 \\
&\leq \mu(\Lambda_2)\int_{\Lambda_2}\left|\left|\int_{\Lambda_1} f(gh)dh\right|\right|^2 dg \\
&= \mu(\Lambda_2)\int_{\Lambda_2}\left\langle\int_{\Lambda_1} f(gh_1)dh_1, \int_{\Lambda_1} f(gh_2)dh_2\right\rangle dg \\
&= \mu(\Lambda_2)\int_{\Lambda_2}\int_{\Lambda_1}\left\langle f(gh_1), \int_{\Lambda_1} f(gh_2)dh_2\right\rangle dh_1dg \\
&= \mu(\Lambda_2)\int_{\Lambda_2}\int_{\Lambda_1}\int_{\Lambda_1} \langle f(gh_1), f(gh_2)\rangle\, dh_2dh_1dg \\
&= \mu(\Lambda_2)\int_{\Lambda_1}\int_{\Lambda_1}\int_{\Lambda_2} \langle f(gh_1), f(gh_2)\rangle\, dgdh_1dh_2
\end{aligned}$$

and note in particular that the part of this argument after the inequality proves that $g \mapsto \left|\left|\int_{\Lambda_1} f(gh)dh\right|\right|^2$ is measurable (and therefore its square root too), which means that Lemma 3.2.2 does indeed apply to this situation. $\square$

Proposition 3.2.4. *Consider the situation in Proposition 3.2.1. Assume that $F : G\times G \to \mathbb{C} : (g,h) \mapsto \langle f(g), f(h)\rangle$ is Borel measurable. Then $\int_\Lambda \langle f(g), f(gh)\rangle\, dg$ exists for all measurable $\Lambda \subset G$ with $\mu(\Lambda) < \infty$, and all $h \in G$. Assume that*

$$\gamma_h := \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\langle f(g), f(gh)\rangle\, dg$$

exists for all $h \in G$. Then

$$\lim_{m\to\infty}\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_n}\int_{\Lambda_n}\int_{\Lambda_m}\langle f(gh_1), f(gh_2)\rangle\, dgdh_1dh_2 = \int_{\Lambda_n}\int_{\Lambda_n}\gamma_{h_1^{-1}h_2}dh_1dh_2$$

for all n, and in particular these integrals exist.

Proof. The triple integral exists by Proposition 3.2.3. Let b be an upper bound for $(g,h) \mapsto |\langle f(g), f(h)\rangle|$, which exists since f is bounded. Fix any $n \in \mathbb{N}$, and set

$$A_m(h_1,h_2) := \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} \langle f(gh_1), f(gh_2)\rangle \, dg$$

for all $h_1, h_2 \in \Lambda_n$ and all m. Note that $A_m(h_1,h_2)$ exists and is a measurable function of (h_1,h_2) by Fubini's Theorem. Since F is Borel, and $G \to G^2 : g \mapsto (g, gh)$ is continuous, the map $G \to \mathbb{C} : g \mapsto \langle f(g), f(gh)\rangle$ is Borel for every $h \in G$. Now,

$$\begin{aligned}\left|A_m(h_1,h_2) - \gamma_{h_1^{-1}h_2}\right| \leq{}& \frac{1}{\mu(\Lambda_m)} \left|\int_{\Lambda_m} \langle f(gh_1), f(gh_2)\rangle \, dg - \int_{\Lambda_m} \left\langle f(g), f(gh_1^{-1}h_2)\right\rangle dg\right| \\ &+ \left|\frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} \left\langle f(g), f(gh_1^{-1}h_2)\right\rangle dg - \gamma_{h_1^{-1}h_2}\right|\end{aligned}$$

for all $h_1 \in G$ and $h_2 \in G$. But since μ is a right invariant measure

$$\begin{aligned}&\frac{1}{\mu(\Lambda_m)} \left|\int_{\Lambda_m} \langle f(gh_1), f(gh_2)\rangle \, dg - \int_{\Lambda_m} \left\langle f(g), f(gh_1^{-1}h_2)\right\rangle dg\right| \\ &= \frac{1}{\mu(\Lambda_m)} \left|\int_{\Lambda_m h_1} \left\langle f(g), f(gh_1^{-1}h_2)\right\rangle dg - \int_{\Lambda_m} \left\langle f(g), f(gh_1^{-1}h_2)\right\rangle dg\right| \\ &= \frac{1}{\mu(\Lambda_m)} \left|\int_{(\Lambda_m h_1)\setminus\Lambda_m} \left\langle f(g), f(gh_1^{-1}h_2)\right\rangle dg - \int_{\Lambda_m\setminus(\Lambda_m h_1)} \left\langle f(g), f(gh_1^{-1}h_2)\right\rangle dg\right| \\ &\leq \frac{1}{\mu(\Lambda_m)} \left[\int_{(\Lambda_m h_1)\setminus\Lambda_m} \left|\left\langle f(g), f(gh_1^{-1}h_2)\right\rangle\right| dg + \int_{\Lambda_m\setminus(\Lambda_m h_1)} \left|\left\langle f(g), f(gh_1^{-1}h_2)\right\rangle\right| dg\right] \\ &= \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m \Delta (\Lambda_m h_1)} \left|\left\langle f(g), f(gh_1^{-1}h_2)\right\rangle\right| dg \\ &\leq \frac{\mu\left(\Lambda_m \Delta(\Lambda_m h_1)\right)}{\mu(\Lambda_m)} b\end{aligned}$$

for all $h_1 \in G$. Hence $\lim_{m\to\infty} A_m(h_1,h_2) = \gamma_{h_1^{-1}h_2}$.

Furthermore, $|A_m(h_1,h_2)| \leq \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} b dg = b$, which implies that the sequence A_m is dominated by $B : \Lambda_n \times \Lambda_n \to \mathbb{R} : (h_1,h_2) \mapsto b$. Hence $\Lambda_n \times \Lambda_n \ni (h_1,h_2) \mapsto \gamma_{h_1^{-1}h_2}$ is in $L^1(\Lambda_n \times \Lambda_n, \mu \times \mu)$ and

$$\lim_{n\to\infty} \int_{\Lambda_n \times \Lambda_n} A_m(h_1,h_2) d\,(h_1,h_2) = \int_{\Lambda_n\times\Lambda_n} \gamma_{h_1^{-1}h_2} d\,(h_1,h_2)$$

by Lebesgue's dominated convergence theorem. The proposition now follows by Fubini's Theorem. □

Now we can finally state the Van der Corput lemma. It is worth pointing out again that the following result continues to hold even if G is not abelian, but still amenable, and μ is right invariant. The placing of the g in (1.8) is then of course important. Lemma 3.2.6 and Proposition 3.2.7 however require μ to be left rather than right invariant when G is not abelian, while they do not directly use property (1.8).

Theorem 3.2.5. *Consider a bounded $f : G \to \mathfrak{H}$, with $\mathfrak{H}$ a Hilbert space, such that $\langle f(\cdot), x\rangle$ and $\langle f(\cdot), f(\cdot)\rangle : G \times G \to \mathbb{C}$ are Borel measurable (for all $x \in \mathfrak{H}$). Assume*

$$\gamma_h := \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle f(g), f(gh)\rangle \, dg$$

exists for all $h \in G$. Also assume that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)^2} \int_{\Lambda_n} \int_{\Lambda_n} \gamma_{h_1^{-1}h_2} dh_1 dh_2 = 0 \tag{3.1}$$

(note that the integral exists by Proposition 3.2.4). Then

$$\lim_{m\to\infty} \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} f d\mu = 0.$$

Proof. By Proposition 3.2.1 and Proposition 3.2.3 we only have to show that for any $\varepsilon > 0$ there is an n and m_0 such that $|A_{n,m}| < \varepsilon$ for all $m > m_0$ where

$$A_{n,m} := \frac{1}{\mu(\Lambda_n)^2} \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_n} \int_{\Lambda_n} \int_{\Lambda_m} \langle f(gh_1), f(gh_2)\rangle \, dg dh_1 dh_2.$$

But this follows from Proposition 3.2.4 and our assumptions, namely

$$\lim_{n\to\infty} \lim_{m\to\infty} A_{n,m} = \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)^2} \int_{\Lambda_n} \int_{\Lambda_n} \gamma_{h_1^{-1}h_2} dh_1 dh_2 = 0.$$

□

We still need a few refinements regarding condition (3.1):

Lemma 3.2.6. *Let $\Lambda \subset G$ be Borel and $\mu(\Lambda) < \infty$, and $S \subset G$ Borel such that $\Lambda^{-1}\Lambda \subset S$. For a Borel $f : G \to \mathbb{R}^+$ we then have*

$$\int_\Lambda \int_\Lambda f(h_1^{-1}h_2) dh_1 dh_2 \le \mu(\Lambda) \int_S f d\mu.$$

Proof. Let χ denote characteristic functions, and set $\varphi : \Lambda \times \Lambda \to G : (h_1, h_2) \mapsto h_1^{-1} h_2$. Then $f \circ \varphi$ is Borel on $\Lambda \times \Lambda$, and therefore measurable in the product σ-algebra on $\Lambda \times \Lambda$ obtained from Λ 's Borel σ-algebra, since φ is continuous. Let $Y \subset \Lambda^{-1}\Lambda$ be Borel in G. For $W \subset G \times G$, let $W_g := \{h : (g, h) \in W\}$. Then, since $\varphi^{-1}(Y)$ is Borel in $\Lambda \times \Lambda$ and hence Borel in $G \times G$, it follows that $\varphi^{-1}(Y)$ is in the product σ-algebra on $G \times G$, hence we can consider $(\mu \times \mu)\left(\varphi^{-1}(Y)\right) = \int_\Lambda \mu\left(\varphi^{-1}(Y)_g\right) dg$. Now

$$\varphi^{-1}(Y) = \left\{(g, gh) : h \in Y, g \in \Lambda \cap \left(\Lambda h^{-1}\right)\right\} \subset \{(g, gh) : h \in Y, g \in \Lambda\} =: V$$

but $V_g = gY$, therefore $\mu\left(\varphi^{-1}(Y)_g\right) \le \mu(V_g) = \mu(gY) = \mu(Y)$, since μ is left invariant. Hence

$$\begin{aligned}
\int_{\Lambda \times \Lambda} \chi_Y \circ \varphi d(\mu \times \mu) &= (\mu \times \mu)\left(\varphi^{-1}(Y)\right) \\
&\le \mu(\Lambda)\mu(Y) \\
&= \mu(\Lambda) \int_S \chi_Y d\mu
\end{aligned}$$

There is an increasing sequence $f_n : S \to \mathbb{R}^+$ of simple functions converging pointwise to f. From the above we know that

$$\int_{\Lambda \times \Lambda} f_n \circ \varphi d(\mu \times \mu) \le \mu(\Lambda) \int_S f_n d\mu$$

and by applying Lebesgue's monotone convergence first on the right and then of the left of this inequality, we obtain

$$\int_\Lambda \int_\Lambda f\left(h_1^{-1} h_2\right) dh_1 dh_2 = \int_{\Lambda \times \Lambda} f \circ \varphi d(\mu \times \mu) \le \mu(\Lambda) \int_S f d\mu$$

as required, where we have used Fubini's theorem, which holds in this case, since f is non-negative. □

Proposition 3.2.7. *Consider a Borel measurable function $\gamma_h : G \to \mathbb{C}$. Also assume that*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n^{-1}\Lambda_n} |\gamma_h|\, dh = 0.$$

Then

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)^2} \int_{\Lambda_n} \int_{\Lambda_n} \gamma_{h_1^{-1}h_2} dh_1 dh_2 = 0$$

if the iterated integral exists for all $n \ge n_0$ for some n_0.

Proof. Since Λ_n is compact, $\Lambda_n \times \Lambda_n$ is also compact and from the continuity and surjectivity of the function $\Lambda_n \times \Lambda_n \to \Lambda_n^{-1}\Lambda_n : (g,h) \mapsto g^{-1}h$, it follows that $\Lambda_n^{-1}\Lambda_n$ is also compact and hence Borel. So

$$\begin{aligned}\left|\frac{1}{\mu(\Lambda_n)^2}\int_{\Lambda_n}\int_{\Lambda_n}\gamma_{h_1^{-1}h_2}dh_1dh_2\right| &\leq \frac{1}{\mu(\Lambda_n)^2}\int_{\Lambda_n}\int_{\Lambda_n}\left|\gamma_{h_1^{-1}h_2}\right|dh_1dh_2\\ &\leq \frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n^{-1}\Lambda_n}|\gamma_h|\,dh\end{aligned}$$

by Lemma 3.2.6. □

3.3 Weak mixing of all orders

Under the assumption that a $*$-dynamical system is asymptotically abelian, we show that weak mixing implies weak mixing of all orders. For elements of A we'll use the notation $\prod_{j=1}^{k} a_j$ to denote the product $a_1 ... a_k$ in this specific order. Our approach is strongly influenced by that of [20] for the case of a measure theoretic dynamical system and the group $\mathbb{Z}$. The proof is by induction, two steps of which are given by the following:

Proposition 3.3.1. *Given $M \subset \mathrm{Hom}(G)$, let (A,ω,τ,G) denote any C^*-dynamical system such that $G \to A : g \mapsto \tau_{\varphi(g)}(b)$ is continuous for all $a,b \in A$, and for all $\varphi \in M$. We are going to work with a collection of such systems, but with G, (Λ_n) and M fixed. Given $k \in \mathbb{N}$, let $\varphi_1,\dots,\varphi_k$ denote elements of M and $a_0,\dots,a_k$ elements of A. Set $\varphi_0(h) = e$ for all $h \in G$.*

Consider the following statements (where the existence of the integrals contained in each statement forms part of that statement):

1[k]: *The integral $\int_{\Lambda_n}\omega\left(\prod_{j=0}^{k}\tau_{\varphi_j(g)}(a_j)\right)dg$ exists for all $n \geq n_0$ for some n_0, and*

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\left|\omega\left(\prod_{j=0}^{k}\tau_{\varphi_j(g)}(a_j)\right)-\prod_{j=0}^{k}\omega(a_j)\right|dg = 0.$$

2[k]: $$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\left(\prod_{j=0}^{k}\tau_{\varphi_j(g)}(a_j)\right)dg = \prod_{j=0}^{k}\omega(a_j).$$

3[k]: *For $\kappa := \prod_{j=1}^{k}\omega(a_j)$, we have*

$$\lim_{n\to\infty}\left\|\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\iota\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\right)dg-\kappa\Omega\right\| = 0.$$

Then

(1) 1[k] *implies* 2[k].

(2) *If* 3[k] *holds for all M-weakly mixing (A, ω, τ, G) which are M-asymptotically abelian relative to (Λ_n), and all $a_1, ..., a_k$ and all $\varphi_1, ..., \varphi_k$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, ..., k\}$, then* 1[k] *also holds for all M-weakly mixing (A, ω, τ, G) which are M-asymptotically abelian relative to (Λ_n) and all $a_0, ..., a_k$ and all $\varphi_1, ..., \varphi_k$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, ..., k\}$.*

Proof. (1) Trivial.

(2) The strong convergence in $3[k]$ implies weak convergence, i.e.

$$\begin{aligned}\lim_{n\to\infty}\left\langle \iota(a_0^*), \frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n} \iota\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\ dg\right)\right\rangle &= \langle \iota(a_0^*), \kappa \cdot \Omega\rangle \\ &= \langle \iota(a_0^*), \kappa \cdot \iota(1)\rangle \\ &= \omega\left(\kappa a_0\right) \\ &= \prod_{j=0}^{k} \omega(a_j).\end{aligned}$$

Furthermore, by the definition of the integral, and from the assumption that $\tau_{\varphi_0(h)} = \tau_e = id$, we have that

$$\begin{aligned}&\lim_{n\to\infty}\left\langle \iota(a_0^*), \frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n} \iota\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\ dg\right)\right\rangle \\ &= \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\left\langle \iota(a_0^*), \iota\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\right)\right\rangle dg \\ &= \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\left(\prod_{j=0}^{k} \tau_{\varphi_j(g)}(a_j)\right)\ dg,\end{aligned}$$

hence

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\left(\prod_{j=0}^{k} \tau_{\varphi_j(g)}(a_j)\right)\ dg = \prod_{j=0}^{k}\omega(a_j) \tag{3.2}$$

and in particular the integral on the left exists for all $n \geq n_0$ for some n_0. It also then holds that

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\overline{\omega\left(\prod_{j=0}^{k} \tau_{\varphi_j(g)}(a_j)\right)}\ dg = \overline{\prod_{j=0}^{k}\omega(a_j)}. \tag{3.3}$$

To complete the argument we need to show that

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\left|\omega\left(\prod_{j=0}^{k}\tau_{\varphi_j(g)}(a_j)\right)\right|^2 dg = \left|\prod_{j=0}^{k}\omega(a_j)\right|^2. \tag{3.4}$$

First note that Proposition 3.1.1 (1 and 2) imply that the product system $(A\otimes\overline{A},\omega\otimes\overline{\omega},\tau\otimes\tau,G)$ is M-weakly mixing. Since $\|a\otimes b\|\leq\|a\|\,\|b\|$, it is also straightforward to verify that $(A\otimes\overline{A},\omega\otimes\overline{\omega},\tau\otimes\tau,G)$ is M-asymptotically abelian relative to (Λ_n). Hence by (3.3), which applies to all systems which are both M-weakly mixing and M-asymptotically abelian, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\otimes\overline{\omega}\left(\prod_{j=0}^{k}(\tau\otimes\tau)_{\varphi_j(g)}\left(a_j\otimes a_j\right)\right)dg\\ &=\prod_{j=0}^{k}\omega\otimes\overline{\omega}(a_j\otimes a_j)\\ &=\left|\prod_{j=0}^{k}\omega(a_j)\right|^2.\end{aligned}$$

Since

$$\left|\omega\left(\prod_{j=0}^{k}\tau_{\varphi_j(g)}(a_j)\right)\right|^2=\omega\otimes\overline{\omega}\left(\prod_{j=0}^{k}(\tau\otimes\tau)_{\varphi_j(g)}\left(a_j\otimes a_j\right)\right),$$

(3.4) follows, proving $1[k]$ by Corollary 1.3.5 and Lemma 1.3.6. □

Note that the only property of M-weak mixing and M-asymptotic abelianness which is used in Proposition 3.3.1's proof, is that if a system is M-weakly mixing, then so is its product system, and similarly for M-asymptotic abelianness. Proposition 3.3.1 would still hold if we just considered M-weakly mixing systems for example, or systems with some abstract property, call it E, as long as the product system is again an E dynamical system. M-weak mixing and M-asymptotic abelianness will be used more directly in subsequent steps.

In order to complete the induction argument, we need $1[1]$, and that if $2[k-1]$ holds for all relevant systems, then the same is true for $3[k]$. The latter requires some more work, and we will need to specialize the M that we will allow. Firstly note that for an abelian group G and any homomorphisms φ_1 and φ_2 of G, the function $\varphi':G\to G$ defined by

$$\varphi'(g):=\varphi_2(g)^{-1}\varphi_1(g) \tag{3.5}$$

is also a homomorphism of G.

Definition 3.3.2. Let $M \subset \text{Hom}(G)$. We call M *translational* if for all $\varphi_1, \varphi_2 \in M$ with $\varphi_1 \neq \varphi_2$, the homomorphism φ' defined by (3.5) is also in M.

Proposition 3.3.3. *Let (A, ω, τ, G) be a C^*-dynamical system which is M-asymptotically abelian relative to (Λ_n), with M translational. Set $\varphi_0(g) = e$ for all $g \in G$. Assume that for some $k \in \mathbb{N}$*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\left(\prod_{j=0}^{k-1} \tau_{\varphi_j(g)}(a_j)\right) dg = \prod_{j=0}^{k-1} \omega(a_j) \tag{3.6}$$

for all $a_0, ..., a_{k-1} \in A$ and $\varphi_1, ..., \varphi_{k-1} \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, ..., k-1\}$, and in particular the existence of the limit is assumed. Now set

$$u_h := \iota\left(\prod_{j=1}^{k} \tau_{\varphi_j(h)}(a_j)\right) - \kappa\Omega$$

for all $h \in G$, where $\kappa := \prod_{j=1}^{k} \omega(a_j)$, for a given set of $a_j \in A$ and $\varphi_j \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, ..., k\}$. Then

$$\gamma_h := \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle u_g, u_{gh}\rangle \, dg$$

exists (where $\langle \cdot, \cdot \rangle$ is taken in H), and

$$\gamma_h = \prod_{j=1}^{k} \omega\left(a_j^*\left(\tau_{\varphi_j(h)}(a_j)\right)\right) - |\kappa|^2$$

for all $h \in G$.

Proof. We have

$$
\begin{aligned}
&\langle u_g, u_{gh}\rangle \\
&= \left\langle \iota\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j) - \kappa\cdot 1\right),\ \iota\left(\prod_{j=1}^{k}\tau_{\varphi_j(gh)}(a_j) - \kappa\cdot 1\right)\right\rangle \\
&= \omega\left(\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j) - \kappa\cdot 1\right)^{*}\left(\prod_{j=1}^{k}\tau_{\varphi_j(gh)}(a_j) - \kappa\cdot 1\right)\right) \\
&= \omega\left(\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\right)^{*}\left(\prod_{j=1}^{k}\tau_{\varphi_j(gh)}(a_j)\right)\right) - \overline{\kappa}\omega\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\right) \\
&\quad - \kappa\omega\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\right)^{*} + |\kappa|^2.
\end{aligned}
$$

We now consider the terms in the last expression each separately:

(a) We see that

$$
\begin{aligned}
&\omega\left(\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\right)^{*}\left(\prod_{j=1}^{k}\tau_{\varphi_j(gh)}(a_j)\right)\right) \\
&= \omega\left(\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\right)^{*}\left(\prod_{j=1}^{k}\tau_{\varphi_j(g)}\left(\tau_{\varphi_j(h)}(a_j)\right)\right)\right) \\
&= \omega\left(\prod_{j=-k}^{k}\tau_{\varphi_{|j|}(g)}(b_j)\right)
\end{aligned}
$$

where $b_j = \begin{cases} a_{|j|}^{*}, & \text{if } -k \le j \le -1; \\ 1, & \text{if } j = 0; \\ \tau_{\varphi_j(h)}(a_j), & \text{if } 1 \le j \le k. \end{cases}$

For $T_{-k}, ..., T_k \in A$ with $\|T_j\| \le c$, one has (see Lemma 7.4 in [28]) that

$$
\left\|\prod_{j=-k}^{k}T_j - T_0\left(\prod_{j=1}^{k}(T_{-j}T_j)\right)\right\| \le c^{2k-1}\sum_{r=1}^{k}\sum_{l=-(r-1)}^{r-1}\|[T_{-r}, T_l]\|
$$

and applying this to $T_j = \tau_{\varphi_{|j|}(g)}(b_j)$ with $c := \max_{-k\le j\le k}\|b_j\|$, and keeping in mind that $\|\tau_g(a)\| = \|a\|$, it follows that

$$
\left\|\prod_{j=-k}^{k}\tau_{\varphi_{|j|}(g)}(b_j) - \prod_{j=1}^{k}\left(\tau_{\varphi_j(g)}(b_{-j}b_j)\right)\right\| \le c^{2k-1}\sum_{r=1}^{k}\sum_{l=-(r-1)}^{r-1}\left\|\left[b_{-r}, \tau_{\varphi_r(g)^{-1}\varphi_{|l|}(g)}(b_l)\right]\right\|.
$$

Since M is translational, and (A, ω, τ, G) is asymptotically abelian relative to (Λ_n), it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\left(\prod_{j=-k}^{k} \tau_{\varphi_{|j|}(g)}(b_j)\right) dg \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(b_{-j}b_j)\right) dg \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\left(\prod_{j=1}^{k} \tau_{\varphi_1(g)}\left(\tau_{\varphi_1(g)^{-1}}\left(\tau_{\varphi_j(g)}(b_{-j}b_j)\right)\right)\right) dg \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\left(\tau_{\varphi_1(g)}\left(\prod_{j=1}^{k} \tau_{\varphi_1(g)^{-1}\varphi_j(g)}(b_{-j}b_j)\right)\right) dg \\
&= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\left(\prod_{j=0}^{k-1} \tau_{\varphi'_j(g)}(b_{-(j+1)}b_{j+1})\right) dg \\
&= \prod_{j=0}^{k-1} \omega\left(b_{-(j+1)}b_{j+1}\right) \\
&= \prod_{j=1}^{k} \omega\left(a_j^* \tau_{\varphi_j(h)}(a_j)\right)
\end{aligned}$$

by using (3.6), where $\varphi'_j(g) := \varphi_1(g)^{-1}\varphi_{j+1}(g)$ for all $g \in G$ and $j = 0, \ldots, k-1$, so $\varphi'_j \in M$ for $j = 1, \ldots, k-1$ since M is translational. Note that $\varphi'_j \neq \varphi'_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k-1\}$ and $\varphi'_0(g) = e$ for all $g \in G$, as required in our assumption. Hence

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\left(\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\right)^* \left(\prod_{j=1}^{k} \tau_{\varphi_j(gh)}(a_j)\right)\right) dg = \prod_{j=1}^{k} \omega\left(a_j^* \tau_{\varphi_j(h)}(a_j)\right).$$

(b) It follows as in (a) that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \overline{\omega\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\right)} dg &= \overline{\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\left(\prod_{j=0}^{k-1} \tau_{\varphi'_j(g)}(a_{j+1})\right) dg} \\
&= \overline{\prod_{j=0}^{k-1} \omega(a_{j+1})} \\
&= \overline{\kappa}
\end{aligned}$$

by assumption.

(c) Lastly, using similar arguments as before,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\left(\prod_{j=1}^{k}\tau_{\varphi_j(gh)}(a_j)\right)dg\\
&=\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\left(\prod_{j=0}^{k-1}\tau_{\varphi_j'(g)}\left(\tau_{\varphi_{j+1}(h)}(a_{j+1})\right)\right)dg\\
&=\prod_{j=0}^{k-1}\omega(\tau_{\varphi_{j+1}(h)}(a_{j+1}))\\
&=\prod_{j=0}^{k-1}\omega(a_{j+1})\\
&=\kappa.
\end{aligned}$$

(d) From (a)-(c)

$$\gamma_h=\prod_{j=1}^{k}\omega\left(a_j^*\left(\tau_{\varphi_j(h)}(a_j)\right)\right)-|\kappa|^2$$

and in particular γ_h exists. □

Next we prove that weak mixing implies weak mixing of all orders. This is where our Van der Corput lemma is finally applied, along with Propositions 3.3.1 and 3.3.3.

Theorem 3.3.4. *Assume that there exists a Følner sequence (Λ_n) in G, satisfying the Tempelman condition, and such that $(\Lambda_n^{-1}\Lambda_n)$ is also a Følner sequence in G. Let $M\subset Hom(G)$ be translational. Let (A,ω,τ,G) be an M-weakly mixing C^*-dynamical system which is M-asymptotically abelian with respect to (Λ_n). Assume furthermore that $G\to A : g\mapsto\tau_{\varphi(g)}(a)$ is continuous in the norm topology on A for all $\varphi\in M$ and all $a\in A$. Then*

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\left|\omega\left(\prod_{j=0}^{k}\tau_{\varphi_j(g)}(a_j)\right)-\prod_{j=0}^{k}\omega(a_j)\right|dg=0$$

for any $a_j\in A$ and any $\varphi_1,...,\varphi_k\in M$ with $\varphi_j\neq\varphi_l$ when $j\neq l$ for $j,l\in\{1,...,k\}$, and with $\varphi_0(g)=e$ for all $g\in G$.

Proof. We need to complete the induction argument started in Proposition 3.3.1, and we will continue using its notation and that of Proposition 3.3.3. Since $G \to A : g \mapsto \tau_{\varphi(g)}(a)$ is continuous, so is $G \to A : g \mapsto \prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)$ in the norm topology on A. We then also have that $G \to H : g \mapsto \iota\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\right)$ is continuous, since $\|\iota(a)\| \leq \|a\|$. It follows that

$$G \times G \to \mathbb{R} : (g,h) \mapsto \left\langle \iota\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\right), \iota\left(\prod_{j=1}^{k} \tau_{\varphi_j(h)}(a_j)\right)\right\rangle$$

is continuous. Keep in mind that $\omega\left(\left(\prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)}\right)^* \left(\prod_{j=1}^{k} f_j \circ T_{\varphi_j(h)}\right)\right) = \left\langle \iota\left(\prod_{j=0}^{k} \tau_{\varphi_j(g)}(a_j)\right), \iota\left(\prod_{j=0}^{k} \tau_{\varphi_j(h)}(a_j)\right)\right\rangle$. Now we write

$$u_g := \iota\left(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\right) - \kappa\Omega$$

for all $g \in G$, where $\kappa := \prod_{j=1}^{k} \omega(a_j)$. It follows that $G \times G \to \mathbb{C} : (g,h) \mapsto \langle u_g, u_h\rangle$ is continuous and therefore Borel measurable. Note that $g \mapsto \langle u_g, x\rangle$ is also Borel measurable for all $x \in H$. Furthermore, $G \to H : g \mapsto u_g$ is bounded. (We need these properties, since we will be applying Theorem 3.2.5 to the function $g \mapsto u_g$.) Since $\mu(\Lambda_n^{-1}\Lambda_n) \leq c\mu(\Lambda_n)$, and we have M-weak mixing relative to $(\Lambda_n^{-1}\Lambda_n)$ by Corollary 3.1.2, it follows that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n^{-1}\Lambda_n} \left|\omega\left(a\tau_{\varphi(g)}(b)\right) - \omega(a)\omega(b)\right| dg = 0 \tag{3.7}$$

for all $a, b \in A$ and $\varphi \in M$. By Proposition 3.3.3, assuming $2[k-1]$ for all M-weakly mixing C*-dynamical systems, which are M-asymptotically abelian relative to (Λ_n) (for the given G and (Λ_n)), and of course for all $a_0, ..., a_{k-1}$ and all $\varphi_1, ..., \varphi_{k-1} \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, ..., k-1\}$, we have

$$\gamma_h := \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle u_g, u_{gh}\rangle\, dg = \prod_{j=1}^{k} \omega\left(a_j^*\left(\tau_{\varphi_j(h)}(a_j)\right)\right) - \prod_{j=1}^{k} |\omega(a_j)|^2$$

for any $a_1, ..., a_k$ and all $\varphi_1, ..., \varphi_k \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, ..., k\}$, for all $h \in G$. Using the following identity (also see Section 4 in [20])

$$\prod_{j=1}^{k} c_j - \prod_{j=1}^{k} d_j = \sum_{j=1}^{k} \left(\prod_{l=1}^{j-1} c_l\right)(c_j - d_j)\left(\prod_{l=j+1}^{k} d_l\right) \tag{3.8}$$

which holds in any algebra and is easily verified by induction, it follows that

$$\int_{\Lambda_m^{-1}\Lambda_m} |\gamma_h|\, dh \leq \sum_{j=1}^{k} A_j \prod_{l=j+1}^{k} |\omega(a_l)|^2 \int_{\Lambda_m^{-1}\Lambda_m} \left|\omega\left(a_j^*\left(\tau_{\varphi_j(h)}(a_j)\right)\right) - |\omega(a_j)|^2\right| dh$$

where $A_j := \sup_{h\in G}\left|\prod_{l=1}^{j-1}\omega\left(a_l^*(\tau_{\varphi_l(h)}(a_l))\right)\right| \leq \prod_{l=1}^{j-1}\|a_l\|^2$.

Note that $\int_{\Lambda_m^{-1}\Lambda_m}|\gamma_h|\,dh$ exists, since the integrand is continuous. Hence

$$\lim_{m\to\infty}\frac{1}{\mu(\Lambda_m)}\int_{\Lambda_m^{-1}\Lambda_m}|\gamma_h|\,dh = 0$$

by (3.7). From Proposition 3.2.7 and Theorem 3.2.5 we then have

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n} u_g dg = 0$$

i.e., $3[k]$ holds for all M-weakly mixing C*-dynamical systems, which are M-asymptotically abelian relative to (Λ_n), and all $a_1, ..., a_k$ and all $\varphi_1, ..., \varphi_k \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, ..., k\}$. But $1[1]$ holds for all $a_0, a_1 \in A$ and all $\varphi \in M$ for all M-weakly mixing C*-dynamical systems, which are M-asymptotically abelian relative to (Λ_n), by Definition 2.1.7(i), completing the induction argument started in Proposition 3.3.1, and proving $1[k]$ for all $k \in \mathbb{N}$. □

Note that if $(\Lambda_n^{-1}\Lambda_n)$ is also Følner in G, then the assumption that the system be M-weakly mixing relative to $(\Lambda_n^{-1}\Lambda_n)$ can be dropped because of Corollary 3.1.2. If $(\Lambda_n^{-1}\Lambda_n)$ does not have the properties required in Theorem 3.3.4, for example if the system is not M-weak mixing relative to $(\Lambda_n^{-1}\Lambda_n)$, but there is some other uniformly Følner sequence (Λ'_n) such that $(\Lambda_n'^{-1}\Lambda'_n)$ does have the required properties, then we can replace (Λ_n) by (Λ'_n) because of Corollary 3.1.2, to get weak mixing of all orders relative to (Λ'_n).

We now briefly consider examples of Følner sequences with the required properties.

In the simple case where $G = \mathbb{Z}$ with the counting measure μ, and $\Lambda_n = \{-n, \ldots, n\}$ which is uniformly Følner in $\mathbb{Z}$, we have $\Lambda_n^{-1}\Lambda_n = \{-2n, \ldots, 2n\}$, so $\mu(\Lambda_n) \leq \mu(\Lambda_n^{-1}\Lambda_n) \leq 2\mu(\Lambda_n)$ for $n \geq 1$, and if the dynamical system is weak mixing relative to $\{\Lambda_n\}$, then it is also weak mixing relative to $\Lambda_n^{-1}\Lambda_n = \Lambda_{2n}$. Hence the conditions of Theorem 3.3.4 are satisfied. Furthermore, if the system is only weak mixing relative to $\{0, \ldots, n\}$ (so we are working over the semigroup $\mathbb{N}\cup\{0\}$) and T is injective, then it is easily seen that it is also weak mixing relative to Λ_n. This implies the usual version of weak mixing of all orders when working on the semigroup $\mathbb{N}\cup\{0\}$, for an injective T.

As another example of a sequence with the properties in Theorem 3.3.4, let Λ_m be the closed ball of radius m in $\mathbb{R}^q$ for any positive integer q. Note that (Λ_m) is a Følner sequence in $\mathbb{R}^q$. Then $\Lambda_m^{-1}\Lambda_m = \Lambda_{2m}$, which means that M-weak mixing relative to (Λ_m), implies M-weak mixing relative to $(\Lambda_m^{-1}\Lambda_m)$, while $\mu(\Lambda_m^{-1}\Lambda_m) = 2^q\mu(\Lambda_m)$, as is required in Theorem 3.3.4.

Concerning the assumption that M is translational, a simple example would be of the following type: Use the group $G = \mathbb{R}^q$. Let M be all $q \times q$ non-zero diagonal real matrices acting as linear operators on $\mathbb{R}^q$. (We exclude the zero matrix simply because this would make M-weak mixing impossible.) Then M is a translational set of homomorphisms of $\mathbb{R}^q$. The same is true if we drop the condition that the matrices be diagonal. Similarly if we work with $\mathbb{Z}^q$ instead of $\mathbb{R}^q$ and use matrices over the integers.

Chapter 4

The Szemerédi property

4.1 Compact systems

The notion of compactness can be extended to noncommutative dynamical systems consisting of a $*$-algebra A, a positive linear functional ω, and an evolution of A over a general semigroup G, as seen in Definition 2.1.11. A generalization of Furstenberg's Theorem to C^*-dynamical systems was initiated in [28] which included a discussion of compact systems in the case of an evolution over $\mathbb{N}$. In this Section we prove the Szemerédi property (see Definition 2.1.17) for a compact system for which A is a C^*-algebra, ω is tracial, i.e. $\omega(ab) = \omega(ba)$ for all $a, b \in \mathfrak{A}$, and G is, as before, a group with a right invariant measure containing a Følner sequence.

In this Section we obtain some recurrence results in seminormed spaces which are used in Theorem 4.1.6 to prove the Szemerédi property for compact systems. Our proof of the Szemerédi property follows the basic structure of the one given in [20], but we have to take into account certain subtleties arising from working in a noncommutative C^*-algebra rather than in the abelian algebra $L^\infty(\nu)$ used in [20], and with more general groups and semigroups than $\mathbb{Z}$ and $\mathbb{N}$. Also, since we work via abstract seminormed spaces, the structure of the proof becomes clearer. We first introduce some notation and terminology.

A linear functional ω on a $*$-algebra A is called positive if $\omega(a^*a) \geq 0$ for all $a \in A$. This allows us to define a seminorm $\|\cdot\|_\omega$ on A by

$$\|a\|_\omega := \sqrt{\omega(a^*a)}$$

for all $a \in A$, as is easily verified using the Cauchy-Schwarz inequality for positive linear functionals.

Proposition 4.1.1. *Let X be a metric space and $B \subset X$ totally bounded* *(see Definition 2.1.10). For any $\varepsilon > 0$ there exists a maximal set $V \subset B$ (maximal in the sense of cardinality or number of elements) that is ε-separated. Furthermore, if $B \neq \varnothing$, then V is finite with $|V| > 0$.*

Proof. If B is empty we are done, so assume that $B \neq \varnothing$. Let $M_{\frac{\varepsilon}{2}} \subset X$ be an $\frac{\varepsilon}{2}$−net for B, i.e.

$$B \subset \bigcup_{x \in M_{\frac{\varepsilon}{2}}} N(x, \frac{\varepsilon}{2}).$$

Let $n := |M_{\frac{\varepsilon}{2}}|$ be the number of elements of $M_{\frac{\varepsilon}{2}}$. If we choose $m > n$ elements of B, the drawer principle (also the pigeonhole or Dirichlet's box principle) states that two of them are in the same ball $N(x, \frac{\varepsilon}{2})$, $x \in M_{\frac{\varepsilon}{2}}$ and hence at distance less than ε from one another. Without loss of generality we can assume that m is finite. Hence any $\frac{\varepsilon}{2}$-separated subset of B contains at most n elements. Let $\mathcal{V}$ be the collection of all ε-separated subsets of B. Let V be any one of $\mathcal{V}$'s elements with maximum cardinality. Also note that $\mathcal{V}$ contains all the 1-point subsets (which are ε-separated) of B. □

Proposition 4.1.2. *Let K be a semigroup, $(X, \|\cdot\|)$ a seminormed space, and $U_g : X \to X$ a linear map for each $g \in K$ such that $U_g U_h = U_{gh}$ and $\|U_g x\| \geq \|x\|$ for all $g, h \in K$ and $x \in X$. Suppose that $B_{x_0} := \{U_g x_0 : g \in K\}$ is totally bounded in $(X, \|\cdot\|)$ for some $x_0 \in X$. Then for each $\varepsilon > 0$, the set*

$$E := \{g \in K : \|U_g x_0 - x_0\| < \varepsilon\}$$

is relatively dense in K.

Proof. Since B_{x_0} is totally bounded in $(X, \|\cdot\|)$, there is a maximal

$$V = \{U_{g_1} x_0, ..., U_{g_r} x_0\},$$

with $U_{g_j} x_0 \neq U_{g_l} x_0$ whenever $j \neq l$, which is ε-separated. But

$$\|U_{g' g g_j} x_0 - U_{g' g g_l} x_0\| \geq \|U_{g_j} x_0 - U_{g_l} x_0\|$$

for any $g, g' \in K$, hence

$$V_{g'g} := \{U_{g' g g_1} x_0, ..., U_{g' g g_r} x_0\}$$

is ε-separated, with r elements. Since $V_{g'g} \subset B_{x_0}$, it is also maximally ε-separated in B_{x_0}. But $U_{g'} x_0 \in B_{x_0}$, therefore

$$\|U_{g g_j} x_0 - x_0\| \leq \|U_{g' g g_j} x_0 - U_{g'} x_0\| < \varepsilon$$

for some $j \in \{1, \ldots, r\}$. Hence, for each $g \in K$ there exists an $h \in \{gg_1, \ldots, gg_r\}$ such that $\|U_h x_0 - x_0\| < \varepsilon$, i.e.

$$E \cap \{gg_1, \ldots, gg_r\} \neq \varnothing$$

for all $g \in K$, and so E is relatively dense in K. □

Corollary 4.1.3. *Let (A, ω, τ, K) be a compact system and let $m_0, ..., m_k \in \mathbb{N} \cup \{0\}$. For any $\varepsilon > 0$ and $a \in A$, the set*

$$E := \{g \in K : \|\tau_{g^{m_j}}(a) - a\|_\omega < \varepsilon \text{ for } j = 0, ..., k\}$$

is then relatively dense in K, where we write $\tau_{g^0}(a) \equiv a$.

Proof. Without loss we can assume that none of the m_j 's are zero. Then the result follows from Proposition 4.1.2 with ε replaced by $\varepsilon / \max\{m_0, \ldots, m_k\}$, since for every $j = 0, ..., k$ we have

$$\begin{aligned}
&\|\tau_{g^{m_j}}(a) - a\|_\omega \\
&\leq \|\tau_{g^{m_j}}(a) - \tau_{g^{m_j-1}}(a)\|_\omega + \|\tau_{g^{m_j-1}}(a) - \tau_{g^{m_j-2}}(a)\|_\omega + \ldots + \|\tau_g(a) - a\|_\omega \\
&= \|\tau_{g^{m_j-1}}[\tau_g(a) - a]\|_\omega + \|\tau_{g^{m_j-2}}[\tau_g(a) - a]\|_\omega + \ldots + \|\tau_g(a) - a\|_\omega \\
&= m_j \|\tau_g(a) - a\|_\omega \\
&< \varepsilon
\end{aligned}$$

for all $g \in K$ for which $\|\tau_g(a) - a\|_\omega < \varepsilon / \max\{m_0, \ldots, m_k\}$. □

A positive linear functional ω on a C*-algebra A is bounded, and without loss we can assume that $||\omega|| = 1$ (the case $\omega = 0$ being trivial), i.e. ω is a *state* on A. By the Cauchy-Schwarz inequality we have

$$|\omega(ab)| \leq ||a^*||_\omega \, ||b||_\omega \leq \sqrt{||aa^*||} \, ||b||_\omega = ||a|| \, ||b||_\omega \, .$$

A *trace* is defined to be a state ω on a C*-algebra $\mathfrak{A}$ such that $\omega(ab) = \omega(ba)$ for all $a, b \in \mathfrak{A}$. Note that from the previous inequality we then have

$$|\omega(abc)| = |\omega(cab)| \leq \|a\| \|b\|_\omega \|c\|$$

for all $a, b, c \in A$. This fact is used in the proof of Lemma 4.1.4. The set of positive elements of A will be denoted by A^+.

Lemma 4.1.4. *Let A be a C^*-algebra and ω a trace on A. Suppose that $b \in A^+$, $\|b\| \leq 1$ and $\omega(b) > 0$. Let $k \in N \cup \{0\}$, then $\omega(b^{k+1}) > 0$ so we can choose $\varepsilon > 0$ such that $\varepsilon < \omega(b^{k+1})$. Consider $c_0, \ldots, c_k \in A$ such that $\|c_j\| \leq 1$ and $\|c_j - b\|_\omega < \varepsilon/(k+1)$ for $j = 0, \ldots, k$. Then*

$$\left|\omega\left(\prod_{j=0}^{k} c_j\right)\right| > \omega\left(b^{k+1}\right) - \varepsilon > 0.$$

Proof. We have $\omega(b^{k+1}) > 0$ by using the Gelfand representation of the abelian C*-algebra B generated by b, restricting ω to B and then using Riesz's theorem to represent ω by a positive measure on the locally compact Hausdorff space appearing in the Gelfand representation. Furthermore,

$$\begin{aligned}
\left|\omega\left(\prod_{j=0}^{k} c_j\right) - \omega(b^{k+1})\right| &= \left|\omega\left(\prod_{j=0}^{k} c_j - \prod_{j=0}^{k} b\right)\right| \\
&= \left|\omega\left(\sum_{j=0}^{k}\left(\prod_{l=0}^{j-1} c_l\right)(c_j - b)\left(\prod_{l=j+1}^{k} b\right)\right)\right| \\
&\leq \sum_{j=0}^{k}\left(\left\|\prod_{l=0}^{j-1} c_l\right\| \|c_j - b\|_\omega \left\|b^{k-j}\right\|\right) \\
&\leq \sum_{j=0}^{k} \|c_j - b\|_\omega \\
&< \varepsilon,
\end{aligned}$$

where we've used (3.8). Hence

$$\left|\omega\left(\prod_{j=0}^{k} c_j\right)\right| > \omega(b^{k+1}) - \varepsilon > 0.$$

□

Corollary 4.1.5. *Let (A, ω, τ, K) be a compact C^*-system with ω a trace. Suppose that $a \in A^+$, and $\omega(a) > 0$. Take any $m_0, \ldots, m_k \in \mathbb{N} \cup \{0\}$ and any $\varepsilon > 0$ with $\varepsilon < \omega\left(a^{k+1}\right)$. Then there exists a relatively dense set E in K such that*

$$\left|\omega\left(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\right)\right| > \omega\left(a^{k+1}\right) - \varepsilon > 0$$

for all $g \in E$.

Proof. Since $\omega(a) > 0$, $||a|| > 0$, so we can set $b := a/\,||a||$. For $c_j := \tau_{g^{m_j}}(b)$ we have $||c_j|| \leq ||b|| = 1$, so from Lemma 4.1.4 it follows that

$$\left|\omega\left(\prod_{j=0}^{k}\tau_{g^{m_j}}(b)\right)\right| > \omega\left(b^{k+1}\right) - \frac{\varepsilon}{\|a\|^{k+1}}$$

for every $g \in K$ for which $\|\tau_{g^{m_j}}(b) - b\|_\omega < \varepsilon/\,\|a\|^{k+1}(k+1)$ for all $j = 0, ..., k$. By Corollary 4.1.3 this set of g 's is relatively dense in K. □

So far in this Section we have not used Følner sequences. However, for the remainder of this Section we again make use of such sequences, so let G and (Λ_n) be as in Section 1.2. We can make a simple refinement without complicating the proofs, namely in the rest of this Section let K be a Borel set in G which forms a semigroup and contains each Λ_n. We then say that (Λ_n) is a Følner sequence in K. (A trivial example is $G = \mathbb{Z}$, $\Lambda_n = \{1, ..., n\}$ and $K = \mathbb{N}$.) This is just to make clear that only semigroup structure is used in this Section. In the next Section, where Theorem 4.1.6 below is applied, we will of course take K to be G. Let Σ denote the σ-algebra of Borel sets of G that are contained in K.

Remark. It is also interesting to note that the arguments below do not require the semigroup K to be abelian, however this would require the existence of a slightly different type of Følner sequence. Namely, if G were non-abelian, and μ right invariant, one would have to assume the existence of a sequence (or a net) of Borel sets (Λ_n) of G (which are contained in K) with $0 < \mu(\Lambda_n) < \infty$ such that $\lim_{n\to\infty}\mu(\Lambda_n\Delta(g\Lambda_n))/\mu(\Lambda_n) = 0$ or all $g \in K$. Note that g is to the left of the set despite μ being right invariant.

Finally we reach the goal of this Section, namely a 'Szemerédi property' for compact C^*-systems:

Theorem 4.1.6. *Let (A, ω, τ, K) be a compact C^*-system with ω a trace and K a Borel measurable semigroup in G such that $\Lambda_n \subset K$ for every n, where G and (Λ_n) are as in Section 1.2. Let $a \in A^+$ with $\omega(a) > 0$. Take any $m_0, \ldots, m_k \in \mathbb{N} \cup \{0\}$. Assume that $g \mapsto \omega\left(\prod_{j=0}^{k}\tau_{g^{m_j}}(a)\right)$ and $g \mapsto \|\tau_{g^{m_j}}(a) - a\|_\omega$ are Σ-measurable on K for $j = 0, 1, \ldots, k$. Then there exists a Følner sequence (Λ'_n) in K such that*

$$\liminf_{n\to\infty}\frac{1}{\mu(\Lambda'_n)}\int_{\Lambda'_n}\left|\omega\left(\prod_{j=0}^{k}\tau_{g^{m_j}}(a)\right)\right|d\mu(g) > 0.$$

Proof. This follows from Lemma 1.2.12(3) and Corollary 4.1.5, since $E = \{g \in K : \|\tau_{g^{m_j}}(a) - a\|_\omega < \varepsilon$ for $j = 0, ..., k\}$ is Σ-measurable. $\square$

Note that if for example K is a topological semigroup and we assume that $g \mapsto \tau_g(A)$ is continuous in $\mathfrak{A}$'s norm, then both $g \mapsto \omega\left(\prod_{j=0}^{k} \tau_{g^{m_j}}(A)\right)$ and $g \mapsto \|\tau_{g^{m_i}}(A) - A\|_\omega$ are continuous and hence Borel measurable.

The Szemerédi property for a compact measure preserving dynamical system (X, Σ, ν, T) with evolution over $\mathbb{N}$ is a special case of this theorem, but note that T need not be invertible in this case. Just let $\omega(f) := \int_X f d\nu$ and $\tau(f) := f \circ T$ for all $f \in \mathfrak{A} := B_\infty(\Sigma)$, let $\tau_n = \tau^n$ for $n \in \mathbb{N}$, set $\Lambda_N := \{1, ..., N\}$ for all $N \in \mathbb{N}$, and let $A = f$ be a positive function in $B_\infty(\Sigma)$ which is not ν-a.e. zero. Keep in mind that the conclusion of Lemma 1.2.12 (and hence that of Theorem 4.1.6) holds for this choice of (Λ_N), as is well known. The condition $||\tau(f)|| \leq ||f||$ follows directly from τ's definition, while $||\tau(f)||_\omega = ||f||_\omega$ expresses the fact that T is measure preserving, namely $\nu \circ T^{-1} = \nu$ as set functions on Σ. More specifically (1.2) is obtained for these assumptions by taking f to be the characteristic function χ_V of a set $V \in \Sigma$ with $\nu(V) > 0$ and setting $m_j = j$.

4.2 Compact factors and ergodic systems

In measure theoretic ergodic theory it is well known that a system is weakly mixing if and only if it contains no nontrivial compact factors. Compactness means that the orbits of the system in the corresponding L^2-space are totally bounded.

In this Section we show that the measure theoretic result can be extended to noncommutative ergodic theory, where the measure space (and its algebra of L^∞-functions) is replaced by a σ-finite von Neumann algebra and a faithful normal state. The two main ingredients of the proof are a so-called "proper value theorem" due to Størmer (Theorem 2.5 in [32]), and the splitting theorem of Jacobs-Deleeuw-Glicksberg (see Section 2.4 in [25]). Once this is done, we prove our final result regarding the Szemerédi property in ergodic systems.

Note that Definitions 2.1.13 to 2.1.17 are used in the results and discussion below. Let $B(H)$ denote the algebra of all bounded linear operators in the Hilbert space H, and let S' denote the commutant of a set $S \subset B(H)$. Let B denote the set of all eigenoperators of α together with the zero operator, and let C be the $*$-algebra generated by B. Set $N = C''$, hence N is a Von Neumann algebra contained in R.

Proposition 4.2.1. *$(N, \omega_\Omega, \alpha)$ is a compact factor of $(R, \omega_\Omega, \alpha)$.*

Proof. From the fact that α_g is a $*$-homomorphism for all $g \in G$, it follows that B is closed under adjoints, products and scalar multiples, if we note that

$$\alpha_g(a^*) = (\alpha_g(a))^* = \lambda_a(g)^* a^* = \overline{\lambda}_a(g) a^*$$

and

$$\alpha_g(\beta ab) = \beta\alpha_g(a)\alpha_g(\beta b) = \beta\lambda_a(g)\lambda_b(g)ab$$

for any scalar β and all $a, b \in B$. Hence clearly

$$C = \left\{ \sum_{j=1}^{n} a_j : a_1, ..., a_n \in B, n > 0 \right\} \tag{4.1}$$

but for $a \in B$ we have $\alpha_h(\alpha_g(a)) = \lambda_a(h)\alpha_g(a)$, hence $\alpha_g(B) \subset B$ and $\alpha_g(C) \subset C$ for all g. By Von Neumann's density theorem C is strongly dense in N, hence there exists a net (a_γ) in C such that $a_\gamma x \to ax$ for all $x \in H$. Therefore

$$\begin{aligned}
\lim_\gamma \langle x, \alpha_g(a_\gamma)y\rangle &= \lim_\gamma \left\langle U_g^* x, a_\gamma U_g^* y \right\rangle \\
&= \left\langle U_g^* x, a U_g^* y \right\rangle \\
&= \langle x, \alpha_g(a)y\rangle
\end{aligned}$$

for all $x, y \in H$ and all $g \in G$. In other words $\alpha_g(a_\gamma)$ converges in the weak operator topology to $\alpha_g(a)$. Clearly the $*$-algebra C contains the identity operator and, like B, it is closed under adjoints. Also $\alpha_g(a_\gamma) \in C \subset C'' = N$ so by the Von Neumann bicommutant theorem $\alpha_g(a) \in N$. This proves that $\alpha_g(N) \subset N$, and therefore $(N, \omega_\Omega, \alpha)$ is a factor of $(R, \omega_\Omega, \alpha)$.

Next we show that $(N, \omega_\Omega, \alpha)$ is compact. First note that for $a \in B$ we have $\alpha_G(a) = \lambda_a(G)a$ or $\alpha_G(a) = \{0\}$, and both these orbits are totally bounded in B with the pseudo metric obtained by restricting $\|\cdot\|_\Omega$ to B, since $\lambda_a(G)$ is a subset of the unit circle (which is compact) in $\mathbb{C}$. From (4.1) we can then conclude that $\alpha_G(a)$ is totally bounded in $(C, \|\cdot\|_\Omega)$ for every $a \in C$. Again by Von Neumann's density theorem for any $a \in N$ and $\varepsilon > 0$ there exists some $b \in C$ such that $\|a - b\|_\Omega = \|a\Omega - b\Omega\| < \varepsilon$. Now, if E is a finite ε-net in $(C, \|\cdot\|_\Omega)$ for $\tau_G(b)$, then from

$$\|\alpha_g(a) - c\|_\Omega \leq \|\alpha_g(a) - \alpha_g(b)\|_\Omega + \|\alpha_g(b) - c\|_\Omega < 2\varepsilon$$

for some $c \in E$ we see that E is a finite 2ε-net for $\alpha_G(a)$, i.e. the latter is totally bounded. $\square$

Lemma 4.2.2. *Let H_k be the set of all elements of a Hilbert space H with totally bounded orbits under U (where U is as in Definition 2.1.15). Then H_k is a Hilbert subspace of H.*

Proof. Let $x, y \in H_k$. Since $U_G x$ and $U_G y$ are totally bounded, for any $\varepsilon > 0$ there are finite sets M and N in H such that for each $g \in G$ there is an $x_g \in M$ and a $y_g \in N$ such that $\|U_g(a) - x_g\| < \varepsilon$ and $\|U_g(b) - y_g\| < \varepsilon$. For a scalar α we then have that

$$\|U_g(\alpha x + y) - (\alpha x_g + y_g)\| \leq |\alpha|\|U_g(x) - x_g\| + \|U_g(y) - y_g\| \leq 2\varepsilon$$

where $\alpha x + y$ is in the finite set $\alpha M + N$. Hence H_k is an inner product space with the inner product of H. To show that H_k is complete, note that as a subspace of H every Cauchy sequence in H_k is convergent. We now show that every convergent sequence in H_k has a limit with a totally bounded orbit under U. Let (x_n) be a convergent sequence in H_k with limit x. Given any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x\| < \frac{\varepsilon}{2}$ for all $n \geq N$. For an x_n with $n \geq N$ total boundedness of its orbit under U implies that there is a finite set $M_{x_n} \subset H_k$ such that for each $g \in G$ there is an $x_{n,g} \in M_{x_n}$ such that $\|U_g x_n - x_{n,g}\| < \frac{\varepsilon}{2}$. From the fact that $\|U_g x_n - U_g x\| \leq \|x_n - x\| < \frac{\varepsilon}{2}$ we have that

$$\|U_g x - x_{n,g}\| \leq \|U_g x_n - x_{n,g}\| + \|U_g x - U_g x_n\| < \varepsilon$$

implying that the orbit of x is totally bounded under U. □

Lemma 4.2.3. *Let H be a Hilbert space. If G is abelian, then H_0 is the set of all elements $x \in H$ whose orbits $U_G x$ in H are totally bounded.*

Proof. This is essentially a special case of general results proven in Section 2.4 of [25]. We show how it follows from those general results. Let H_k be as in Lemma 4.2.2.

Let E be the set of all eigenvectors of U, so $H_0 = \overline{\text{span} E}$. Clearly $U_G x$ is totally bounded for every $x \in E$, since $\lambda_x(G)$ is a subset of the unit circle in $\mathbb{C}$. From this it follows that $U_G x$ is totally bounded for every $x \in H_0$. I.e. $H_0 \subset H_k$.

Now suppose that $H_0 \neq H_k$. Since we have from Lemma 4.2.2 that H_k is a Hilbert subspace of H, it follows that there is an $x \in H_k \backslash \{0\}$ which is orthogonal to H_0. So $x \in H_v := H \ominus H_0$, but H_v is the space of so-called "flight vectors" which means that there is an S in the weak operator closure $\overline{U_G}$ of U_G in $B(H)$ such that $Sx = 0$. This is the essence of the splitting theorem in Section 2.4 of [25] as applied to a unitary group on a Hilbert space. Now since any closed ball in H is weakly compact we have that $U_G y$

is relatively weakly compact for every $y \in H$. Hence according to Lemma 2.4.2 in [25] $\overline{U_G x} = \overline{U_G x}^w$ where $\overline{U_G x}^w$ denotes the weak closure of $U_G x$ in H. The norm closure $\overline{U_G x}$ of the totally bounded set $U_G x$ is compact and therefore weakly compact and hence weakly closed, so $\overline{U_G x}^w \subset \overline{U_G x}$. We then have

$$0 = Sx \in \overline{U_G} x = \overline{U_G x}^w \subset \overline{U_G x},$$

contradicting $x \neq 0$ and the fact that U is a unitary group and hence norm-preserving. □

Proposition 4.2.4. *(1) If $(R, \omega_\Omega, \alpha)$ is ergodic but not weakly mixing, then the factor $(N, \omega_\Omega, \alpha)$ is nontrivial.*

(2) If $(R, \omega_\Omega, \alpha)$ is weakly mixing and G is abelian, then every element $a \in M$ with a totally bounded orbit $\alpha_G(a)$ lies in $\mathbb{C}1$. In particular $(R, \omega_\Omega, \alpha)$ has no nontrivial compact factor.

Proof. (1) Since Ω is cyclic and separating for R, we have from Theorem 2.5 in [32] that the map $a \mapsto a\Omega$ is a bijection from the set of eigenoperators of α to the set of eigenvectors of U. We then also note that our definition of ergodicity of $(R, \omega_\Omega, \alpha)$ is equivalent to that of [32], Section 2, namely $\alpha_g(a) = a$ for all $a \in R$ and all $g \in G$ implies that $a \in \mathbb{C}1$. Since H_0 strictly contains $\mathbb{C}\Omega$ by Definition 6.3, it follows that B strictly contains $\mathbb{C}1$, hence N strictly contains $\mathbb{C}1$. Therefore $(N, \omega_\Omega, \alpha)$ is indeed nontrivial.

(2) Consider any $a \in R$ with totally bounded orbit, then from $\|\alpha_g(a) - b\|_\Omega = \|U_g a\Omega - b\Omega\|$ we see that $U_G a\Omega$ is totally bounded in H. So $a\Omega \in \mathbb{C}\Omega$ by Lemma 4.2.3 and Definition 2.1.15. Hence $a\Omega = \lambda\Omega$, i.e. $(a - \lambda 1)\Omega = 0$ for some $\lambda \in \mathbb{C}$. Since Ω is separating for R, we have $a - \lambda 1 = 0$, so $a \in \mathbb{C}1$. In particular any compact factor of $(R, \omega_\Omega, \alpha)$ must be contained in $\mathbb{C}1$. □

Using these results, we can now prove

Theorem 4.2.5. *Let (A, ω, τ, G) be an ergodic W^*-dynamical system. Then (A, ω, τ, G) is weakly mixing if and only if it has no nontrivial compact factor.*

Proof. Let (H, π, Ω) be the GNS representation of (A, ω) and $(R, \omega_\Omega, \alpha)$ the corresponding represented system. We show that a nontrivial compact factor in (A, ω, τ, G) gives one in $(R, \omega_\Omega, \alpha)$, and vice versa. First suppose that (N, ω, τ) is a nontrivial compact factor of (A, ω, τ, G). Consider the system $(\pi(N), \omega_\Omega, \alpha)$. As noted earlier (after Definition 2.1.14), π is a faithful $*$-homomorphism i.e. a $*$-isomorphism $A \to R$. Since N is a $*$-subalgebra of A,

$\pi(N)$ is a $*$-subalgebra of R. For $g \in G$ let $a \in \alpha_g(\pi(N))$, i.e. $a = \alpha_g(\pi(n))$ for some $n \in N$. So

$$a = \alpha_g(\pi(n)) = \pi(\tau_g(n)) \in \pi(N)$$

since $\tau_g(N) \subset N$. Therefore $(\pi(N), \omega_\Omega, \alpha)$ is a factor of $(R, \omega_\Omega, \alpha)$. Furthermore, we have that each orbit $\tau_G(a) = \{\tau_g(a) : g \in G\}$ is totally bounded in $(N, \|\cdot\|_\omega)$. Hence for any $\varepsilon > 0$ there is a finite subset M of N such that for each $g \in G$ there is an $a_g \in M$ such that $\|\tau_g(a) - a_g\|_\omega < \varepsilon$. Consider the orbit $\alpha_G(\pi(a)) = \pi(\tau_g(a))$ in $\pi(N)$. We then have

$$\|\pi(\tau_g(a)) - \pi(a_g)\|_\Omega \leq \|\tau_g(a) - a_g\|_\Omega = \|\tau_g(a) - a_g\|_\omega < \varepsilon.$$

Since the set $\pi(M)$ is finite it is clear that $(\pi(N), \omega_\Omega, \alpha)$ is totally bounded in $(N, \|\cdot\|_\Omega)$. Since N strictly contains $\mathbb{C}1$, clearly $\pi(N)$ must strictly contain $\mathbb{C}1$. It similarly follows that a nontrivial compact factor in $(R, \omega_\Omega, \alpha)$ yields one in (A, ω, τ, G). Finally we note that, per definition, a (A, ω, τ, G) is weakly mixing if and only if $(R, \omega_\Omega, \alpha)$ is weakly mixing. The Theorem then follows from Propositions 4.2.1 and 4.2.4. $\square$

Theorem 4.2.6. *Suppose that (A, ω, τ, G) is a nontrivial (i.e. $A \neq \mathbb{C}$) ergodic W^*-dynamical system, with G locally compact, second countable, and containing a Følner sequence (Λ_n) satisfying the Tempelman condition and such that $(\Lambda_n^{-1}\Lambda_n)$ is also a Følner sequence. Assume that (A, ω, τ, G) is asymptotically abelian relative to (Λ_n). Let ω be a trace and $g \mapsto \tau_g(a)$ be continuous for every $a \in A$. Suppose that for each $m \in \mathbb{Z}\backslash\{0\}$ there exist two Følner sequences (Γ_n) and (Γ'_n) and a $c > 0$ such that*

$$\frac{1}{\mu(\Gamma_n)} \int_{\Gamma_n} f(g^m)\, dg \leq \frac{c}{\mu(\Gamma'_n)} \int_{\Gamma'_n} f(g) dg$$

for all Borel measurable $f : G \to [0, \infty)$ and all n. Then the Szemerédi property holds for a nontrivial factor of (A, ω, τ, G).

Proof. Set $m_0 := 0$ and $g^0 := e$. Let $a \in A$ with $\omega(a) > 0$. From Theorem 4.2.5 it follows that (A, ω, τ, G) is either weakly mixing or it has a nontrivial compact factor. If (A, ω, τ, G) is weakly mixing then (A, ω, τ, G) is M-weakly mixing, where $M := \{\varphi_m : m \in \mathbb{Z}\backslash\{0\}\}$ and $\varphi_m : G \to G : g \mapsto g^m$. This can be seen from

$$\frac{1}{\mu(\Gamma_n)} \int_{\Gamma_n} |\omega(a\tau_{g^m}(b)) - \omega(a)\omega(b)|\, dg \leq \frac{c}{\mu(\Gamma'_n)} \int_{\Gamma'_n} |\omega(a\tau_g(b)) - \omega(a)\omega(b)|\, dg \to 0$$

as $n \to \infty$. From Corollary 3.1.2 it then follows that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\omega(a\tau_{g^m}(b)) - \omega(a)\omega(b)|\, dg = 0.$$

Since M is translational, it now follows from Theorem 3.3.4 that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left| \omega\left(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\right) - \omega(a)^{k+1} \right| dg = 0.$$

and therefore

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left| \omega\left(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\right) \right| dg = \omega(a)^{k+1} > 0.$$

If (A, ω, τ, G) is not weakly mixing, it has a nontrivial compact factor (N, ω, τ, G). Continuity of $g \mapsto \tau_g(a)$ implies that $g \mapsto \omega\left(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\right)$ and $g \mapsto \|\tau_{g^{m_j}}(a) - a\|_\omega$ are also continuous. The Szemerédi property for (N, ω, τ, G) then follows directly from Theorem 4.1.6. $\square$

We note that, since ω is a faithful normal trace, the theorem above refers to finite Von Neumann algebras (see [23] Section 8.1). It is easily seen that for $G = \mathbb{Z}^q$ and $G = \mathbb{R}^q$ such (Γ_n) and (Γ'_n) exist: For $G = \mathbb{Z}^q$ let $\Gamma_n = \{1, \ldots, n\}^q$ and $\Gamma'_n = \{1, 2, \ldots, n|m|\}$, $m \in \mathbb{Z}\backslash\{0\}$. It can then be seen that (Γ_n) and (Γ'_n) are Følner sequences satisfying the requirements of the theorem. For $\mathbb{R}^q$, take $(\Gamma_n) = [0, n]^q$ and $(\Gamma'_n) = |m|\Gamma_n$.

The assumption of asymptotic abelianness is rather stringent and in further work it should be investigated if and how one can progress without it. If it is not possible to get rid of this assumption, one might attempt to at least soften the assumptions regarding asymptotic abelianness, e.g. that only a C^*-dynamical system is asymptotically abelian and that the W^*-algebra in the theorem above is the one obtained from such a system.

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