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Levels of Thought in Geometry of Pre-service Mathematics Educators according to the van Hiele Model

by
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LIST OF ABBREVIATIONS

FET	Further Education and Training (i.e. grades 10-12)
NCTM	National Council of Teachers of Mathematics
UP	University of Pretoria
PME	Pre-service mathematics education
Matric	Matriculation: final year of schooling in South Africa
TIMSS	Trends in International Mathematics and Science Study

CHAPTER ONE

INTRODUCTION AND CONTEXTUALISATION

1.1 PROBLEM STATEMENT

In the planning of this study, two factors played an important role. The first was the experience of the researcher in nearly twenty years of teaching geometry at high school level; the second was the experience of the researcher in teaching mathematics and the methodology of mathematics to PME (Pre-service Mathematics Education) students at the University of Pretoria. In comparing the two sets of experiences, it became clear that a link between the problems experienced by high school learners and the understanding or lack thereof of Euclidian geometry in PME students would bear investigating.

While geometry implies the study of shape and space from a mathematical viewpoint, Euclidian geometry may be described as “bodies of knowledge consisting of statements justified by proof, which depend on mathematical axioms and underlying logic” (Kotzé, 2007, p. 22). The study of geometry has formed part of most secondary school curricula since such schools came into being, but research concerning the learning and teaching of Euclidian geometry really only became the focus of attention a mere fifty years ago. Most learners experience more difficulty with the solution of geometry riders in the sense of proof construction than with any other section of the current mathematics syllabus. Senk (1989) found that,

> Although teaching students to write proofs had been an important goal of the geometry curriculum for the college bound in the United States for more than a century, contemporary students rank doing proofs in geometry among the least important, most disliked, and most difficult of school mathematics topics. (p. 309)

The general technique observed in the way South African Further Education and Training *learners* (FET: grades 10 to 12) deal with geometry is the following: learn the theorems really well (off by heart); leave the riders until the rest of the exam has been completed; try to do them if time allows. The van Hiele model is a developmental hierarchy which describes the level of competence of a student of geometry as progress is made in understanding and insight, as is explained in Chapter Two. According to the van Hiele Theory of Levels of Thought in Geometry, students who are situated below level 3 can do proofs in no way other than memorisation. Thus FET type geometry proof construction, requiring learners to be on Level 3, is doomed to failure since many students obviously function no higher than the upper reaches of Level 2 (Senk, 1989; Burger & Shaughnessy, 1986). Blanco (2001) speaks of the difference between the definition and the representation of a concept and it would appear that this is exactly where the difficulty lies: learners are encouraged to master the definition, but cannot be brought to insight into the representation of the concept. Azcarate (1997) finding in her research that memorisation is the preferred method for handling geometry and that little success ensued says, "Therefore, memorising the definition of a concept is no guarantee of understanding its meaning" (p. 29). Euclidian geometry riders are one of the main contributing factors to the fear and anxiety generally associated with mathematics in the minds of learners in the FET phase. Jenkins (1968) reflects that: "Shortly after matriculation, the college freshman discovers to his dismay that he has gained very little insight into the axiomatic systems which is at the foundation of not only geometry but also much of mathematics." (p. 35)

The general technique observed in the way many *educators* in South Africa deal with geometry is the following: make the learners learn the theorems off by heart (test this in an all-or-nothing assessment); do perhaps two riders per theorem with the learners in class; give them many riders to solve at home; make sure that they have at least tried to solve them by the next day; show them how to do them in class. Blanco (2001) calls this "abusive dependence on the textbook" (p.10). The problematic nature of this methodology is explained by Goldenberg (1999) who states:

> Students cannot possibly reinvent thousands of years of mathematics in a few years of high school. Some things must be told or explained. …But learning, to be creative, like learning anything else, requires opportunity to try out and exercise the skill... One must also devote a significant part of one's experience to the creative and disciplined act of doing the discovering" (p. 200).

Both national and international literature show that, in an effort to bring learners to a point where they can achieve success in the traditional assessment or national examination used to test geometric proficiency, conceptual knowledge has been the price paid for procedural efficiency. This has certainly been the case in South Africa.

It may well be that one of the problems underlying this type of teaching is a deficiency in "teachers' knowledge of student cognition in geometry" (Swafford, Jones & Thornton 1997, p. 468). There may also be gaps in the content knowledge of the educators themselves, which are frequently accompanied by a lack of confidence and even a fear of this aspect of mathematics education.

Many of the teachers of Euclidian geometry have problems themselves with the solution of geometric riders – having, in many cases, been taught by educators with similar problems. Blanco confirms this: "They also have deeply rooted conceptions about the teaching/learning of mathematics deriving from their own experience as primary and secondary pupils, and which present contradictions with the new school-level mathematical culture" (Blanco, 2001). He found during the course of his research that many of the errors made by the preset mathematics teachers were "based on the teaching/ learning process that they went through in primary school." Although it may be argued that pedagogical knowledge should help to overcome this problem, content knowledge gaps in the mind of the educators "affects their ability to respond to students' dilemmas involving the hows and the whys of learning mathematics." (Vistro-Yu, 2005, p. 2)

Preset mathematics students at the University of Pretoria (UP) enter their mathematics courses with their Euclidian geometry issues unresolved. Speaking for tertiary institutions, Lorenzo Blanco says, "Our intention is that the activities that we develop might generate simultaneously mathematical knowledge and knowledge of teaching/learning of geometry" (Blanco, 2001). Unfortunately, in this endeavour there remains very little time for the solving of problems whose origins are deep-seated and of long standing. Jenkins, writing about the same issue in 1968, found that, "a large majority of teachers had never had a college geometry course prior to their teaching geometry in high school. A background in algebra, trigonometry, analytic geometry, and calculus was present in varying degrees, but essentially a void existed with regard to one of the most exacting mathematical disciplines, one which they would surely be required to teach" (p. 35). It is the contention of this study that this exact situation, described by Jenkins nearly forty years ago, persists in our context.

1.2 CONTEXT OF THE STUDY

Geometry teaching and learning has long been a sore point in South Africa and indeed, in world education for a multiplicity of reasons. Hans Freudenthal (1971) speaks of the history of issues and conflicts regarding geometry and the teaching thereof:

> For long times mathematics has been synonymous with geometry. In fact, there existed other branches too … which, however, were not much more than collections of

> haphazard, badly founded rules, whereas geometry was a perfect logical system, where everything rigorously followed from definitions and axioms… Today mathematicians are prone to reject traditional geometry, because it is not a rigorously deductive system." (p. 417)

The South African Minister of Education, Mrs Naledi Pandor, like her predecessors, has repeatedly emphasised governmental concern about the level of mathematics learning and teaching in this country. The TIMSS'99 report serves to confirm that government's concern is not misplaced:

> South African pupils performed poorly when compared to other participating countries. The average score of 275 points out of 800 points is well below the international average of 487 points. The result is significantly below the average scores of all other participating countries, including the two other African countries of Morocco and Tunisia as well as that of other developing or newly developed countries such as Malaysia, the Philippines, Indonesia and Chile." (Howie, 2002, p. 608)

A test was administered to Grade 8 learners in which their understanding of mathematics in general, as well as of the kind of geometry that might be expected at their level of experience was assessed. A cursory review of exemplar items and learners' responses to questions in the test, contained within the report, reveals the following (Howie, 2002):

Table 1

Percentage of correct responses by South African learners and the international average score

Question number	South African average % correct	International average % correct
1	37	74
2	37	80
3	42	77
4	5	44
5	30	65
6	9	44
7	1	24
8	12	46
9	3	43
10	15	42

When represented graphically as in Figure 1, the comparison between the performance of South African learners and their counterparts in other countries presents a stark picture. South African learners of mathematics are clearly well behind other countries in terms of their mathematical performance. However, more pertinent to the object of this research is the general finding of the TIMSS'99 report with regard to performance in geometric questions. Although this research is concerned particularly with geometric understanding at FET level, it is important to remember that geometry is already introduced at elementary school level, so the learners tested for the TIMSS'99 should have been subjected to a certain amount of geometry teaching.

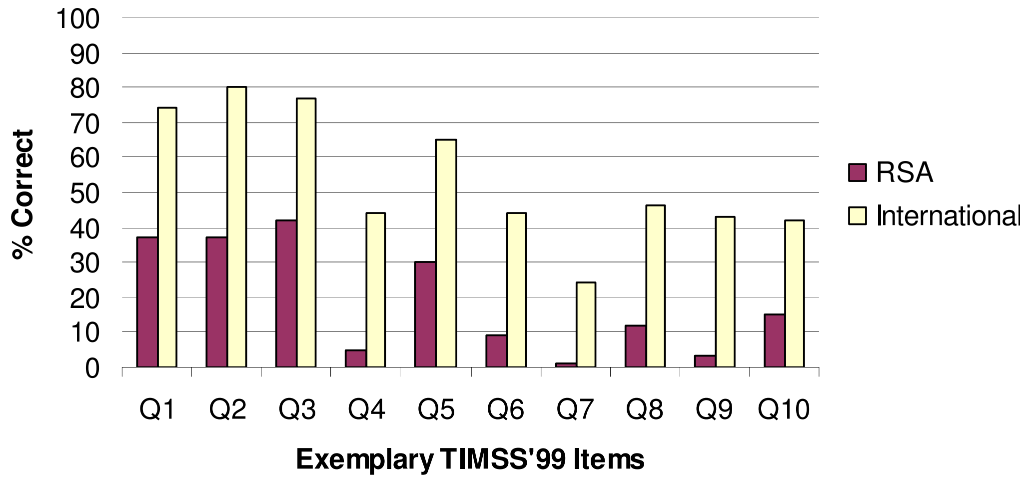


Figure 1. Comparison of performance of South African learners with the international average in selected items in mathematics in TIMSS'99

The last three of the ten questions were specifically geometric in nature. While these questions did not yield the very worst results, they certainly lie in the lower half of the range. At the same time it must be remembered that this test was completed by Grade 8 learners who have not yet been introduced to true Euclidian geometry in all its complexity. It may thus safely be assumed that with results already poor at this level, matters can only get worse.

Howie and Plomp (2002) look at mathematical literacy as a general rule in school leavers in South Africa. They examined a sample of Grade 12 learners from 90 different schools in South Africa, both urban and rural, of all ethnic groups and from all nine provinces. Their research reveals that the learners generally perform poorly in terms of reasoning skills and in the ability to communicate their reasoning accurately. This they ascribe to the fact that the majority of learners in South Africa are not taught in their mother tongue and so they often lack the language proficiency to both understand what is asked and to communicate what they do understand (Howie & Plomp 2002, p. 613). It is my contention that, in terms of geometry, although the situation is exacerbated by the language problem, there is a fundamental lack of skill in reasoning and proving logically which can be attributed to inadequate teaching of those very skills. Howie and Plomp conclude that South African learners leave school generally with a very low level of mathematical literacy and that they are generally unable to put into practice in real life applications that which they did learn. Significantly, these researchers found that "there is little difference between grade 8 and grade 12 in the pupils' basic mathematical literacy level despite the fact that more than 80% of the pupils received four additional years of tuition in mathematics." (ibid, p. 614)

1.3 RATIONALE FOR THE STUDY

The experience of the researcher and an initial exploration through general discussion of geometry riders with mathematics education students at the University of Pretoria reveal an intrinsic lack of confidence associated with a lack of insight and skills in the solution of geometric riders. Consequently, there are people who are going into the teaching profession, specifically as teachers of mathematics, whose skills and abilities in the solution of geometric riders are inadequate for the purpose of teaching those very same skills.

Although the school version of Euclidian geometry in its traditional form of theorem recognition and proof construction has been omitted from the syllabi of such countries as The Netherlands and the USA, it is believed that the life skills that are honed by such geometric reasoning remain relevant, whatever the syllabus. In fact, in the USA "recent reform recommendations have advocated increased emphasis on geometry instruction at all levels." (Swafford, Jones & Thornton, 1997, p. 467). The value of studying knowledge of geometry and the associated proof construction is affirmed by Hanna (1998), who during the course of research found, Further evidence of the importance accorded to proof in school geometry is the benefit which it is expected to bring beyond the borders of that subject. The consensus seems to be that the key goals of geometry instruction are the development of thinking abilities, of spatial intuition about the world, of knowledge necessary to study more mathematics and of the ability to interpret mathematical arguments." (p. 5)

Stephen Hawking said in an interview in 1988, "Equations are just the boring part of mathematics. I attempt to see things in terms of geometry." Geometric proof construction is a practical, space-orientated problem, which needs to be solved with insight and innovation, using the acquired tools. The logic, ability to reason and insight exercise demanded by this discipline render its pursuit worthwhile since these skills are not only essential in all the mathematical disciplines, but also in life itself. Hans Freudenthal famously said, "Geometry is grasping space… that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it." (cited in the National Council of Teachers of Mathematics, 1989, p. 48). Suydam (1985, p. 481) described the goals of teaching geometry as follows: to

- Develop logical thinking abilities
- Develop spatial intuition about the real world
- Impart the knowledge needed to study more mathematics
- Teach the reading and interpretation of mathematical arguments

However, if educators are negatively disposed towards Euclidian geometry, largely because of the difficulties they themselves experience in solving riders, there is little hope of young learners acquiring an aptitude for and enjoyment of such riders. Hence the aim of this research was to ascertain the depth of this problem amongst preset mathematics students in the Faculty of Education, University of Pretoria by assessing them both before and after they have been exposed to further teaching on the topic and by evaluating the students' attitudes towards the topic. In doing so, this study would provide insight into the reasons for the problems experienced by learners of geometry at high school level, as well as an understanding of the measures required at tertiary level to bring about change in the teaching of Euclidian geometry.

1.4 RESEARCH QUESTIONS

In view of the background against which this investigation takes place, the following questions formed the framework in which this inquiry could be structured, starting with an investigation of what the Preset Mathematics Education (PME) students in the FET phase currently understand and can apply, basing this on their experience of high school teaching and learning in this regard, and finally exploring how they felt about learning geometry.

1. How do FET phase mathematics pre-service education students apply their content knowledge of Euclidian geometry theorems to solve riders (including doing proofs)?

This question addresses the level of content knowledge that FET phase mathematics preset students have and to what extent this prepares them to teach the topic effectively upon qualifying. It is expected that students leaving matric having successfully completed the mathematics course will have attained Level 3 of the van Hiele model. This means that they are able to do formal deductions and can efficiently apply the theorems they have acquired to solve problem questions. During the geometry module presented to the preset mathematics students in their third year of academic study, Levels 0 to 3 are reinforced. Extensive exploration of thinking on Level 4 of the van Hiele model is beyond the parameters of what is required by UP for the training of FET level mathematics teachers.

The concern addressed here is that it remains somewhat beyond the realms of possibility for the general level of geometric ability in learners to improve when their educators lack understanding and insight into the problem presentations of the very subject matter they are trying to teach. It is an incontestable fact that no one can teach beyond their own understanding. Surely then, this question must be investigated as a possible key factor in the Euclidian difficulties experienced in so many mathematics classrooms. Furthermore, according to the Department of Education (2001), only 50% of practicing mathematics teachers have specialised in mathematics during their training and were “targeted for in-service training to address the lack of subject knowledge” (p.13) Since these educators have therefore been teaching with only their matric knowledge of mathematics to guide them, this begs the question of what level of knowledge they had achieved at that level that rendered it necessary for the Department to allocate millions of rands for in-service training.

In order to answer the question above, a number of specific questions have been constructed. These are the following:

a) To what extent is the content knowledge of FET phase mathematics pre-service education students sufficient in terms of the expected van Hiele levels before the start of the UP geometry module?

b) To what extent does the content knowledge of FET phase mathematics pre-service students attained during the mathematics course prepare them adequately to teach the topic effectively upon successful completion of the geometry module?

2. Was the preset FET trainees’ experience of learning Euclidian geometry at high school conducive toward prompting their progression from one van Hiele level to the next?

This question deals specifically with the notion that inadequate learning as experienced in the FET phase has the effect that learners leave school without having progressed to the expected level of geometric understanding (in terms of the van Hiele levels). The emphasis here lies on the experience of learning at high school by each of these students. It is not the aim of this study to investigate the teaching methodology or level of expertise of the educators who taught the students involved in this inquiry. The concern here is with the student's cognition of geometry proof construction and how they perceived their side of the learning/teaching process.

The focus is thus on how the student personally perceived the teaching and learning he/she underwent at FET level and whether they were encouraged to acquire rote (or otherwise) knowledge of theorems and to develop skills in solving riders. Such an inquiry serves to confirm or disprove the notion of a vicious, self-perpetuating circle in which the ill-taught become the ill-teachers.

Specific questions in this regard are:

a) What was the experience of the students at school in relation to the teaching and learning of Euclidian geometry?

b) How was the acquisition of knowledge of theorems related to the acquisition of skills in solving riders?

3. How do PME students feel about Euclidian geometry?

As the literature study reveals, the predominant emotions in most general mathematics students internationally when it comes to Euclidian geometry include very little confidence, concern, and an overriding fear of failure. This study limits itself to ascertaining whether this is also the case with these South African PME students and to identifying the concerns experienced by students in this regard in as much as they are relevant to these students' confidence and ability to teach the subject.

1.5 DEFINITION OF TERMS

Geometry content knowledge: Jones (2002) distinguishes between the two knowledge domains in this context:

> Subject matter knowledge (which includes key facts, concepts, principles, and explanatory frameworks of a discipline, as well as the rules of evidence used to guide inquiry in the field), and pedagogical content knowledge (which consists of an understanding of how to present specific topics in ways appropriate to the students being taught). (p. 96)

For the purpose of this research, geometry content knowledge is exactly what Jones defines as subject matter knowledge. He refers to Ma's term – profound understanding of fundamental mathematics (1999) - and expands on that with "the depth, breadth, and thoroughness of the knowledge that is required to be an accomplished teacher…" (p. 96)

Deductive reasoning: a method of taking what is known or accepted to be true and using it to make further logical inferences.

Axiomatic System: when such deductive reasoning takes place within an organised and logical structure, containing its own terms, definitions and axioms, we may speak of an axiomatic system. Euclidian geometry is such a system, confining itself to dealing with mathematical concepts as applied to space, such as measurement, and relationships of points, lines and angles. Basic truths which are used in such a system are primitive in the sense that they are universally accepted and require no proof. These basic truths can now be used to deduce other useful facts or to prove other useful propositions, or theorems.

Theorem: A proposition that can be accepted on the grounds of logical deduction and the use of previously agreed upon assumptions or axioms. In the context of South African education, theorems are the basic tools that have to be learnt by the students in order for them to use such tools to solve geometric problems.

Riders: These are geometric problem-solving situations in which, in order to arrive at an answer or conclusion, theorems and other axiomatic facts are required to be applied. H E Lockwood, writing in 1936, spoke of riders in the following glowing terms:

> Now if there is any part of school mathematics in which the mind is free to use all its resources, it is in rider-work in geometry: the successful solver must be able to explore his figure with an observant and roving eye, must be able to imagine how it may be varied, must spot relationships, must jump instinctively for the right construction, must argue backwards as well as forwards, before finally arranging his ideas in logical sequence. In America, riders are called "originals", and for the average boy they are certainly the nearest possible approach to original work. A very restricted kind of original work, perhaps, but on that account sufficiently easy for the average boy to tackle. Rider solving is an art; that is to say, an activity free within certain well-defined limits. (p. 93)

The term “riders” is no longer in common use, except in South Africa, where solving a rider and constructing a proof are often synonymous.

Proofs /proof construction: Schoenfeld (1988) defines “proof” as follows:

> For mathematicians, a "proof" is a coherent chain of argumentation in which one or more conclusions are deduced, in accord with certain well specified rules of deduction, from two sets of "givens:" (1) a set of hypotheses, and (2) a set of "accepted facts" consisting of either axioms or results that are known to have been proven true. (p. 155)

So, in South African secondary schools, it has been required for a student to learn a theorem and then to apply that knowledge to solve riders, which very often entail proof construction.

1.6 STRUCTURE OF THE DISSERTATION

In order to position this study coherently within the context of existing research, an in-depth literature study was done, the result of which is recorded in Chapter Two. This is followed in Chapter Three by a description of the logistics of the research for this study, as well as a discussion of the methodological norms and how the data were analysed. The next chapter presents the results of the study, following the processing and analysis of the data. Finally, Chapter Five summarises the findings and offers recommendations derived from these findings.

CHAPTER TWO

LITERATURE REVIEW

If we accept the assumption that "Mathematics is a science of pattern and order" (NCTM 1989, p. 31), it should not preclude the notion of philosophy in terms of a search for truth. The "science" aspect lies in the need for inductive reasoning; to find out what is not at first clear and to define the truth to which that reasoning leads. The "order" aspect lies in the use of deductive reasoning which uses the truths found inductively, and applies them to that which needs to be defined and ordered. "How do mathematicians establish truth?" ask Clements and Battista (1992). "They use proof, logical, deductive reasoning based on axioms" (p.437). In this context, it is useful momentarily to consider the basic tenets of both Piaget and van Hiele. Battista and Clements summarise these theories as follows:

> Piaget's theory, on the one hand, describes how thinking in general progresses from being non-reflective and unsystematic, to empirical, and finally to logical-deductive. The theory of van Hiele, on the other hand, deals specifically with geometric thought as it develops through several levels of sophistication under the influence of a school curriculum. (ibid, p. 50)

According to Kotzé (2007), Piaget's argument can be put like this: there is a "maturation process" (p. 22) which takes a learner through acquisition, representation and characterisation of spatial concepts. Van Hiele, however, suggested progress through thinking on sequential levels as a result of experience. This experience is almost entirely dependent on instruction (Larew, 1999, p. 6). However, both these great educational thinkers recognised that this kind of gradual proceeding must by its very definition take time. And herein, in fact, lies the crux of the problem: when students are required to produce logical and systematic proofs before they have attained the understanding that accompanies that level of thinking, rote-learning becomes the student's only option. Battista and Clements (1995) put this most succinctly: "Because students cannot bypass levels of understanding, prematurely dealing with formal proof can lead students only to attempts at memorisation and to confusion about the purpose of proof" (p. 50). They conclude:

> Furthermore, both theories suggest that students can understand and explicitly work with axiomatic systems only after they have reached the highest levels in both hierarchies. Thus, the explicit study of axiomatic systems is unlikely to be productive for the vast majority of students in high school geometry. (ibid, p. 50)

The "explicit study of axiomatic systems" is not attempted in any kind of depth in high schools in South Africa because it is tacitly understood that reaching "the highest levels" of which Battista and Clements speak does not happen by the end of matric. In fact, van Hiele himself declared that schooling did not need to take anyone beyond the level of formal deduction (Level 3).

Euclidian geometry, in particular, lends itself to the demonstration of such logical and deductive reasoning. In order to solve a rider efficiently, the student must not only understand what is being asked, but must have a sound knowledge of the pre-established "truths" or axioms which constitute the tools with which to solve the problem. The logical thought patterns which are applied to the problem need to be formulated in such a way that they can be written down succinctly and sequentially.

Against this general background, reviewing the national and international literature is structured around the three research questions.

2.1 CONTENT KNOWLEDGE OF PRESET STUDENTS IN EUCLIDIAN GEOMETRY AS APPLIED TO PROOF CONSTRUCTION

"All knowledge consists of internal or mental representations of ideas our mind has constructed," states van de Walle (2004, p. 25). According to Clements and Battista (1992), this begins at an early age:

> Evidence supports a constructivist position on how children learn spatial and geometric ideas. It appears that there is a progressive construction of geometric concepts from the perceptual to the conceptual plane as well as developmental sequences in which children build increasingly integrated and synthesized geometric schemata. (p. 457)

Content knowledge of Euclidian geometry which should have been synthesized into such "geometric schemata" in the high school student's mind, is traditionally demonstrated through the construction of proofs.

The crux of the problem lies in this cognitive connection between what is rote learnt and how that knowledge is applied. Schoenfeld (1986) speaks of a "sectorialization":

> Pupils are competent when they deduce and they are competent when they construct, but they often sectorialize their knowledge…So a large sector of their knowledge remains unused …An inappropriate sectorialization of activities of deduction and activities of construction is a direct consequence of teaching. (p. 226)

Fischbein refers to a "fusion between concept and figure" (1993, p. 143) and comes to the conclusion that "the process of building figural concepts in the student's mind should not be considered a spontaneous effect of geometry courses" (ibid, p. 156). There is thus a distinction to be made between what a learner knows and whether or how he/she applies that knowledge.

Put simply, in the case of Euclidian geometry, "to know" does not necessarily imply "can do"; the converse is also true: being able to follow step-by-step procedure prompted by a set of clues does not imply "knowing" or understanding. Schoenfeld (1988) explains as follows:

> Despite gaining proficiency at certain kinds of procedures, the students gained at best a fragmented sense of the subject matter and understood few if any of the connections that tie together the procedures that they had studied. (p. 145)

Wertheimer (as cited in Schoenfeld, 1988, p. 148) famously illustrated this tendency with his "parallelogram problem".

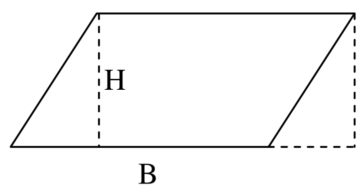


Figure 2. Wertheimer's parallelogram problem.

Learners in the classroom observed by Wertheimer were set the problem of determining the area of a parallelogram with base B and altitude H. They had been taught that by cutting off a specific triangle from one side of the parallelogram and moving it to the other side, the parallelogram becomes a rectangle. They knew very well how to calculate the area of a rectangle. Wertheimer described the lesson as successful, and found that the learners could present their argument correctly from a mathematical point of view. However, when these same learners were subsequently asked to find the area of a parallelogram that looked a little different, or of a figure similar to a parallelogram whose area could be similarly calculated, they were at a loss. Schoenfeld (1988) explains:

> Wertheimer argues that although they had memorized the proof, they had failed to understand the reason that it worked; although they had memorized the formula, they

> used it without deep understanding. With that understanding, he argues, the students would have been able to answer his questions without difficulty; without it they could solve certain well specified exercises but in reality had acquired only the superficial appearance of competence. (p. 149)

This "superficial appearance of competence" may be ascribed to a sound procedural knowledge, accompanied by flimsy conceptual knowledge. John van de Walle (2004) defines these two types of knowledge as follows:

> Conceptual knowledge consists of relationships constructed internally and connected to already existing ideas. It is the type of knowledge Piaget referred to as logico-mathematical knowledge. Procedural knowledge of mathematics is knowledge of the rules and the procedures that one uses in carrying out routine mathematical tasks and of the symbolism that is used to represent mathematics (p. 25)

He also describes "understanding" in terms of the linkages that are created between existing ideas and new ones.

> Understanding can be described as a measure of the quality and quantity of connections an idea has with existing ideas. Understanding depends on the existence of appropriate ideas and on the creation of new connections… The understood idea is associated with many other existing ideas in a meaningful network of concepts and procedures." (p. 27)

He refers to this network as relational understanding. However, it is possible for ideas to be acquired that remain unconnected or isolated. Such understanding may be denoted as instrumental understanding. Knowledge acquired through this kind of understanding, says van de Walle, is learned by rote, usually through drill and repeated practice. While recognising the value and significance of procedural knowledge, he speaks of the importance of moving from a procedural approach to a conceptual one. If teaching leads to only procedural understanding, the danger is that "rules without reasons" are being taught (ibid. p. 28). This, while it is to be deprecated, is not unusual: "Unfortunately, it is much more common to find children who simply do not possess the conceptual knowledge that supports a procedure" (ibid. p. 29). Van de Walle claims that it is a much simpler matter to teach procedure with repetition and practice, than to teach for relational understanding which requires much more effort from the educator. However, the benefits make the effort worthwhile:

> Relational understanding is intrinsically rewarding, enhances memory, requires that less be remembered, helps with learning new concepts and procedures, improves problem-solving abilities, can be self-generative, and has a positive effect on attitudes and beliefs. (p. 29)

Weber, working with undergraduate and doctoral students in the eastern United States, encountered the very lack of connection, as described by van de Walle, between content knowledge and understanding demonstrable through conceptual application. He determined that firstly they did not have a truly accurate idea of what a valid proof is and so could not construct one (Weber 2001, p. 101). Secondly, they did not fully understand a concept or theorem and so they "systematically misapply it" (ibid, p. 102). This does not mean that they are unable to recognise the appropriateness of a particular theorem; in fact, says Weber, "Simply recognizing a fact or a theorem does not guarantee one can apply it properly." (ibid, p.102) However, most significantly of all, he found that:

> While this research has provided rich data, there is a large and important class of failed proof attempts that it cannot explain. Students often fail to construct a proof because

> they reach an *impasse where they simply do not know what to do*. (ibid, p.102) (Emphasis added)

Having arrived at this conclusion, he states that upon further investigation he found that very little is documented with regard to studies about why students simply battle with proofs, even when they have the necessary knowledge at their fingertips. The achievement or lack thereof cannot be accounted for on the grounds of van Hiele levels alone since, as is confirmed by Mayberry (1983), Usiskin (1982) and Burger & Shaughnessy (1986), a student can be on different van Hiele levels simultaneously with regard to certain concepts. Weber found that undergraduates in particular were prone to putting down irrelevant facts in their desperate search for evidence, whereas effective provers possessed strategic knowledge which they could recall and apply appropriately. He concludes with a daunting challenge: "Since strategic knowledge is heuristic, designing activities that will lead students to acquire this knowledge will be a formidable task." (ibid, p.116) In essence, what Weber calls strategic knowledge is described by van de Walle as relational understanding, and its absence might very well explain some of the impasse situations Weber observed in proving activities.

However, in research conducted by French psychologist, Duval (2006), relational or conceptual understanding was in fact *not* absent, and yet the student could not solve the problem. He says that it must be remembered that sketches or figures use

> a system of representation that is independent of the statements and of the mathematical properties to which they refer. That would mean that what one sees in a figure depends on factors of *visual organization*: it is these factors that determine the discrimination, that is the recognition, of certain one-, two- and three-dimensional forms in a figure and exclude the discrimination of other possible configurations and sub-figures in the same

> figure. Now “seeing” in geometry frequently requires that one be able to recognize one or another of these other possible configurations and sub-configurations. (p.119) (Emphasis added)

This “visual organisation” may well be a significant factor in the cognitive link between procedural and relational understanding in working with Euclidian geometry. In geometry, very often the “givens” Schoenfeld (1988) speaks of are represented in a sketch, but this visual representation is not always useful in helping the student to bring his/her content knowledge into play. As Bagni (1998, p. 174) points out, “visualization could be excluded from deduction activities and pupils could make little use of a very important learning possibility”. In other words, looking at the sketch or even the related theorems does not mean that the learner necessarily makes the cognitive connection with the deductive activity he/she is performing in constructing a proof. Duval recognizes the “complexity of the mathematical use of figures” and the “non-natural character for most students of the act of ‘seeing’ in geometry” (2006, p. 119). Despite the fact that the sketch which might accompany a rider covertly or overtly presents all the information necessary for its solution, most students find solving such riders extremely difficult. Duval lays his finger on the problem: “Good conceptual comprehension ought to lead to seeing in a figure what has to be seen in order to find there the elements for solving a problem” (ibid, p. 119). However, “good conceptual comprehension” is frequently lacking, the emphasis in tuition having been on the learning of step-by-step procedure.

Apart from this, learners also have to contend with the “multitasking” aspect of geometry proofs. As Balacheff (1988) puts it, “The practice of proof requires reasoning *and* a specific state of knowledge at one and the same time.” (p. 229) (Emphasis added). Duval groups Balacheff’s “reasoning” and “state of knowledge” into one type of cognitive process, but also identifies two other types of cognitive processes which are “closely connected and their synergy is cognitively necessary for proficiency in geometry" (Duval, as cited in Jones & Bills, 1998, p. 124):

- Visualisation processes: for example the visual representation of a geometrical statement, or the heuristic exploration of a complex geometrical situation.
- Construction processes: using tools
- Reasoning processes: particularly discursive processes for the extension of knowledge, for explanation, for proof.

Duval contends that, in order to promote the development of geometric reasoning, these three processes must be developed separately. Until recently, the use of construction in the development of geometric reasoning was largely underemphasised in the South African curriculum. Jones and Bills describe the imagery created by construction as "conceptually generative" (2006, p. 127).

It may thus be concluded that it is an international as well as a local phenomenon that the geometry content knowledge of students upon leaving high school and subsequently commencing some sort of study in mathematics, is inadequate. Rote learnt theorems are by definition generally unrelated to application in terms of proof construction, and are thus easily forgotten. Consequently, the solution of riders, which is dependent on a working knowledge of theorems, remains problematic. In fact, proof construction skills are not easily acquired, since the insight required is more caught than taught. According to research done by van der Sandt and Niewoudt (2005), "Numerous international studies have addressed the state of teachers' subject knowledge, but research on South African teachers' mathematical subject knowledge is sparse. Some research has been done, but none of the research focused solely on geometry …"

(p. 109). However, it stands to reason that, with an inadequate content knowledge to begin with, unless serious work in this regard takes place during the course of a teaching qualification, many educators are entering the profession ill-equipped to teach this subject. Muijs and Reynolds (2002) found that there is a significant and direct correlation between the educator's conceptual knowledge of mathematics and the quality of instruction which that educator is capable of delivering.

2.2 THE EXPERIENCE OF PME STUDENTS IN THEIR STUDY OF GEOMETRY AT HIGH SCHOOL AS RELATED TO THEIR PROGRESS THROUGH THE VAN HIELE LEVELS

In spite of rhetoric in the media and various curriculum innovations in modern countries all over the world, the mathematics classroom remains fairly unchanged, according to educational psychologist, Jo Boaler (2000). Researching this area of education in America and England, she declares the following about the mathematics classroom:

> What is perhaps most remarkable about this particular community of practices is how little is has changed in most countries over the last hundred years. For most students, mathematics continues to be a teacher-dominated practice, with a substantial amount of self-directed work undertaken from either the textbook, board work or individual worksheets. It has been heavily reliant on formal pencil-and-paper testing, particularly in the secondary school. (Mathematics as a community of practice, para. 5)

The situation described by Schoenfeld (1988) in America nearly twenty years ago is one which mirrors current mathematics education in this country.

> For each of the years K-12 (and beyond; calculus instruction in college is pretty much the same), there was an agreed-upon body of knowledge, consisting of facts and procedures, that comprised the curriculum. In each course, the task of the teacher was to

> get students to master the curriculum. That meant that subject matter was presented, explained, and rehearsed; students practiced it until they got it (if they were lucky). There was little sense of exploration, or of the possibility that the students could make sense of the mathematics for themselves. Instead, the students were presented the material in bite-sized pieces so that it would be easy for them to master…step-by-step procedure. (p. 159)

In South Africa, like in America, final assessment examinations are predictable in terms of both their content and the way questions are posed. It is therefore perfectly understandable and logical that instruction, the perceived purpose of which is to achieve success in these examinations, is generally designed for the purpose of assessment success. Schoenfeld (1988) explains:

> The primary goal of instruction, therefore, was to have students do well on the exam. The curriculum and the examinations were well established and quite consistent from year to year. Thus the amount of attention to give to each topic, and the way to teach it (for "mastery" as measured by the exams), were essentially prescribed. The curriculum contained a dozen "required" proofs, one of which appeared on the Regents exam and was worth 10 points (of 100). (p. 152)

This tendency is confined neither to South Africa, nor to the American education system of twenty years ago. Researching mathematics education in six United States and six United Kingdom secondary schools, Boaler (2000) and her team found that the students interviewed from four of the schools in each of the two countries all described the same typical mathematics classroom practice: the lesson would begin with homework review, methods would be explained on the board, and then new questions would be assigned to the students to do. The result is that these students all described mathematics as a "procedural, rule-bound subject" (Method, para. 5) Boaler quotes one of the British students as follows:

> "It's all about formulas. If you know how to use it then you've got it made. Even if you don't quite understand the concept, if you're able to figure out all the parts of the formula; if you have the formula then you can do it."

It is therefore not strange that preset teaching students often do not "own" geometry, since they rote-learnt theorems at school without acquiring their own grasp of the discipline. In fact, students may not even understand the problems that they do solve (Schoenfeld, 1988, p. 160)! It is not true, however, that all geometry educators simply encourage rote learning. As pointed out by Schoenfeld (1988), good teachers advocate understanding rather than blind memorisation, if for no reason other than when memory fails, understanding can help to fill in the missing step. However, "in truth .. this rhetoric – in which the teacher truly believed – was contradicted by what took place in the classroom. The classroom structure provided reinforcement for memorisation, and the reward structure promoted it" (p. 159).

Since rote learning very often implies inadequate insight and understanding, what Larew (1999) found in her research makes perfect sense: "If a student is forced to skip levels of understanding by learning information in one level in a rote manner, then this lack of mastery at one level becomes an obstacle to learning at the next" (p. 13). She discovered that this was commonly the case in high school geometry courses and that students were consequently unable to operate successfully on a deductive reasoning level. She explained this phenomenon as follows:

> The students are unsuccessful at this deductive reasoning level because they have little intrinsic understanding at the analysis of properties and informal deduction levels [Van Hiele Levels 1 and 2]. This lack of intrinsic understanding means that concepts within these levels such as the relationships that exist between figures and parts of figures and one-step deductive conclusions, which are necessary skills for proof-writing, cannot be extrinsically used in the deductive reasoning level. (p. 13)

This statement is in fact corroborated by the research of Gutiérrez, Jaime and Fortuny (1991, p. 249) on the levels of geometric understanding in, amongst others, a group of twenty preset primary school teachers specialising in science. They found that only two of these teachers had achieved mastery of van Hiele Levels 0-2. The testing took place after these students had all completed a mathematics course for one year. This statistic suggests that if these are the levels that were achieved after a year's tertiary training in mathematics, the level of geometric acquisition upon leaving school was even poorer. Swafford, Jones and Thornton (1997) found that 80% of the educators in their research operated on Level 3 or lower of the van Hiele Levels.

By contrast, as de Villiers (1996) points out, if the teaching that takes place in a classroom is at a van Hiele level that is above the current level of understanding of the learners in that classroom, learning becomes very problematic. In fact, "The main reason for the failure of the traditional geometry curriculum was attributed by the Van Hieles to the fact that the curriculum was presented at a higher level than those of the pupils" (De Villiers, 1996). The learners do not understand what the teacher is saying, and the teacher cannot understand why the learners do not understand. It must therefore be also be considered that there are cases in which the educator, while having an adequate knowledge of the subject matter at hand, is unable or even unaware of the need to present it at a level which can be grasped by the learners.

In South Africa, however, particularly in the sort of schools researched by Mogari, it is more probable that teachers were, themselves, not in a position to bring their learners up to a level of successful deductive reasoning. In a study he did in South Africa with Grade 10 learners, he found that there was a disturbing tendency toward low performance when it came to logical deductive reasoning. He ascribed this to a lack of well-developed problem solving skills, a poor understanding of geometric concepts and fuzzy notions about proof structure and development (2003, p. 69). He concludes that learners generally find Euclidian geometry more problematic than other mathematics topics, but thinks that this may be ascribed to the fact that his sample of learners came from rural, deprived schools where good teachers and technological support are hard to find (ibid, p. 70).

Technological support, however, is not the panacea. de Villiers has completed a considerable body of research on the use of computer programmes like Geometer Sketchpad to facilitate geometric understanding. He found that although a deeper level of understanding was generated when a learner actually saw the rules in action on the screen in front of him/ her, it still did not provide the kind of insight which would allow the learner to answer the question, "Why?"

> Although most students seem to have no further need for conviction once they explore geometric conjectures in dynamic geometry environments like Cabri or Sketchpad, it is not difficult to solicit further curiosity by asking them why they think a particular result is true. Challenge them to try to explain it. Students quickly admit that inductive

> experimental verification merely confirms; it gives no satisfactory insight or understanding." (de Villiers 1997, p. 23)

De Villiers (1997) comes to much the same conclusion as did Piaget and van Hiele: "To present the fundamental function of proof as explanation and discovery requires that students be introduced *early* to the art of problem posing and allowed sufficient opportunity for exploration, conjecturing, refuting, reformulating and explaining" (p. 23) (Emphasis added).

Even so, it must be recognized that some learners do not have insight into what might be called, the "bigger picture". They "view deductive proofs in geometry as proofs for a single case, the case that is pictured in the associated diagram." (Chazan, 1993, p.362). In other words, they are unable to recognize the pattern that is universally true of all such situations as are represented by the diagram they see before them. Other learners, while able to state quite clearly what they believed, experienced grave difficulties in explaining *why* what they thought was true (ibid, p.371).

The explanation for this phenomenon may well lie in this statement by Romburg and Carpenter:

> The traditional classroom focuses on competition, management, and group aptitudes; the mathematics taught is assumed to be a fixed body of knowledge, and it is taught under the assumption that learners absorb what has been covered (as cited in Schoenfeld, 1988, p 147).

Kotzé (2007) researched the understanding of space and shape with a group of Grade 10 mathematics teachers who were required to do an assessment instrument upon the completion of an Advanced Certificate in Education: Mathematics course. They then had to administer the same instrument to the learners they taught at their respective schools. She found that "Correspondence between the achievement of teachers and learners could be observed: the achievement of both groups revealed a need to develop spatial sense in geometry" (p. 33). Most interestingly, she found that the teachers in general overestimated their abilities in the subject. Learners cannot achieve success when the content knowledge of their teachers is not what is required to be able to teach the syllabus. Says Kotzé, "Teacher subject knowledge is crucial. This study confirmed that such knowledge is a good predictor of learner achievement" (ibid, p. 33).

Apart from teachers' subject knowledge being a factor in learner achievement, Schoenfeld's research revealed that there were far-reaching effects of the type of tuition that the learners in his study received: they developed, often inadvertently, ideas about the nature of mathematics that "were likely to impede their acquisition and use of other mathematical knowledge" (1988, p. 145). Geometry in particular, by its very nature, lends itself to the creation of misconceptions about memorisation as a valid substitute for understanding.

Mistretta, working in Wagner College, New York, came to the conclusion that while "geometry is a vital part of the curriculum …Unfortunately, many students develop misconceptions, and others fail to go beyond simple visualisation of geometric figures" (Mistretta 2000, p. 365). In her sample of 8th grade students, she found that 61% of the learners felt that geometry was "complicated and confusing" (ibid, p. 369) and that all they could do was to memorise theorems and formulae without understanding why or for what purpose they were doing so. Having submitted them to a special geometry course, the learners responded positively, saying that once they understood the why's and the wherefores of what they were doing, geometry seemed easier (ibid, p. 378)

Evidence thus suggests that unless learning is based on understanding instead of uninformed memory, with the van Hiele levels very specifically born in mind, it is unlikely that conceptual understanding of proof construction will develop. In fact, it is generally the case both in South Africa and abroad, that Euclidian geometry is learnt in terms of rules and procedures, and that proof and problem-solving skills are rarely acquired. The research of Gutiérrez, Jaime and Fortuny (1991) as well as Swafford, Jones and Thornton (1997) also suggests that practicing teachers of mathematics are themselves not functioning on an adequate level of van Hiele (at least Level 3) and that this has a direct impact on the pupils' experience of learning.

2.3 WHAT PME STUDENTS FEEL ABOUT EUCLIDIAN GEOMETRY

Mathematics by its very nature generates emotion, ranging from anger, frustration, puzzlement and despair through to satisfaction, joy and elation, the latter two rather rarely. Schoenfeld (1988) gives an account of a personal experience: an instructor he once had was explaining a particular theorem, but stopped at the point where a formula was required.

> "I never remember this formula," she said, "but it's so easy to derive that you don't need it anyway." Then she showed us how to derive the formula. What she showed us made sense. To this day, I can't remember the formula, but I can derive it, either when I need it (which is rare) or because the thought of it brings back pleasant memories. The idea that was brought home in that class -- that mathematics really makes sense and that you can figure something out if you need to -- was exhilarating. It is (or should be) part of the pleasure of learning mathematics. (p. 158)

Crucial to the pleasure he experienced was the conviction that "you can figure something out if you need to". However, if students believe that such reasoning lies beyond them because the body of knowledge in which they are working is just to deep and complex to make the effort worthwhile, that kind of pleasure will remain elusive. A British student interviewed by Boaler (2000) said, “I used to enjoy it, but I don’t enjoy it anymore because I don’t understand it. I don’t understand what I’m doing … But I enjoy it when I can actually do it, but when I don’t understand it I just get really annoyed with it” (Method, para. 6).

Although the philosophy of mathematics is not the subject of this study, the idea of the absolute truth of each mathematical and, in particular, each geometric fact is relevant in that research shows that students believe that geometry is absolutely factual, completely discovered and entire in its encompassment of its own truth. Boaler found that high school students in both England and the United States believe mathematics to be “rigid and inflexible, and in particular, that it is a subject that leaves no room for negotiation of meaning” (2000, Abstract). Interestingly, she found that their beliefs about learning mathematics corresponded directly with the extent to which mathematical discussions were commonly held in their classes as opposed to individual work on questions out of the textbook. Students in the latter type of classroom described mathematics as “absolute, concrete and always having one right answer” (2000, Method, para. 5), while the others, even though they also used the same or similar textbooks, were encouraged to discuss questions and the meanings of possible answers with each other and consequently saw mathematics as a “field of enquiry that they could discuss and explore” (Method, para. 8). In fact, says Boaler, “the procedural nature [of mathematics teaching] denied them [the learners] access to understanding” (2000, Enjoyment and Identification, para. 1).

Thus mathematics, and in particular geometry, remain inaccessible in terms of insight into its depth, and only its fringes, selected Euclidian theorems and related proofs for example, can be rote learnt and used for marks. American students interviewed from the six schools in her study in America impressed upon Boaler (2000) their dislike of mathematics despite being relatively successful in their results because “their perceptions of the subject as abstract, absolute and procedural conflicted with their notions of self, of who they wanted to be” (Enjoyment and Identification, para. 1).

According to Schoenfeld, “In these students' experience, proofs had always served as confirmation of information that someone (usually the teacher or mathematicians at large) already knew to be true” (1988, p. 154). van Hiele claimed that the intuitive basis for proof construction starts with “a pupil’s statement that belief in the truth of some assertion is connected with belief in the truth of other assertions. The notion of this connection is intuitive” (1986, p.124).

It may therefore be deduced that should there be no such belief, or should there have been no opportunity to acquire this belief, proof construction will remain a mystery in the minds of many learners. This intuitive aspect is largely neglected; in the words of Romburg and Carpenter: “[Mathematics and science were seldom] taught as scientific inquiry -- all subjects were presented as what experts had found to be true” (as cited in Schoenfeld, 1988, p. 161).

When memorisation of someone else’s “truth” is what is required to achieve successful results in an examination, alongside of some indefinable intuitive skill that one lacks, it is understandable that studying geometry at school is often fraught with emotion. Mogari (2003) identifies four dimensions of emotion in relation to geometry: enjoyment, motivation, perceived importance of geometry and freedom from fear of geometry. His particular focus is the correlation between attitude and achievement. In his search for similar studies he found that some academics discovered that students whose attitudes were on the extreme ends of the scale showed a close relationship between attitude and achievement, while students whose attitudes were in the middle range of emotion showed a poor link between attitude and achievement. In his own research, working with Grade 10 pupils from eight high schools in a mostly rural, greatly impoverished and disadvantaged milieu in South Africa, he came to the conclusion that "there was no statistically significant relationship between achievement and attitude and between achievement and each of the three components of attitude with an exception of motivation, which showed a statistically significant relationship" (ibid, p. 68). He admits that his findings are in contrast with those of other researchers in the same field. This he explains by saying that opting to continue with mathematics at the end of grade 9 does not necessarily mean that the learner likes/hates the subject, but that the subject is required for the career the learner wants to pursue. This would then also explain the motivation statistic. Boaler, (2000) interviewing high school students from England and the United States, came across the same sort of attitude:

> A much more salient factor in determining students' attitudes towards mathematics was that they did not see success at mathematics as in any way relevant to their developing identities, except insofar as success at mathematics allowed access to future education and careers. (Abstract)

So, research has shown that learners in general believe that geometry is just difficult, and because rider solution depends to some extent on insight and intuition, for which procedural knowledge alone is inadequate, it may remain inaccessible. Students believe geometry to be a body of knowledge that is deep and complex, rigid and absolute, abstract, and with no possibility of negotiating meaning. Nevertheless, in many schools, mathematics with its accompanying geometry component is often chosen as a subject for matric or its equivalent qualification, not because the learner particularly enjoys or excels in it, but because it is a requirement for certain career choices. For most students, doing geometry provides little pleasure; instead, words such as annoyance, frustration and fear come more readily to mind.

It is against this background of concerns about the Euclidian geometry content knowledge of the PME students at the University of Pretoria, their experience of learning geometry at high school and how they feel about geometry that a decision had to be made about the conceptual framework which would inform this study. The literature study clearly reveals that the teaching and learning of geometry have been fraught with unresolved questions for many years.

2.4 CONCEPTUAL FRAMEWORK

What is the theory underpinning the effective learning of geometry? According to Alva Walker Stamper (1909) the aims of geometry teaching have for most of mathematical history been either practical or logical ie for purposes such as architecture (Egypt, Babylon) or toward the perfection of the human soul (Greece). Euclid thrived and became part of the English higher education, "education that fits for the so-called higher callings" (ibid, p. 702). In Holland systematic teaching of logic in geometry began only in the eighteenth century. Little observable change occurred in the teaching of geometry until the advent of the van Hieles in the 1958. Pegg and Davey (1998) summarise the intention and content of the van Hiele model as follows:

> In summary, the van Hiele theory is directed at improving teaching by organizing instruction to take into account students' thinking, which is described by a hierarchical series of levels. According to the theory, if students' levels of thinking are addressed in the teaching process, students have ownership of the encountered material and the development of insight (the ability to act adequately with intention in a new situation) is

> enhanced. For the van Hieles, the main purpose of instruction was the development of such insight. (p. 110)

Pegg and Davey also point out that the van Hiele theory is more pedagogical than psychological (p. 111). This implies that the teacher, according to the van Hieles, is required to do what is necessary to guide the student into the acquisition of insight and understanding, in other words, conceptual knowledge. Pierre van Hiele (1986), writing subsequent to the demise of his wife, did not underestimate the enormity of this task: "It takes nearly two years of continual education to have the pupils experience the intrinsic value of deduction, and still more time is necessary to understand the intrinsic meaning of this concept" (p. 64).

It was this revolutionary insight into the pedagogy of geometry teaching that made an impact on the Russian researchers who sought to understand the problems associated with geometry education in school. De Villiers (1996) notes that,

> In the late sixties Russian (Soviet) researchers undertook a comprehensive analysis of both the intuitive and the systematisation phases in order to try and find an answer to the disturbing question of why pupils who were making good progress in other school subjects, showed little progress in geometry. In their analysis, the Van Hiele theory played a major part.

Gutiérrez (1992) points out that since Wirszup reported on the changes in the Soviet Curriculum in geometry in the late 60's, the van Hiele Model became the focus of world attention in terms of geometry teaching. Piaget published two works relating to the learning of geometry, The Child's Conception of Space (1956) and The Child's Conception of Geometry (1960), but, as Pegg and Davey point out, "little impact on classroom practice has resulted" (1998, p. 109). The van Hiele theory, however, has made a significant impact on the world in terms of geometry education, particularly after it became known internationally what its impact was on Russian mathematics education.

Larew (1999) refers to a third major theoretical perspective on geometric thinking apart from Piaget and van Hiele: James Greeno, a cognitive scientist, proposed a model of geometry problem-solving which offers “precision to models of geometric thinking, especially in the arena of direct instruction for problem-solving strategies” (Greeno, as cited in Larew, 1999, p. 6). Clements and Battista (1992) suggest that the cognitive scientists’ model may either be seen as advocating educator-centric procedural activities, in which case it stands in opposition to the findings contained in the literature reviewed in this research, or as an educator-as-facilitator-of-learning theory, in which case it is similar to what is proposed by van Hiele. However, since this study is concerned with the geometry development of PME students, the van Hiele model was selected as the most suitable conceptual framework. This was done for much the same reasons as identified by Larew (1999): the van Hiele model is not age-related, as is Piaget’s, and is therefore applicable to university students; the van Hiele model is experience-related and looks at the influence of instruction, as does this study.

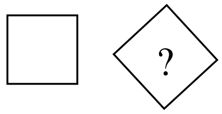
2.4.1 THE VAN HIELE MODEL

The structure of levels as developed by Pierre van Hiele and Dina van Hiele-Geldoff inform the conceptual framework of this study. For the purpose of this study, the original naming of the levels as 0 through to 4 is used. The levels, simply put by John van de Walle (2004), are as follows:

Level 0 - VISUALISATION

Thinking subject: shapes and what they look like.

The student recognises a shape by its appearance and can group together shapes that look more or less alike. Thus a square is a square because it looks like a square.



Is the second figure still a square?

Thinking result: shapes can be grouped according to what they look like.

Level 1 – ANALYSIS:

Thinking subject: classes of shapes rather than individuals.

The student can think about what makes a rectangle a rectangle and not a cube: the reasons for which certain shapes are grouped together becomes clearer as the student identifies properties of a shape type.

Thinking result: shapes can be grouped according to their properties.

Level 2 – INFORMAL DEDUCTION:

Thinking subject: the properties of shapes.

The student is now able to make deductions about the properties and can follow logical argument in simple deductive reasoning. So he can now figure out that if 2 adjacent sides of a parallelogram are equal, and at least one corner is a right angle, it's a square.

Thinking result: relationships among the properties of shapes.

Level 3 – DEDUCTION

Thinking subject: relationships among the properties of shapes.

The student is now able to go beyond just the properties, placing them in a structure where given information can be used to derive further information. He can now use logic rather than intuition most of the time.

Thinking result: axiomatic systems for thinking deductively.

Level 4 – RIGOR

Thinking subject: axiomatic systems for thinking deductively.

The student now thinks about the axiomatic systems themselves, making comparisons and appreciating their differences.

Thinking result: comparisons and contrasts between different axiomatic systems of geometry.

Usiskin (University of Chicago) did an extensive investigation in 1982 for a project called the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) of the writings of Pierre and Dina van Hiele in order to determine accurately the descriptions of behaviour at each of the levels. The following van Hiele quotations were selected by Usiskin (1982, p. 9) as determining the behaviour at a level:

Level 0:

"When one has shown to a child of six, a six year old child, what a rhombus is, what a rectangle is, what a square is, what a parallelogram is, he is able to produce those figures without error on a geoboard of Gattegno, even in difficult situations."

"…and the rectangle seems different to him from a square."

"a child does not recognise a parallelogram in a rhombus."

"the rhombus is not a parallelogram. The rhombus appears …as something quite different."

From this it is clear that the person operating exclusively on Level 0 can distinguish a particular shape from amongst others that are similar in some ways. However, no identification of properties or characteristics of certain parts of a shape is achieved.

Level 1:

"… a pupil who knows the properties of the rhombus and can name them, will also have a basic understanding of the isosceles triangle = semi-rhombus."

"The figures are the supports of their properties."

"The figures are identified by their properties. If one is told that the figure traced on the blackboard possesses four right angles, it is a rectangle, even if the figure is not traced very carefully."

"The properties are not yet organised in such a way that a square is identified as being a rectangle."

"Once it has been decided that a structure is and 'isosceles triangle' the child will also know that a certain number of governing properties must be present, without having to memorise them in this special case."

Level 2:

"Pupils …can understand what is meant by 'proof' in geometry. They have arrived at the second level of thinking."

"The intrinsic significance of deduction is not understood by the student."

"The square is recognised as being a rectangle because at this level definitions of figures come into play."

"The child knows how to reason in accordance with a deductive logical system …this is not however, identical with reasoning on the strength of formal logic."

"I can learn a definition by heart. No level. I can understand that definitions may be necessary: second level."

Level 3:

"He will reach the third level of thinking when he starts manipulating the intrinsic characteristics of relations. For example: if he can distinguish between a proposition and its converse."

"We can start studying a deductive system of propositions, ie the way in which the interdependency of relations is affected. Definitions and propositions now come within the pupils' intellectual horizon."

"The mind is occupied with the significance of deduction, of the converse of a theorem, of an axiom, of the conditions necessary and sufficient."

Level 4:

"A comparative study of the various deductive systems within the field of geometrical relations is …reserved for those who have reached the fourth level."

"Finally at the fourth level (hardly attainable in secondary teaching) logical thinking itself can become the subject matter.'

"One doesn't ask such questions as: what are the points, lines, surfaces, etc.? …Figures are defines only by symbols connected by relationships. To find the specific meaning of the symbols, one must turn to lower levels where the specific meaning of these symbols can be seen."

Gutiérrez, Jaime and Fortuny (1991, p. 238) look at the acquisition of the van Hiele levels of geometric understanding and come to the conclusion that "the acquisition of a specific level does not happen instantaneously or very quickly, but rather can take several months or even years." As a direct conclusion of this research, they point out that there are degrees of acquisition of any one level. In other words, learners leaving matric with a mathematics "qualification" to their names, may well only have begun or may be halfway through the understanding required by, say, Level 3. They also emphasise the significance of not only *understanding*, but being able to *apply* what they understand: "we should take into account their capacity to *use* each one of the van Hiele levels" (ibid, p. 238) (Emphasis added). If learners are taught effectively, the following process may be observed

- Learners are taught about the thinking and methods of a new level. They try to use them, but are not successful, and revert to the lower level of reasoning.

- With increased exposure and experience in the new level, they begin to use its methods of thinking and doing more accurately, but they still fall back on the previous level when they run into problems.

- With further experience, thinking according to the new level becomes customary, but they still revert from time to time as difficulties arise.

- Ultimately they are able to use the current level continuously and effectively without recourse to lower levels of reasoning. (p. 238-239)

In view of the population on which the research in this study was carried out, geometric proficiency of Level 0 was not tested at all, since students of mathematics at tertiary level must by definition have mastered this level.

2.4.2 Structure within the Conceptual Framework for this Study

While the process described by Gutiérrez, Jaime and Fortuny (1991) is certainly ideal and to be striven towards in the effective acquisition of geometric ability, it is dependent upon pedagogic input. According to discussion held in a seminar organised under the auspices of Institute for Advanced Study of Princeton, in 2001 (Ferrini-Mundy et al),

> A common issue was the lack of connection between what took place in many teacher preparation programs and the reality of the classroom. This seems to reflect a mismatch between what prospective teachers are being taught and the expectations and needs of the classrooms. The majority of the group felt teacher education was important, however, offering reasons such as building confidence as a teacher, learning other knowledge necessary for improving teaching, and establishing teaching as a profession.

In the context of this study, however, it is not as relevant to look at what, per se, is being presented to the PME students as pedagogic knowledge, as to investigate whether these students are on a level of understanding which enables them to teach effectively. In their research, van der Sandt and Nieuwoudt (2005) came to the following conclusion:

Results seem to indicate that prospective teachers exit their school career with higher geometric understanding than after three years of mathematics content and methodology training or after four years of methodology training. The deterioration in prospective teachers' geometry content knowledge indicates that pre-service education does not assist in maintaining or improving levels of geometric understanding." (p. 109)

Simply put, knowing how to teach mathematics in general in no way compensates for a lack of content knowledge of geometry in particular. It is therefore essential that any tertiary training for secondary school mathematics teachers includes a sound and well-taught course in which geometric thinking is developed to the point where a competence on at least Level 3 of the van Hiele model is clearly demonstrable in the students' ability to do formal deductions. For this reason this study operates within the structure of an Input-Process-Output model, which may be represented visually as follows:

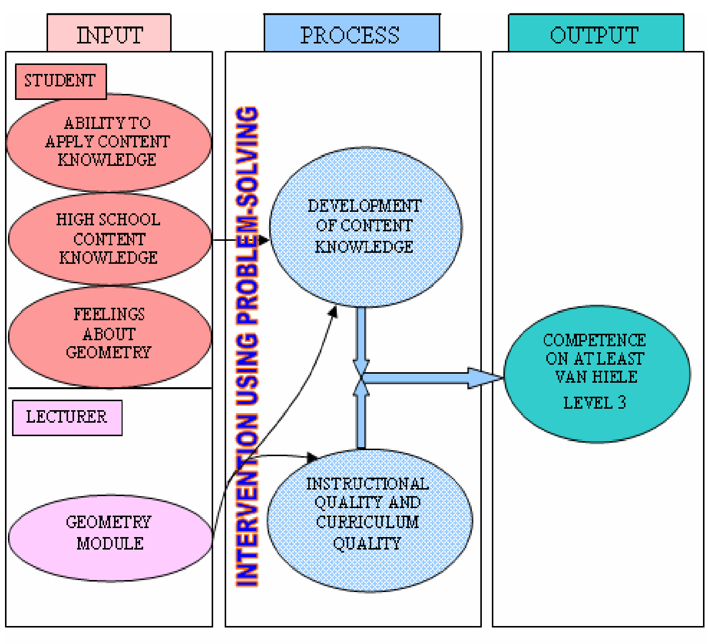


Figure 3. Input-Process-Output structure

The first column of the model describes the elements which are involved in the process that is being studied. The first block refers to the PME students as one of the two elements in the process. The PME students exit school with a certain geometric content knowledge and an accompanying perception of the learning that took place to acquire that knowledge. At the same time, conceptual content knowledge should have been acquired which would enable application of that knowledge in problem-solving situations. Literature has shown that certain emotions are generated during the course of this acquisition, which, while not controlled per se by their high school experiences, certainly influence the student's geometric problem solving ability. This then answers the question suggested by De Feiter et al (1995): "An important question to be asked is whether factors under the control of educators can explain the previously mentioned considerable variation in educational achievement of students." (p. 28)

The second block of the model's first column deals with the University's side of the factors involved in the process. A module on geometry is offered to the third year PME students. Both the content of the module and the excellence of the lecturer in terms of mathematical and pedagogic quality are mentioned here only insofar as it is necessary for the purpose of this research to eliminate the possibility of misgivings with regard to failings on the side of these students' tertiary education. Since the second sub-question of the first research question deals specifically with the content knowledge of FET phase mathematics pre-service students attained during the mathematics course and whether they are subsequently adequately prepared to teach the topic effectively upon qualification, it is relevant to examine the effectiveness of UP's input into the process.

The central column depicts the process in which the elements described above are actively involved. Specifically with reference to geometry, the third year PME students at UP complete a course in which the methodology of teaching of geometry is expounded, in addition to the module which deals with geometric content. This module is presented specifically to develop problem-solving strategies in the minds of the students. It deals with various aspects of geometry, including Euclidian, without following the traditional lesson format of theory instruction followed by exercises. Instead, problems are presented and discussed in class, gradually and almost imperceptibly taking students from one level to the next in terms of the development of their thinking. In fact, the process suggested by Gutiérrez, Jaime and Fortuny (1991, p. 238-239) provides the structure of the module. Students are taught about the thinking and methods of a new level. They experience difficulties and fluctuate in their thinking between the new level and the previous one, which is more comfortable for them. However, they gradually acquire a familiarity with the new level, reverting to the previous one with decreasing frequency. Eventually the new level acquires the status the previous level once had, and they habitually think in terms of the understanding acquired on the new level. Thus, over a period of one year during their third year of academic study as PME students, development of pedagogical and content knowledge is specifically targeted.

The third column represents the desired end product of the process: teachers who have acquired the necessary knowledge and expertise to teach mathematics in general and geometry in particular, efficiently.

CHAPTER THREE

RESEARCH DESIGN AND METHODS

The research began with an in-depth literature review, with a dual purpose in mind: firstly, to investigate the international trends in the teaching of Euclidian geometry; and secondly, to conduct an enquiry into the research that has already been done in this field in South Africa. In the light of this research, a case study design was chosen as the most appropriate to successfully investigate the problem and to serve the purpose of this study. According to Edwards and Talbot (1999),

> The case is a unit of analysis: it can be an individual, a family, a work team, a resource, an institution, an intervention. Each case has within it a set of inter-relationships which both bind it together and shape it, but also interact with the external world. (p. 51)

Ethical clearance was received from UP (see Addendum E), approving this study's adherence to such ethical tenets as "the privacy and dignity of individuals should be respected; informed consent to participate in the test should be sought." (Cohen, Manion & Morrison, 2000, p. 335)

3.1 RESEARCH DESIGN

This is a case study which describes the third year preset mathematics students at the University of Pretoria who are studying to be teachers of mathematics in the FET phase. Merriam (1998) concluded that the single most significant characteristic of this type of research is the clear delimitation of the object of the study, as is the object of this research. Laws and McLeod (n.d.) call it the boundedness of the topic or the finiteness of the data. Merriam (1998, p. 33) also particularly recommends case study as a suitable design for analysis of process – what Laws and McLeod refer to as "causal explanation" (p. 7). The Input-Process-Output

Model presented in Chapter Two and which addresses the research questions, clearly shows that analysis of process is a significant part of this study.

Yin defines case study as an empirical enquiry that explores a “contemporary phenomenon within its real-life context when the boundaries between the phenomenon and context are not clearly evident and in which multiple sources of evidence are used.” (1994, p. 23) The first research question of this study uses a van Hiele test as a source of evidence, while the second and third questions specifically require access (through interviews) to subjective factors in order to be answered, all for the purpose of placing the phenomenon studied under as close a scrutiny as possible. Although the research proposed in this study conforms to this description, it will take the form of a combination of two types of case studies as identified by Yin: it will be both descriptive and explanatory. While a descriptive case study presents a complete description of a phenomenon within its context, an explanatory case study allows “an examination of the processes of change.” (Edwards & Talbot, 1999, p. 52)

This research uses quantitative data such as scores on an assessment instrument that “yield specific numbers that can be statistically analyzed” (Cresswell, 2005, p. 510) in conjunction with qualitative data in the form of open interviews. Cresswell confirms the usefulness of this method when quantitative procedures are followed by qualitative ones “to obtain more detailed specific information than can be gained from the results of statistical tests” (ibid, p. 510) and recommends the explanatory design when the researcher aims to:

> Collect quantitative and qualitative information sequentially or in two phases…. The rationale for this approach is that the quantitative data and results provide a general picture of the research problem; more analysis, specifically through qualitative data collection, is needed to refine, extend, or explain the general picture. (p. 515)

Creswell (2005) speaks of the central phenomenon in qualitative research (p. 133), adding that "the qualitative researcher seeks to explore and understand one single phenomenon, and to do so requires considering all of the multiple external forces that shape this phenomenon" (ibid, p. 134). Creswell (2003, p. 6) also said that "Stating a knowledge claim means that researchers start a project with certain assumptions about *how* they will learn and *what* they will learn during their inquiry" (Emphasis added). This research began with the assumption or central phenomenon, based on the personal experience of the researcher, that the group of students referred to above would manifest the typical difficulties experienced by mathematics students in the proving of Euclidian geometry riders and that this would be confirmed by testing and interviewing the group.

No courses on Euclidian geometry are presented to this group of students in their first and second years of study. Thus, when they were first tested prior to the commencement of their third year module on geometry, their existing knowledge and experience without the benefit of any tertiary institution input was examined. They were then re-tested after completion of the newly introduced semester course in geometry which takes place during the first semester of their third year of academic study. Interviews were also conducted with a sample from this group before and after the geometry course. Lawson and Chinnappan (2000) indicate the usefulness of verbal discussions around a problem-solving situation:

> A teacher who is marking a student's homework or examination script might a times wish that the student were present to explain a particular move … because the verbal explanation might reveal something more about the student's knowledge state than can be identified in the written actions. (p. 28)

The research was structured around the following three questions which were designed to find out what they know and how they apply this knowledge:

How do FET phase mathematics pre-service education students apply their content knowledge of Euclidian geometry theorems to solve riders (including doing proofs)?

A test was designed and administered to PME students to assess their ability to prove geometric riders and reveal their understanding of the premises of proof and the theorems involved. In terms of the theoretical framework, the question being addressed was whether these students could solve problems on van Hiele Level 3, where the principles of formal deduction should be firmly entrenched. This study tests the ability of the PME students to apply their content knowledge before and after the geometry module.

Was the preset FET trainees' experience of learning Euclidian geometry at high school conducive in promoting their progress in terms of the van Hiele levels?

Since the third year mathematics students wrote the pre-test without having done any geometry courses since leaving high school, the test results in fact indicated where these students were positioned in terms of geometry when they left matric. The only factor which may have influenced the results is the erosion that time (approximately 2 years) may have achieved in their recollection of Euclidian geometry theorems and their applications on riders. However, testing alone does not fully answer the question since "experience of learning" also implies perceptions and the recollection of how geometry was taught while they were at school. These issues are addressed specifically, along with the question that follows, during semi-structured interviews with a selected sample of students.

How do the PME students feel about Euclidian geometry?

This information was accessed and studied by means of the interviews mentioned above and gives insight into the nature and complexity of the commonly expressed fears regarding geometry and its application.

3.2 RESEARCH METHODS

Three questions influenced the choice of method used in this research: who was to provide the data as participants in the study; how was the data to be collected and what were the procedures to be followed?

3.2.1 SAMPLE AND PARTICIPANTS IN THE STUDY

The target population was the preset mathematics education (PME) students of 2007 at the University of Pretoria. Only students from the Faculty of Education at the University of Pretoria were tested and involved in this study, firstly for the sake of convenience, since the researcher had direct access to these students as their lecturer, and secondly because the university is known for its wide range and extensive racial and demographic diversity in its student body.

This diversity also implies that the students came from a variety of high schools including rural and urban, private and government, well-resourced and under-resourced institutions, as shown in Table 2. The students forming the sample studied, are mainly Afrikaans speaking and female (sixteen out of the thirty-two), with the second largest demographic group being male and speaking an African language. All of these students passed matric mathematics with a final mark of 50% or more on the Higher Grade (a distinction which was phased out in 2008 and denoting the more difficult of two levels of mathematics that could be taken at matric level in an ordinary high school).

Table 2

Distribution according to medium of tuition in high school and gender of students in sample group

MEDIUM	RURAL	CITY	PRIVATE	TOTAL
Afrikaans				
Female	6	9	1	16
Male	1	2	0	3
				19
English				
Female			2	2
Male				
				2
African language				
Female	2	1	1	4
Male	7			7
				11
TOTAL	16	12	4	32

From the target population, a sample consisting of the entire class of third year mathematics education students (FET phase) was tested at the commencement of the third year geometry module, and again four months later upon completion of the geometry module at the end of the first semester of their third year. This specific sample was selected because these were the teacher trainees, preparing to teach mathematics to grades 10 - 12 learners, a school phase in which approximately 30% of the syllabus consisted of Euclidian geometry (Euclidian geometry is optional for FET learners for the next three years according to the new syllabus), specifically, proving of riders. Of this group of forty-three, eleven, for various private reasons (such as illness or logistical problems), were unable to write the post-test, and were therefore omitted from the calculations used in this research. A purposive sub-sample of five students was selected from the sample of thirty-two for interviewing. Criteria for the selection of interviewees were an approximately equal representation in terms of gender; an equitable distribution in terms of race (white, coloured, and black), language (English, Afrikaans), and range (from failure to distinction) of performance in the pre-test. This distribution is shown in Table 3.

Table 3

Distribution according to race, language, gender, school and respondent ranking in the pre-test of the sub-sample of interviewees

Language, gender and respondent position		White		Coloured		Black	
		Rural School	City School	Rural School	City School	Rural School	City School
Afrikaans	Female	1					
	Male		1	1			
Respondent ranking in pre-test		1	16	22			
English/ African language	Female						1
	Male					1	
Respondent ranking in pre-test						5	43
TOTAL							5

3.2.2 DATA COLLECTION: STRATEGIES AND INSTRUMENTS

An assessment instrument as well as an interview schedule was designed for this study. The pencil-and-paper test was designed as a criterion-referenced mastery test. Gronlund (1998) describes such tests as being "designed to describe which learning tasks a student can and cannot perform rather than to discriminate among students" (p.127). This particular choice of assessment was deemed suitable because this study does not aim to determine which of the respondents was the most gifted or had the most insight into geometry riders. Rather, it was to try to ascertain where their depth of understanding lay in terms of the van Hiele levels.

3.2.2.1 The Pilot Study

The pilot test contained only twenty-four questions, and was conducted with thirty-eight third year FET mathematics pre-service students during the second semester of their third year, in 2006. They had completed the geometry module in June of that year, the year in which this module was, in fact, introduced. The purpose of the pilot study was to determine where the instrument needed improving and what its validity and reliability statistics looked like.

The group was divided into four quarters according to performance. Analysis of their performance in terms of the van Hiele Levels yielded the results depicted in Table 4.

Table 4

Percentages of correct answers per quarter in terms of van Hiele Levels, pilot

Pilot	Van Hiele Level 1	Van Hiele Level 2	Van Hiele Level 3
Top Quarter	79%	60%	75%
Second Quarter	68%	48%	63%
Third Quarter	56%	43%	52%
Lowest Quarter	40%	30%	39%

As is clear from this table, items which accessed thinking on Level 2 were consistently more poorly done than Level 3 items, but were also consistently more poorly done than Level 1 items. This begs the question: Does the problem lie with all the Level 2 questions, or with all the Level 2 question distracters, or with general understanding of Level 2 concepts? Questions 11 through to 15 (Level 2 items) of Usiskin's van Hiele Test, together with their distracters, were similar in content, structure and wording to those used in the pilot. In fact, all the questions in the pilot were similar, level by level, to those in Usiskin's test. In view of the fact that Usiskin's questions had been widely tested and used in many subsequent studies, it was concluded that, instead of changing the items of this instrument and their distracters, it would be more useful to add more questions, level by level, so that a broader base for analysis was created.

The result of allocation of the students to van Hiele levels based on their performance in the pilot testing are depicted in Figure 4. Bearing in mind that the geometry module had been completed just three months prior to the administration of the pilot test, it is interesting to note that just more than half of the students can be categorized on Level 3, and that there are still five students on Level 0. Thus, while the majority of the students are operational on Level 3, 52% of the group reason on levels lower than that.

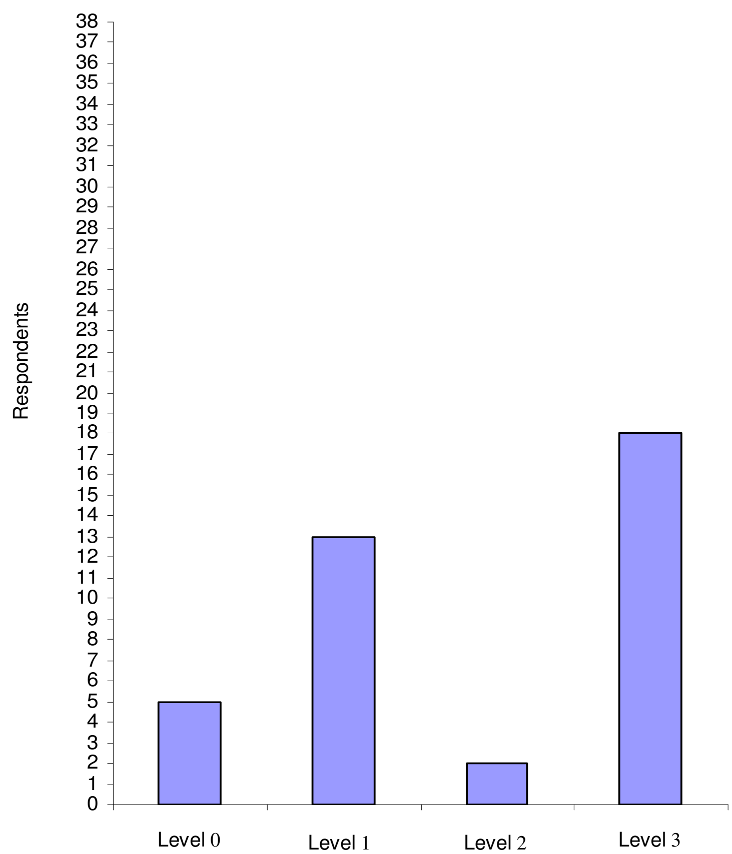


Figure 4. Respondent numbers per level in the pilot test

A traditional item analysis of the pilot was not deemed useful since its results would not used to eliminate or modify items and distracters because, as Gronlund says (1998), "Since criterion-referenced tests are designed to describe what learning tasks a student can perform rather than to discriminate among students, the traditional indexes if item difficulty and item discriminating power are of little value" (p. 127).

3.2.2.2 Design and Development of the Assessment instrument

The paper-and-pencil test (see Addendum F) was designed and developed by the researcher, with questions posed in English and Afrikaans according to the university's policy. Individual items in this test were designed or adapted by the researcher to comply with the requirements of this study ie items which reveal the students' level of understanding in terms of van Hiele. See Addendum A. Eight of the questions (Items 19, 20, 21, 22, 25, 26, 27 and 28) were taken from the test administered in the CDASSG project in 1981 in which 2 700 students in the United States were assessed in terms of the van Hiele levels. This project was conceived by Zalman Usiskin and Sharon Senk (Usiskin, 1982). These eight questions were placed in between the other questions, thus necessitating a change of the numbering of the questions from the pilot to the pre-test.

The same paper-and-pencil test was administered prior to the intervention as after the intervention. It was not the aim of this study to test the validity of the van Hiele theory since this has been reliably done (Mayberry 1983, Burger & Shaughnessy 1986, Senk 1989, Gutierrez et al 1991) but rather to use it as a tool in the attempt to expose the status of geometric understanding in our students. The test items were created and selected in such a way that no knowledge beyond what lay within the reach of the South African high school syllabus was accessed.

The test was designed with the following characteristics:

1. The length of time allocated for completion was 60 minutes, which complied with the students' timetable.

2. The total possible score was 32. This total was based on a one-mark-or-none principle.

3. The first twenty-eight questions were multiple-choice type items.

According to Gronlund (1998), "Multiple-choice items are the most widely used and highly regarded of the selection-type items" (p. 53). Each correct answer was worth one mark. Two types of question were posed: some were knowledge and comprehension items, used to "measure the degree to which previously learned material has been remembered" (Gronlund, 1998, p. 55) and "determine whether the students have grasped the meaning of the material without requiring them to apply it" (ibid, p. 56). The recall of the wording and/or the meaning of the Euclidian theorems that form part of the FET syllabus was accessed through incomplete statement type questions. The second type of question consisted of application items, which, according to Gronlund, make students "demonstrate that they not only grasp the meaning of information but can also apply it to concrete situations that are new to them" (ibid, p. 59). These items took the form of best-answer selection and each of these questions was accompanied by a rider-type diagram upon which the responses were based. Certain riders were posed in such a way as to test whether there were problems in understanding the question rather than in the application of knowledge. An even number of items with the accompanying mark allocation was made with the Split-Half method of testing for reliability in mind.

4. The remaining two questions, each with an allocation one mark per sub-question, fell into the category of supply items. Each of these two questions was divided into two sub-questions, the first of which was a short answer item, and the second of which was an extended-response question. In each case, the short-answer provided clues and information necessary for the solution of the extended-response question, as is customary in tests dealing with geometry riders.

5. The items were classified according to levels 1 through 3 of van Hiele. It was decided that Level 0 would not be tested because this is the most elementary of all the levels and depends upon the recognition of shape. Since the students in question were third year mathematics pre-service teachers, the assumption was made that this level was attained by all long before their entry into secondary school.

6. Questions 1, 3, 8, 9, 12, 27, 28, 29 and 31 are all on Level 1.

Questions 2, 4, 5, 11, 14, 16, 17, 19, 20, 21, and 22 are all on Level 2.

Questions 6, 7, 10, 13, 15, 18, 23, 24, 25, 26, 30 and 32 are all on Level 3.

The order of the items in terms of van Hiele levels is random so that the students are required to think on different levels in no specific sequence, demonstrating their ability to rapidly access reasoning power on various levels. There are more Level 2 items than there are Level 1 items: Level 2 is not clearly addressed in the South African curriculum which formed the basis of these students' schooling, and therefore a broader analysis base on this level would provide evidence in answering Research Question 2 regarding progress through the van Hiele levels engendered by high school geometry education. There are also more Level 3 items than there are Level 2 ones: Research Question 1 in fact requires extensive evidence on this level, since it addresses the ability to do riders and proof construction.

3.2.2.3 Interview schedule

Subsequent to the marking of the pre-test and selection of the interview candidates, semi-structured focus group interviews were conducted with the sub-sample altogether, and were videotaped for later transcription. The same sub-sample was interviewed in the same way after the scoring of the post-test. "Focus groups are advantageous when the interaction among interviewees will likely yield the best information and when interviewees are similar to and cooperative with each other" (Creswell, 2005, p. 215). Although the sub-sample was selected to represent diversity in many ways, the group members were similar in terms of the course they were studying and the fact that they all wanted to be teachers of mathematics at FET level. They could also empathize with each other concerning their high school experiences of geometry learning, all of which happened within the context of South African education.

The interviews consisted of both closed- and open-ended questions, since "Open-ended questions develop trust, are perceived as less threatening, allow an unrestrained or free response, and may be more useful with articulate users" (Richardson, n.d), while closed-ended questions, used to precede an open-ended question, allow for a quicker categorization of the responses eg "Did you feel that the Euclidian geometry section of the matric final was more difficult to do than the other sections?" The strongly agree/ agree/ undecided/ disagree/ strongly disagree type of response that this sort of question elicits facilitated the analysis of the data provided by the sub-question that followed, which dealt with the response in a more detailed manner. The closed-ended/open-ended technique helps the interviewee to focus his/her thinking and is of use in the avoidance of rambling answers. Eight questions were asked in the pre-test and five in the post-test, following Creswell's principle that "a few questions place emphasis on learning information from participants, rather than learning what the researcher seeks to know" (2005, p. 137). Thus the interview protocol allowed the researcher to explore the feelings of the interviewees (eg Did you FEEL that …) as well as the actual experiences of the interviewees which emerged during the discussion that ensued after the posing of each question.

3.3 DATA ANALYSIS PROCEDURES

The raw data provided by the assessment instrument was processed using EXCEL, and was analysed in the light of Research Questions 1 and 2. Each item was also examined in terms its difficulty, discriminating power and effectiveness of the distracters.

3.3.1 ANALYSIS OF THE ASSESSMENT DATA

It was decided to use EXCEL as the tool for the analysis of the raw data. This choice was made because of the accessibility and ease of use of this package, as well as the accuracy and complexity of calculations. Dr Del Siegle, from the University of Connecticut, has created an EXCEL package in which formulae for the calculation of correlation coefficients are preset, including the calculation of the Cronbach Alpha. The advantage of using this particular spreadsheet package is that the correlation coefficients are immediately reflected on the sheet itself, obviating the need to copy and paste data into other calculation sheets. The SPSS programme was used to check the reliability of these results.

Although forty-three students wrote the pre-test, only thirty-two were available to write the post-test. For easy referencing, the thirty-two respondents (R) were numbered according to their performance, so the student with the highest score in the pre-test is coded R1, and the one with the lowest score is R43. It was decided that sufficiently accurate analyses could be done on the thirty-two respondents who wrote both tests. The code names given in the pre-test remained as they were for the post-test as well. Thus, for example, Respondent 3 retained that title despite not being the third best performer in the post-test. The scores were arranged in four performance quarters, each containing the scores of eight students, in descending order, so, for example, the first student in the first quarter is R1 and the eighth student in the fourth quarter is R43.

The data from the pre- and post-test was analysed strictly in relation to research questions 1 and 2, so that it could be determined whether the data did in fact provide the answers that were sought. In addition to this analysis strategy, the items in the pencil-and-paper test were each analyzed according to the criteria suggested by Gronlund (1998, p. 124):

3.4.1.1 The difficulty of the item

3.4.1.2 The discriminating power of the item

3.4.1.3 The effectiveness of each alternative.

While the first two of Gronlund's criteria were each addressed under the pre- and post-test subdivisions, the effectiveness of the distracters was analysed separately to facilitate the examination of anomalies. Finally, an analysis was done of a side-by-side comparison of the pre- and post-test results, in terms of the overall responses as well as the responses per level. In order to assess sensitivity to instruction, a *t*-test analysis was also done, thus determining statistically the impact of the intervention.

3.3.1.1 The Difficulty of the Item

The primary criterion was determining the difficulty of each item according to the van Hiele model. This entailed a careful examination of the criteria governing the placement of a question in a particular van Hiele level. While the eight items taken from the CDASSG instrument were on levels predetermined by Usiskin and Senk, the remaining twenty-four items in the pre- and post-test were discussed with a geometry education expert and their levels established in comparison with similar items both in the CDASSG test and in the test designed by Mayberry (1983). Once this was done, each item was carefully considered in terms of its complexity within that level, the criterion here being that no item was to be more difficult than may be reasonably expected in a matric final examination paper. Finally, the difficulty index (percentage) was calculated per item.

It was decided not to place the questions strictly in sequence according to levels: thus question 14 (a Level 2 item), for example, lies between two Level 3 items. This strategy was chosen in order to force students to move rapidly between levels in terms of their thought processes, thus signifying the facility of their motility between levels. So, while this strategy did not in any way add to the difficulty of the individual items, it did increase the demand for critical thinking.

3.3.1.2 The Discriminating Power of the Item

Gronlund (1998) makes it very clear that, in a test which is criterion-referenced, it is possible that all the candidates give correct answers for a particular question, or in fact, that none do. Analysis of the item’s discriminating power in this case would therefore serve only to show which students are functional on a particular level of the van Hiele model and which are not. At the same time, it had to be taken into account that the complexity of thinking increases as development takes place through a level, thus items may vary in difficulty within a level, depending on what the concepts are that are being accessed by a particular question. Therefore, it was necessary to analyze the data in terms of this criterion, so that the placement of the students’ understanding in the van Hiele levels could be refined.

In order to make sense of the raw scores it was decided to place them in three categories: Mostly correctly answered (sixty-five percent or more of the responses correct); Moderately correctly answered (from forty to sixty-four percent correct answers) and Mostly incorrectly answered (less than forty percent correct answers). This information was tabled as respondent performance analyses. By doing this it became easy to see at a glance which questions were generally beyond the students' level and which were within the reach allowed them by their level of understanding.

3.3.1.3 The Effectiveness of Each Alternative

In some items, alternatives were created in such a way that, while they did not offer the correct response to the question, they were in fact indicators of thinking at a level lower than that accessed by the question. In other words, some of the alternatives were wrong, offering geometric information meaningless in that specific context, but on the same van Hiele level as the correct answer, while others were wrong, but revealed reasoning on a lower van Hiele level than the correct answer. Since a one-mark-or-nothing scoring method was chosen, it was simple matter to extract from the score spreadsheet the frequency of choice of different distracters and then to investigate what made a particular alternative attractive to a particular set of students. Instead of examining and analyzing each distracter of every question, it was decided that more significant information would be gleaned by looking closely at anomalistic situations in terms of distracters chosen.

3.3.2 Analysis of the Interview Data

The interviews were transcribed and then analyzed according to what Creswell (2005) calls a "bottom-up approach… [which] consists of developing a general sense of the data, and then coding description and themes about the central phenomenon" (p. 231). Themes emerged during the process of analysis. Findings from the interviews are reported in the form of a narrative discussion of the information that emerged through responses to each question. Key words were then selected out of the discussion per interview question to facilitate the identification of themes which would act as answers to Research Question 3 as well as explanations of performance in the pre- and post-tests.

Five students were selected for interviewing based primarily on their performance in the pre-test. Interviewee M (Respondent 1: white and female) was the top candidate with a score of twenty out of thirty-two. Interviewee G (Respondent 5: black and female), had a total score of sixteen. Interviewee E (Respondent 16: white and male) scored twelve out of thirty-two, Interviewee W (Respondent 22: coloured and male), ten, and Interviewee D (Respondent 43: black and male) five out of a possible thirty-two. Interviewee D was also the lowest scorer of all in the test. Interviewees are known simply by the first letter of their name.

3.4 METHODOLOGICAL NORMS

According to Gronlund (1998) there are two important questions to ask about an assessment procedure, the first dealing with validity and the second with reliability: “1) to what extent will the interpretation of the results be appropriate, meaningful and useful? And 2) to what extent will the results be free from errors?” (p. 199) One quarter of the items in this assessment were used and tested in the CDASSG project, in which two thousand seven hundred respondents participated.

3.4.1 VALIDITY

As Schell so succinctly states, “Accurate assessment of what students know is a difficult process” (Schell, 1998 p. 2). Geometry, particularly in terms of its requirement for proof construction, is by definition a problem-solving activity. In order to assess this activity, it has been found necessary to use written tests. As Lawson and Chinnappan (2000) have found, “At all levels of education, teachers' analyses of problem-solving behaviour depend heavily on evidence gathered from students' written actions" (p. 28). However, it is acknowledged that no test offers "validity in any absolute sense. Rather, the test scores are valid for some uses and not valid for others" (Thorndike et al, 1991, p. 123).

With regard to the test instrument used in this research, validity is inferred on the grounds of evidence that is construct–related, since the core purpose of this study is to examine geometric thinking levels as described by van Hiele. Geometry by its very nature accesses the construct of the ability to reason. Construct-related evidence was provided in the strict placement of the instrument within the theoretical framework of the van Hiele levels. Through Levels 1 to 3 the ability to reason is the central phenomenon against which all the items in the test were measured.

Criterion-related evidence of validity is also obtained by using this test in that performance in this test is used to "estimate current performance on some criterion" (Gronlund, 1998, p. 204). The criterion under scrutiny here is the ability to construct proof and solve problems on van Hiele Level 3. According to Bond (1996), "CRTs (criterion-referenced tests) report how well students are doing relative to a pre-determined performance level on a specified set of educational goals or outcomes".

At the same time, content-related evidence was produced because FET curriculum content was used. Gronlund (1998) defines content–related evidence as "a matter of determining whether the sample of tasks is representative of the larger domain of tasks it is supposed to represent" (p. 202). He continues to say that "a valid interpretation of the assessment results assumes that the assessment was properly prepared, administered and scored" (ibid, p. 203). Expert judgment was brought to bear upon the instrument used for this research by involving a senior member of the mathematics department and an expert in geometry, specifically, in an advisory capacity in both the design of the assessment instrument and the analysis of its results. In the case of the pencil-and-paper test which was used for this research, such evidence of validity was assured by the use of two norms which were applied in the creation of the items: the theorems and their applications as provided by the FET syllabus, and the exigencies of the van Hiele levels. Through the simultaneous consideration of both of these standard frameworks, the danger of irrelevant or non-functioning items was eliminated. However, as Gronlund (1998) points out,

> A set of items in a criterion-referenced mastery test, for example, might be answered correctly by all students (zero discriminating power) and still be effective items. If the items closely match an important learning outcome, the results simply tell us that here is an outcome that all the students have mastered. (p. 127)

Further concerns in this regard such as inadequate time allowed or an ineffective arrangement of questions were exposed for correction through the piloting of this test with the students who were in their third year of academic study and had completed the geometry intervention which was introduced in 2006.

3.4.2 Reliability

Systematic errors in terms of the raw score of each individual tested are ascribed by Gronlund (1998, p. 211) to “inadequate testing practices” and were eliminated after the field test which was used to pilot the instrument. Measurement error such as memory inconsistencies, motivation and concentration fluctuations, carelessness in marking and guessing when selecting answers (ibid, p. 211) are impossible to eliminate.

It has already been stated that the traditional methods of calculating reliability are not particularly effective when applied to a criterion-referenced test, and usually produce low correlation estimates (Gronlund, 1998):

> "As noted earlier, the traditional methods for computing reliability require score variability (that is, a spread of scores) and are therefore useful mainly with norm-referenced tests. When used with criterion-referenced tests, they are likely to produce misleading results. Since criterion-references tests are not designed to emphasize differences among individuals, they typically have a limited variability. This restricted spread of scores will result in low correlation estimates of reliability, even if the consistency of our test results is adequate for the use to be made of them. (p. 215)

However, the reliability coefficients of the instrument used in this study were obtained by using a variation of the test-retest method as well as internal-consistency methods in both the pre- and post-test. One of the limitations of the test-retest method is the fact that it is influenced by what Gronlund (1998) calls the "day-to-day stability of the students' responses" (p. 212). It may just be, for example, that a student or group of students simply experience a bad day when the re-test is administered, thus affecting the reliability! In this study, the same instrument was used upon the same population group, with an intervening time lapse of approximately four months. However, during that space of time these students participated in the intervention, the new geometry module, taught by a geometry expert. The consistency that was sought after therefore lies in the fact that students who are good at geometry ought to have received the same high marks before and after the intervention: in theory then, the first quarter of top performers in the pre-test would remain, for the most part, in the first quarter of the post-test. At the same time, students who were generally poor performers in geometry would have performed poorly in the pre-test, but should, in theory, have done better in the post-test, having been taught properly by an exponent of the van Hiele model. Therefore, it was to be expected that there would be significant changes in the constitution of the lowest quarter in the post-test.

Since a correlation coefficient is indicative of the "degree of relationship between two sets of measures" (Gronlund, 1998, p. 205), a positive relationship would mean that high scores in one measure are associated with high scores in the other. By contrast, a negative relationship would have high scores in one measure associated with low scores in the other. Thus a perfect positive relationship has a score of 1.00, a perfect negative relationship has a score of -1.00 with no relationship at all being shown by a 0.00 correlation. Thorndike (1991) quotes a value of 0.85 as the test-retest correlation on a Mathematics Composite Test, but does not claim that this is the best coefficient for such a test. In fact, he asks the question, "What is the minimum reliability that is acceptable?" (p.116) He states that,

> The appraisal of any new procedure must always be in terms of other procedures with which it is in competition. Thus a high school mathematics test with a reliability coefficient of .80 would look relatively unattractive if test with reliabilities of .85 or .90 were readily available at similar cost. On the other hand, a procedure for judging leadership that had a reliability of no more than .60 might look very attractive if the alternative was a set of ratings having a reliability of .45 or .50. (p. 116)

In the case of this study, Usiskin's assessment instrument is the obvious "other procedure" because of its similarity to the instrument used in this study, and the fact that eight of the thirty-two questions were in fact taken from the CDASSG test. The average reliability coefficient for questions on Levels 1, 2 and 3 in the CDASSG test was .41.

Despite concerns about predictably low correlations (less than .65) when using internal consistency methods on such a test as this, it was decided to calculate Cronbach's Alpha in order to analyze the variance across the items. Cronbach's Alpha is most appropriately used when the items measure different substantive areas within a single construct, such as in this case, when the single latent variable that is being measured is the ability to reason in a geometric context. Another suitable means of reliability testing of the instrument used in this study is arguably the split-half method: Thorndike (1991) calls this procedure "a sensible one because items of similar form, content, or difficulty are likely to be grouped together in the test" (p. 105). In this study it was possible to rearrange the questions in such a way that pairs of odd numbered items accessing the same sort of knowledge or insight and lying on the same van Hiele level could be scored separately from pairs of even numbered items matched in the same way. These two sets of scores could then be correlated. The reliability coefficient for the total test could then be determined by using the Spearman-Brown prophecy formula, which, according to Thorndike (1991) "has the advantage over internal consistency methods that it does not assume homogeneous content across all items, only between the two halves" (p. 105).

It was decided that, should the field test of the instrument reveal that the reliability coefficient was low, items similar to those in the test would be inserted to increase the length of the test, in an effort to increase the reliability coefficient. This was in fact exactly what transpired. In the pilot twenty-four questions were used and tested on thirty-eight subjects. The results of this analysis are shown in Table 5.

Table 5

Reliability coefficients for the pilot testing of the instrument

Test	Coefficient
Cronbach's Alpha	0.64
Split-Half (odd-even) Correlation	0.67
Spearman-Brown Prophecy	0.80

So it was decided to add more questions for the following reasons: not only would the reliability coefficient be improved by so doing, but a more sound evaluation per level would thus be possible. Thus, eight questions from the CDASSG test were added. However, the modified instrument could not be successfully tested on the same subjects within so short an interval. Interestingly, the reliabilities calculated by Usiskin's team of the first four levels of the CDASSG test were 0.31, 0.44, 0.49 and 0.13 in the pre-test and 0.39, 0.55, 0.56 and 0.30 in the post-test. In their efforts to develop a test most stringently following the guidelines the van Hieles laid down per level, Usiskin (1982) and his team found that the van Hiele's descriptors of behaviours were sufficient in both quantity and detail to make the creation of test questions easy. However, the team found that constructing questions to accurately test Level 3 was considerably more difficult, while Level 4 was denoted as being of questionable testability.

Reliability coefficients calculated on the pre-test revealed surprising figures: there was in fact a marked decrease from the pilot across all three coefficients. However, in the post-test all three coefficients were higher than those computed for both the pilot and the pre-test. This is clearly shown in Table 6.

Table 6

Reliability coefficients for the pre-test and post-test

Pre-test	
Test	Coefficient
Cronbach's Alpha	0.61
Split-Half (odd-even) Correlation	0.48
Spearman-Brown Prophecy	0.64

Post-test	
Test	Coefficient
Cronbach's Alpha	0.68
Split-Half (odd-even) Correlation	0.74
Spearman-Brown Prophecy	0.85

Two closely linked reasons are suggested for this: firstly, the students may have a partial recall of what was possibly, at best, a partial understanding of certain concepts within each level; secondly, as a result, they may be driven to guessing. This inconsistency of understanding is examined more closely in Chapter 4. However, in the post-test the reliability is clearly remarkably improved. This may well be ascribed to the effect of the intervention which brought about a greater consistency in understanding in terms of the van Hiele levels as demonstrated through the more even performance of the individual respondents in the post-test.

The marking of the supply item section of the instrument was moderated by a geometry education expert. This ensured that this section of the instrument in which there was a possibility of subjective scoring was as reliably scored as possible.

3.4.3 Trustworthiness of the Interview Data

The interviews were conducted in a non-threatening, "safe" environment in which participants were assured of the confidentiality of the discussion as well as the fact that their academic results were not affected by the interviews.

While it is assumed that the participants were truthful because they would have nothing to gain by not being honest, Talja (n.d.) confirms that "The reliability of research results does not depend on the trustworthiness of participants' answers, since even a speaker who lies applies cultural forms and interpretative resources which, in themselves, are neither true or false, but simply exist." (p. 12) In effect, data of this kind is seldom completely free of bias, and can therefore not be seen as an absolutely accurate description of reality. However, since the aim of the interviews was largely to access the feelings of the participants, inaccuracy of this nature was not deemed a problem.

CHAPTER FOUR

RESULTS OF THE STUDY

The data provided by this study was collected through two sources: the pencil-and-paper test, and two sets of interviews. The test was administered a total of three times, and group interviews were conducted twice. Analysis of the interview data began with the study of the transcript of the first interview, in March, 2007.

4.1 PME STUDENTS' CONTENT KNOWLEDGE OF EUCLIDIAN GEOMETRY

4.1.1 THE PRE-TEST

The pre-test was conducted with the forty-three PME students of 2007, of whom the eleven who were unable to write the post-test were not included in the data analysis. The highest overall score was 59% and the lowest 16%, while the group average was 35%. Figure 5 clearly demonstrates the range of correct responses over the thirty-two items.

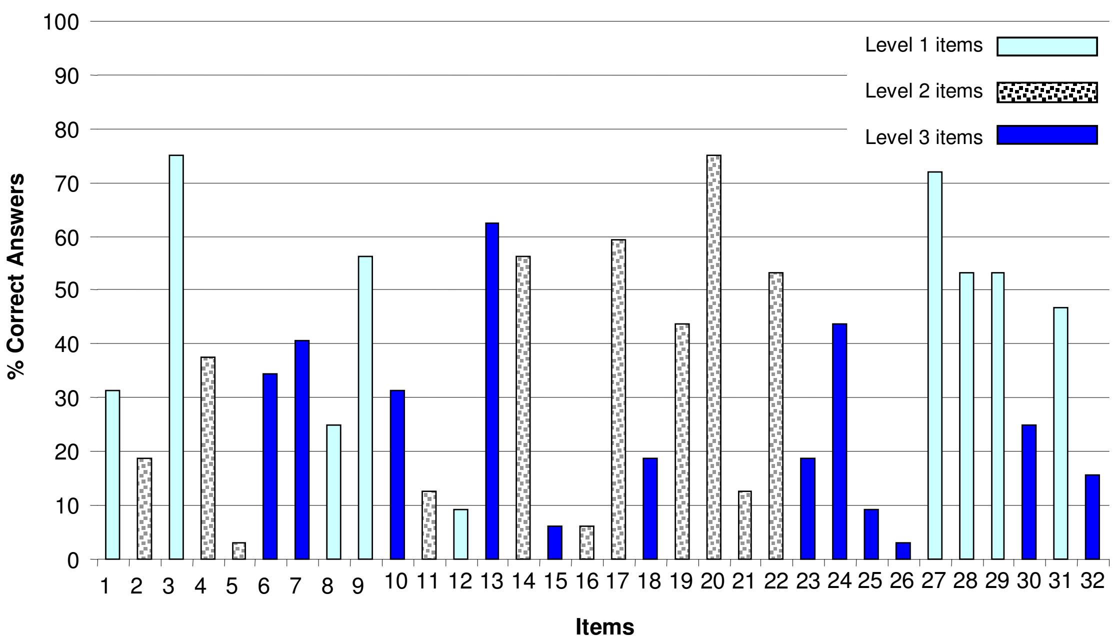


Figure 5. Overall performance per item in the pre-test.

The performance of the students follows the trends as stated in the literature. Although certain questions were well answered (above 60% correct), most of the items lie well below 50%, with items 5, 11, 12, 15, 16, 25 and 26 proving to have been the most incorrectly answered. The standard error of measurement of the students' scores is 0.67, calculated against a standard deviation of 3.74. This implies that the "band of uncertainty" (Thorndike, 1991, p. 109) with regard to the interpretation of these scores is a narrow one.

As discussed in Chapter Three, the assessment instrument was designed in such a way that the questions were not arranged according to their ranking on the van Hiele levels, so as not to predispose the students to thinking that they were working from the easiest to the most difficult questions, and to prompt them to demonstrate their ability to rapidly access reasoning power on various levels. As can be seen in Figure 4, this random distribution of items in terms of van Hiele levels is manifested in a random distribution of performances across the chart. Nevertheless, the highest distribution of scores lies below the 40% line. This means that the performance on all three levels of van Hiele is low. A table presenting a full item analysis can be found in Addendum B.

In order to analyse the performance of the students in terms of the van Hiele model, it is necessary to examine the performance of the students in quarters, where the top quarter consists of the eight highest scoring respondents, through to the fourth quarter made up of the lowest scoring respondents. This distribution is shown in Table 7. From this table can be seen that the top-scoring eight students could only achieve an average score of 44% on Level 3, while the eight lowest scoring students achieved an average score of only 35% on Level 1. Only the top eight students scored a pass average on Level 2, which was in fact the highest level they were able to achieve.

Table 7

Percentages of correct answers per quarter in terms of van Hiele levels, pre-test.

Pre-test	Van Hiele Level 1	Van Hiele Level 2	Van Hiele Level 3	Average
Top Quarter	60%	52%	44%	51%
Second Quarter	50%	38%	24%	36%
Third Quarter	43%	26%	23%	30%
Lowest Quarter	35%	22%	12%	21%
Average	47%	35%	26%	35%

There is a definite descending trend from Level 1 through to Level 3, as predicted in the literature. In fact, the group average lies below 50% on all three levels. It is nevertheless important to take note of the very low percentages of achievement on Level 3. This group's average performance on this level is a mere 26%, which indicates that their ability to reason in a formally deductive way has not been developed to a point where this can be done successfully or consistently. This is confirmed by the allocation of levels to the students according to their performance, as shown in Figure 6.

Allocation of levels was done on the following principle: if a respondent scored 50% or more for the questions on a particular level, he/she was deemed to have "passed" that level, and was thus categorised as competent on that level; if a respondent skipped a level, showing competence on Levels 1 and 3, for example, but not on Level 2, that student was categorised as competent on Level 1 only. Thus the students who are deemed competent on Level 3, are also competent on all previous levels. For the purpose of this particular analysis, Level 0 had to be introduced as a category on the chart, since half of the students did not qualify to be categorised on Level 1 in the pre-test. Figure 6 clearly shows that at the time of the pre-test the vast majority of students did not manifest any competence on either of Level 2 or Level 3. Interestingly, Level 2 was attained by almost the same number of students as was Level 3.

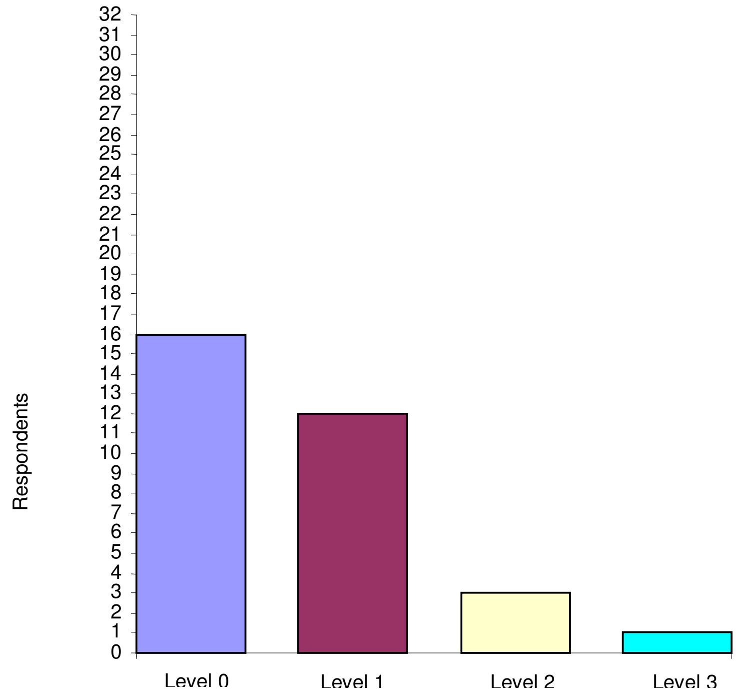


Figure 6. Student numbers per level in the pre-test

It is, however, also useful to analyse the distribution of items across the three categories which denote how well they were answered (Table 8). This particular analysis is necessary as opposed to a simple discussion of items per van Hiele level because this analysis reveals that certain types of content knowledge, as allocated by van Hiele to certain levels, are in fact more difficult than other types *within the same level*; students can be at different depths of understanding within *one* level, depending on the specific knowledge that is being accessed. Thus there are several Levels 1 and 2 questions in the most poorly answered category. Generally, the students seemed to find the items difficult, since twenty-nine of the thirty-two items received less than twenty correct answers. The average standard error of measurement *per item* is 0.07, indicating a very narrow band of uncertainty. See Addendum A. Only three items were answered correctly by more than 60% of the respondents. A significant anomaly here can be seen in the presence of a Level 2 question in the most well answered category, since one would expect only Level 1 items to lie in this category. Item 20 deals with the properties of equilateral and isosceles triangles and achieved an overall score of twenty-four correct answers out of the possible thirty-two.

Table 8

Analysis of performance on items in the pre-test

Van Hiele level of item	Items mostly correctly answered (more than 20 correct answers)	Van Hiele level of item	Items with between 10 and 20 correct answers	Van Hiele level of item	Items mostly incorrectly answered (less than 10 correct answers)
1	3	1	1	2	2
2	20	2	4	2	5
1	27	3	6	1	8
		3	7	2	11
		1	9	1	12
		3	10	3	15
		3	13	2	16
		2	14	3	18
		2	17	2	21
		2	19	3	23
		2	22	3	25
		3	24	3	26
		1	28	3	30
		1	29	3	32
		1	31		
Total:	3		15		14

There is an even distribution of items between the second and third categories, with 15 and 14 items respectively. In the second category there is a very even distribution of van Hiele level items: 5 each on Levels 1, 2 and 3. As may have been expected, most of the items that were very poorly answered lay on Level 3, but contrary to expectation, two of the items in this category lay on Level 1. These specific questions dealt with property recognition and classification according to properties, and so required a high proficiency in property recognition of different shapes.

In theory then, the students should have found the items on Level 1 the easiest, and the items on Level 3 the most difficult. Thus, a simple predictable outcome here would have been that all the questions mostly correctly answered were on Level 1, and all those mostly incorrectly answered were on Level 3. However, although there is a general tendency toward this trend, it is clearly not absolutely the case. There are at least four possible reasons for this (discussed more fully in section 4.1.4.2 when examining anomalies):

i) Memory may allow a random pattern of recall, which may also be dependent on what was particularly drilled at school or not.

ii) Language may be a factor: some of the students in the sample group have poor English language skills, and certain items required more complex geometric terms than others.

iii) Some of the distracters in certain of the questions may have been too strong.

iv) It is possible for a person to be on different levels with regard to different *types* of applications, simultaneously.

It is however obvious that the majority of mostly incorrectly answered questions did, in fact, belong to Level 3. Most interestingly, the questions ostensibly most difficult to answer correctly were items 5 and 26, both Multiple Choice items, each answered correctly by only one person. Item 26, although it is categorized by Usiskin as a Level 3 question, requires a modicum of Level 4 thought. Item 5 tests the understanding of the principle of class inclusion. Respondent 28 (R28) who lies in the bottom quarter in terms of overall performance, answered question 5 correctly, while Respondent 12 (R12), lying in the second quarter, answered question 26 correctly. However, neither respondent answered the same question correctly in the post-test, so it is possible that his/her correct answer in the pre-test was a guess and not based upon knowledge. Items that were correctly answered by only two people were questions 15 and 16, on Levels 3 and 2 respectively. Both of these were applications of the deductive kind, accompanied by sketches and a set of given information. The two questions most well answered were items 3 and 20, both of which, although on Levels 1 and 2 respectively, had to do with properties of isosceles and equilateral triangles. Both of these questions were answered correctly by twenty-four respondents; Item 3 evenly spread through the first three quarters; Item 20 evenly spread through all four quarters. The only other question that was nearly as well answered (twenty-three correct answers) was Item 27, which deals with the properties of two isosceles triangles forming a kite.

By analyzing the respondent performance in this way, Research Question 1a) is addressed: the content knowledge of these students in terms of the van Hiele levels is definitely not sufficient. Only three questions were really well answered, and none of these was on Level 3, the required level for the students. The remaining twenty-nine items were consistently poorly answered, clearly indicating that no efficient application of content knowledge has taken place at all.

Following the method suggested by Gronlund (1998, p. 124), the difficulty index was calculated using the percentage of correct answers in the first and fourth quarters (see Addendum A). This index indicates the average percentage of correct answers in the test, using only the top and bottom quarters. It is important to note that this index shows the difficulty of each item as experienced by *this* particular set of respondents, not the difficulty of each item in a general context. The higher percentages thus indicate items whose difficulty was lower. The discriminating power of each item was also calculated. Since the pre-test was written by the students with only their high school geometry knowledge to depend on, it was to be expected that the inconsistencies in understanding within a particular level, would influence these indices. The average difficulty percentage across the items is 36,5 % : the test was not found to be an easy one; with an average discriminating power of 0.2 across the items, it seems clear that neither the top performers nor the lowest performers were able to achieve notable success.

There are only four questions with a difficulty index above 75% (Items 13, 17, 20 and 27). In the category with difficulty indices lying between 74% and 50%, only six questions (Items 3, 9, 22, 28, 29 and 31) are to be found. The remaining twenty-two questions have difficulty indices which all lie below 40%.

Predictably, the discriminating power of these items proved to be low, once again demonstrating the fact that a test of this kind, which aims not to differentiate between able and less able students, but to expose geometric understanding against the background of a hierarchical structure, cannot be accurately judged in terms of the traditional use of these two indices. Nevertheless, these indices are useful in revealing phenomena both within the van Hiele Model (Level 2 content appears to require insight more suited to a higher level) and within the way geometry is taught (rote learning can disguise a lack of insight). The first of these two phenomena is revealed in that the expected result in the test would be that the Level 3 items yield the highest discriminating power, and the Level 1 items the lowest, since it would be logical to expect that the students who are in the First Quarter would do better at the more difficult Level 3 questions than those in the Fourth Quarter, while the Level 1 questions would be found more-or-less accessible to all. However, Addendum A reveals that the questions with the highest discriminating power are Items 19, 21 and 22, *all Level 2* Multiple Choice questions, while the lowest discriminating power is to be found in Items 16 (Multiple Choice) and 31 (Free Response Item), Level 2 *and* Level 1 questions respectively. Items 19, 21 and 22 deal with the notion of inclusion, which, while being on Level 2 according to van Hiele, requires a certain amount of higher order thinking and is not well covered in the South African geometry syllabus. The second of these two phenomena is demonstrated in the fact that both Items 16 and 31 had a negative discriminating power, having yielded better results in the lowest quarter than in the highest. This may be the result of the lingering effect of rote learning, in the case of Item 31, of the wording of a theorem; in the case of Item 16, of the functioning of altitudes in a triangle.

4.1.2 THE POST-TEST

The post-test was conducted with thirty-two of the PME students who had written the pre-test. The highest overall score was 78% and the lowest 19%, while the group average was 55%. The performance per item is represented in Figure 7. From this representation it can clearly be seen that the items in which the highest scores have been achieved both lie on Level 1, while the items with the three most poorly answered items are two on Level 3 and one on Level 2. There is an overall improvement in the answering of the questions, compared to Figure 5.

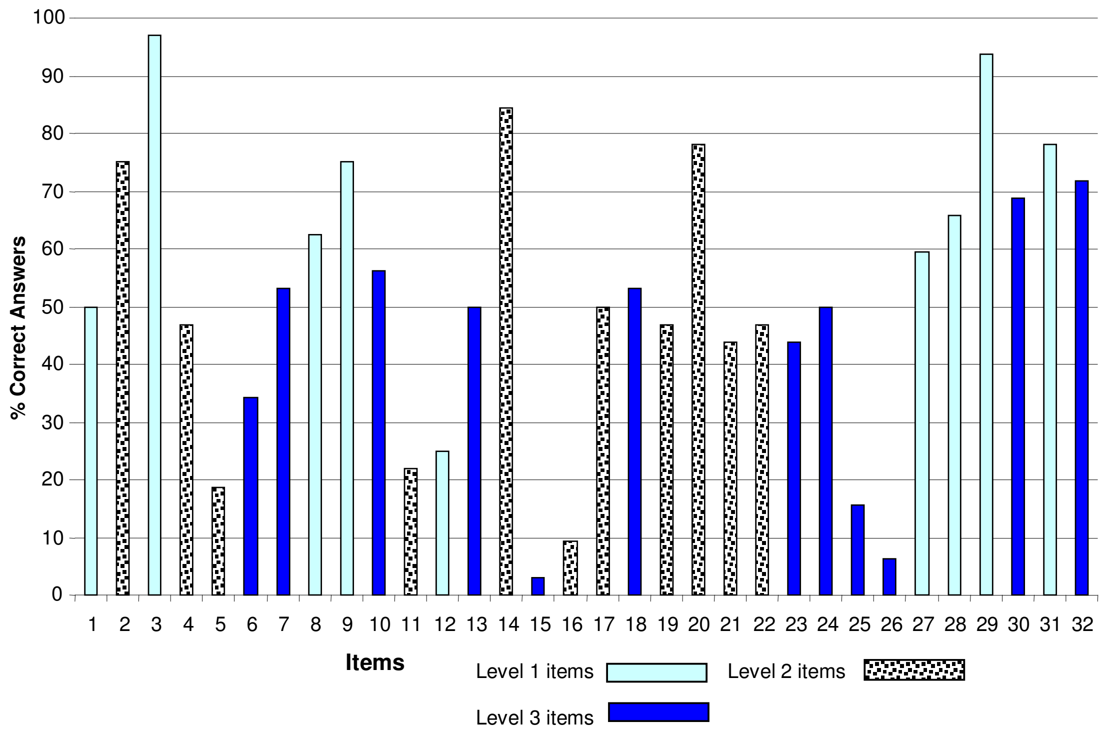


Figure 7. Overall performance per item in the post-test.

Examining the post-test data, it was found that a different picture was revealed. Less than half of the items now lie below the 50% correct line. The items in the pre-test which were the most incorrectly answered (items 5, 11, 12, 15, 16, 25 and 26) remain in that category, but the actual scores on these items has now doubled. In the pre-test there were no items that received more than 80% correct answers, whereas in the post-test there are three, two of which are over 90%. The standard error of measurement of the students' scores in the post-test is 0.75, calculated against a standard deviation of 4.27. This implies that the "band of uncertainty" (Thorndike, 1991, p. 109) with regard to the interpretation of these scores, although slightly wider than in the pre-test, is still a narrow one. Comparing the data presented in Figure 5 with those presented in Figure 7, a distinct improvement in the percentage of correct answers per item is visible. This means that there is an improvement across the van Hiele levels as well. A table presenting a full item analysis can be found in Addendum B.

Figure 8 clearly depicts the general shift in level that has taken place. There are now six of the thirty-two students who demonstrate competence on all three levels that were tested.

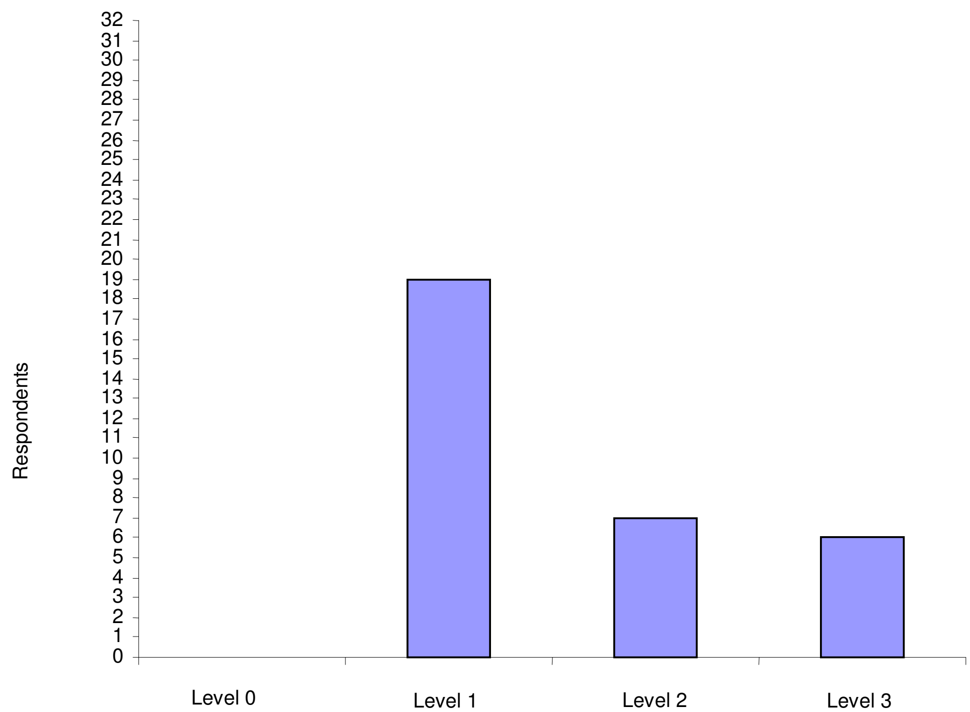


Figure 8. Student numbers per level in the post-test

However, the overwhelming majority of students can still only show competence on Level 1. There is very little difference between Levels 2 and 3 when it comes to student performance: in fact, there is very nearly parity in the number of students competent on these levels: seven students now lie on Level 2 and six on Level 3.

Bearing in mind that the students in this sample should be comfortable working on at least Level 3 in order to teach geometry effectively upon qualification, it is essential to examine the percentage of correctly answered questions per quarter in each of the levels *after* the intervention. The quarters are now made up of different respondents, since many respondents were displaced because of the improved performance of several students. Interestingly, Respondent 1 now lies in the second place, while Respondent 43 now lies in seventeenth place, instead of last.

Table 9

Percentages of correct answers per quarter in terms of van Hiele Levels, post-test compared with pre-test

Post-test	Van Hiele Level 1		Van Hiele Level 2		Van Hiele Level 3		Average	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
First Quarter	60%	78%	52%	63%	44%	63%	51%	72%
Second Quarter	50%	65%	38%	61%	24%	41%	36%	58%
Third Quarter	43%	65%	26%	46%	23%	35%	30%	51%
Fourth Quarter	35%	43%	22%	31%	12%	30%	21%	37%
Average	47%	63%	35%	50%	26%	42%	35%	55%

A vast improvement has taken place: the overall score average has improved by 20%. Although the third and fourth quarters are still not operating comfortably on Level 3, the percentage increase in these two quarters from 23% and 12% respectively in the pre-test shows clearly that these students are indeed capable of having their insight and understanding of geometry increased through effective tuition. It may therefore be suggested that the low percentages seen in the pre-test, which could be ascribed to inadequate teaching at high school as well as the effects of a lapse in time, have been improved by the presentation of a sound geometry module at tertiary level.

Analysis of the distribution of items across the three categories denoting how well they were answered, reveals the information presented in Table 10.

Table 10

Respondent performance analysis for post-test

Van Hiele level of item	Items mostly correctly answered (more than 20 correct answers)	Van Hiele level of item	Items with between 10 and 20 correct answers	Van Hiele level of item	Items mostly incorrectly answered (Less than 10 correct answers)
2	2	1	1	2	5
1	3	2	4	2	11
1	9	3	6	1	12
2	14	3	7	3	15
2	20	1	8	2	16
1	28	3	10	3	25
1	29	3	13	3	26
3	30	2	17		
1	31	3	18		
3	32	2	19		
		2	21		
		2	22		
		3	23		
		3	24		
		1	27		
Total:	10		15		7

Most of the data lies in the middle category where between ten and twenty items were answered correctly. However, we see a significant reduction of items in the last category in which most of the answers were incorrect, from fourteen in the pre-test to seven in the post-test, as well as a significant increase in the first category, from three in the pre-test to ten in the post-test. This may interpreted as a clear indication of an increase in insight as a direct result of the intervention, both by the creation of new insight, and the refreshing of forgotten knowledge. The average standard error of measurement *per item* is 0.08, indicating a very narrow band of uncertainty. See Addendum B. It is also significant that all the questions that were categorized as the most poorly answered in the post-test were in the same category in the pre-test. This would imply that these questions remained challenging despite the understanding acquired or refreshed during the course of the intervention.

An item analysis of the post-test confirms that a change has taken place (see Addendum B). The average difficulty of the items has improved from 36.5% in the pre-test, to 50.8% in the post-test. Also, the discriminating power of the items has changed from 0.2 to 0.3. See Addendum B. Once again it must be borne in mind that these indices are calculated using only the first and last quarters.

There are now six questions (Items 3, 20, 29, 30, 31 and 32) with a difficulty index above seventy-five percent, while there are eleven questions (Items 1, 5, 11, 12, 15, 16, 19, 22, 24, 25 and 26) with a difficulty index of below fifty percent. Therefore, twenty-one out of the thirty-two questions have a difficulty index of 50% or more, yielding an average difficulty index of 50,8%. This may be taken to mean that, while the test is still found to be difficult, it now lies within the reach of most of the respondents.

4.1.3 COMPARISON BETWEEN PRE-TEST AND POST-TEST RESULTS

It is however useful, in order to gauge the effect of the intervention, to look at different representations of the same data: where the overall response per level (including the allocation of levels to the students) in the pre-test is measured against the same statistics in the post-test; where each student's response per level in the pre-test in considered alongside of the post-test. This particular analysis provides an answer to Research Question 1a) juxtaposed to Research Question 1b) in that van Hiele level results prior to the intervention are compared with the same results after the intervention.

4.1.3.1 Overall response comparison

Table 11 represents a side-by-side comparison of the overall response per level in both the pre- and the post-test. There is an indisputable improvement in response after the intervention. Of significance is the smaller change in the responses dealing with thinking on Level 2 thus indicating that the categorisation of certain activities (such as class inclusion) as indicative of a Level 2 competence remains questionable.

Table 11

Number of correct responses per level in the pre- and post-test.

	Level 1			Level 2			Level 3		
	Score	Total possible points	%	Score	Total possible points	%	Score	Total possible points	%
Pre-test	135	288	47	121	352	34	98	384	26
Post-test	194	288	67	167	352	47	162	384	45
Difference	59		20%	46		13%	64		19%

The results demonstrated in this comparison may be analysed as follows: clearly, Level 1, which, since its establishment in the learner's mind begins in approximately Grade 3 and continues to play an important role to matric level, should be the most thoroughly rooted in any mathematics student's understanding. In fact, one might well say that this level should form part of such a student's geometric instincts. Thus, Level 1 questions produced the best results in the pre-test. The difference of 59 points in score on this level in the post-test may well be ascribed to a refreshing of the memory with regard to the finer details contained within this level, rather than to the creation of an understanding of the basics during the course of the intervention. Level 2 was not as well answered as Level 1, in all probability because understanding of work on this level includes, according to van Hiele, quite complex class inclusions. Level 3 questions, not at all well answered in the pre-test, were answered with almost the same indication of competence as the Level 2 questions in the post-test. This confirms the research conducted by Kotzé (2007, p. 33) in which she found that respondents (Grade 10 mathematics teachers and their learners completing the same assessment instrument) in fact performed *better* on Level 3 than on Level 2. She asks how such anomalistic reasoning patterns may be developed: I propose that the phenomenon she observed may be ascribed to the way these teachers were, themselves, taught – by rote learning – and that this in fact disguised their inadequate *deductive* skills on Level 3. However, the greatest improvement in performance can be seen in Level 3, where there is the highest score difference from pre- to post-test. Such an improvement is directly related to an improved ability with regard to deductive reasoning, the essence of Level 3 work.

While Table 11 indicates the correct responses per level, it does not show the performance of the students per se in terms of the levels. When their achievement or lack thereof is presented side by side, as in Figure 9, the differences in performance can clearly be seen. Half the students had to be categorised on Level 0 in the pre-test. However, in the post-test there were no longer any students on Level 0. By the time the post-test was administered, of the sixteen students who had performed on Level 0 in the pre-test, nine had moved onto Level 1 in the post-test, two onto Level 2, and four onto Level 3, the latter without mastering the questions on Level 2.

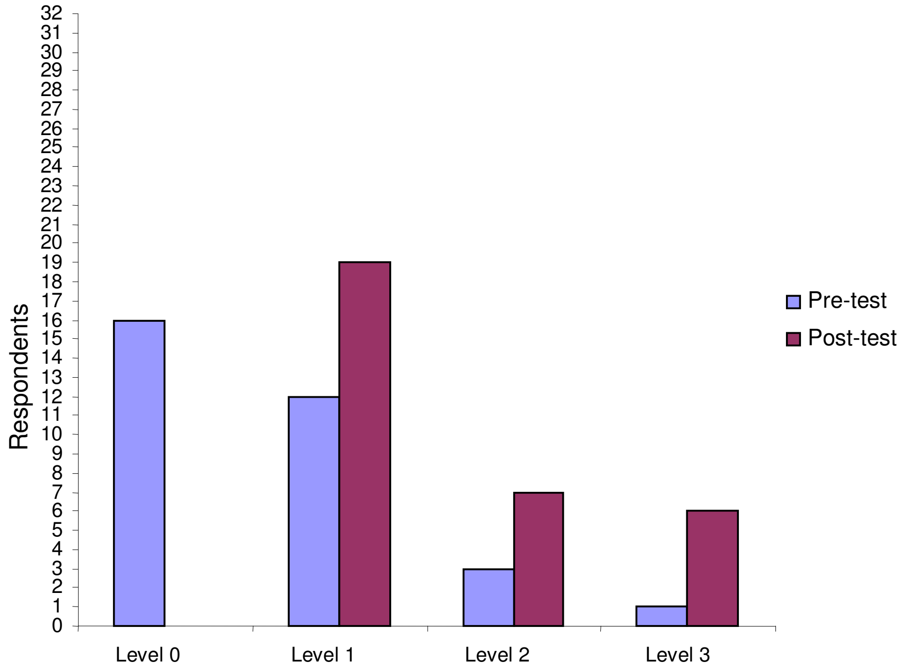


Figure 9. Comparison of respondents numbers per level in the pre- and post-tests.

This statistic corroborates the findings of Mayberry (1983) and Senk (1989) who both discovered that students can be on different levels with regard to specific concepts. In this research the questions dealing with class inclusion, a Level 2 concept according to van Hiele, were uniformly poorly done. For the purpose of this study, students who missed a level were categorised according to the lower level which they *did* pass. This decision was made because such students *do* manifest competence appertaining to the previous level while doing the questions on the skipped level.

4.1.3.2 Comparison of students' responses per level.

A graphic representation of how the students fared per level shows clearly that distinct progress has been made through the van Hiele levels as a result of the intervention. Of particular interest are the anomalies which occur in each representation. Bearing in mind that responses to at least nine items are grouped as the score per level for each student, such anomalies require closer investigation.

Level 1

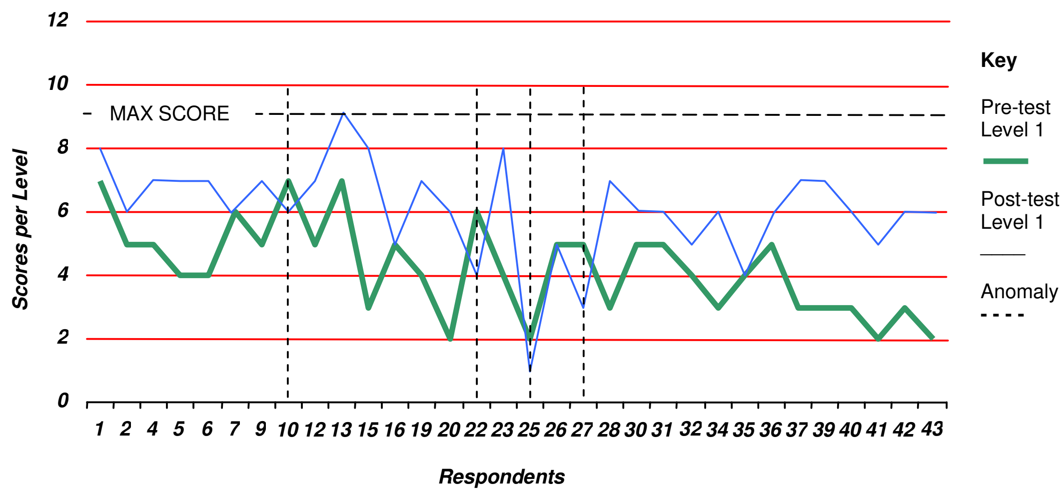


Figure 10. Participants' test scores on Level 1 in the pre-test and the post-test, n = 32.

The general tendency in Figure 10 is that the post-test line lies clearly above the pre-test line. In the fourth quarter (Respondents 35 to 43) this tendency is most marked. This is particularly significant and can possibly be explained by the fact that memory is refreshed and earlier insights re-awakened by the Level 1 work that was done during the module. The exceptionally poor results that had been achieved by these respondents in the pre-test were therefore not entirely attributable to a lack of potential to understand the work done on this level. Four anomalies are to be noted here. The performance of Respondents 10, 22, 25 and 27 is worse in the post-test than in the pre-test. Examining the choices made by these four students on Level 1 during the two tests reveals the following information.

Respondent 10:

	Ques 1	Ques 3	Ques 8	Ques 9	Ques 12	Ques 27	Ques 28	Ques 29	Ques 31	Total Level 1
Pre-test	1	1	0	1	1	1	1	1	0	7
Post-test	0	1	0	0	1	1	1	1	1	6

In Question 1 this respondent was unsure of the answer in the pre-test, circling the correct answer, but drawing a line through it which s/he later erased. This uncertainty solidified into a wrong choice in the post-test.

Question 9 has a language implication. The vocabulary used is mathematical and not elementary. Upon investigation it was found that neither of the languages used in the test was the mother tongue of this student. It is possible that the correct answer chosen in the pre-test was not by design, but based on understanding a distracter which was shorter and in simpler terms than the other choices.

It is noteworthy that Item 31, a theorem recall question, was correctly answered the second time round. Also, the same incorrect choice was not made for Item 8, indicative of a less than perfect understanding of the work being done on this level.

Respondent 22:

	Ques 1	Ques 3	Ques 8	Ques 9	Ques 12	Ques 27	Ques 28	Ques 29	Ques 31	Total Level 1
Pre-test	0	1	1	1	0	1	0	1	1	6
Post-test	0	1	0	1	0	0	0	1	1	4

This student also answered two more questions incorrectly in the post-test, but did not improve in any of the incorrect answers of the pre-test. S/he also did not select the same incorrect answer in Items 1, 12, or 28. This seems to indicate that the selection of different (incorrect) answers for Items 8 and 27 is not the result of understanding being trammelled, but was, instead, an unfortunate random event.

Respondent 25:

	Ques 1	Ques 3	Ques 8	Ques 9	Ques 12	Ques 27	Ques 28	Ques 29	Ques 31	Total Level 1
Pre-test	0	0	0	0	0	1	0	1	0	2
Post-test	0	0	0	0	0	0	0	1	0	1

Language is clearly a factor in the performance of this student in both the pre- and pos-test, rather than simply mal-comprehension. This student was in fact able to do both application riders (Items 30 and 32) in the post-test, both Level 3 questions, while still unable to word the theorem recall question in item 31 correctly. This student's home language was not represented in the test.

Respondent 27:

	Ques 1	Ques 3	Ques 8	Ques 9	Ques 12	Ques 27	Ques 28	Ques 29	Ques 31	Total Level 1
Pre-test	0	1	0	0	0	1	1	1	1	5
Post-test	0	1	0	0	0	0	0	1	1	3

This student, in Items 8 and 12 selected the same incorrect answer both times. This seems to imply a lack of understanding of the work presented on this level which could well explain the lapse from correctness in Items 27 and 28.

Level 2

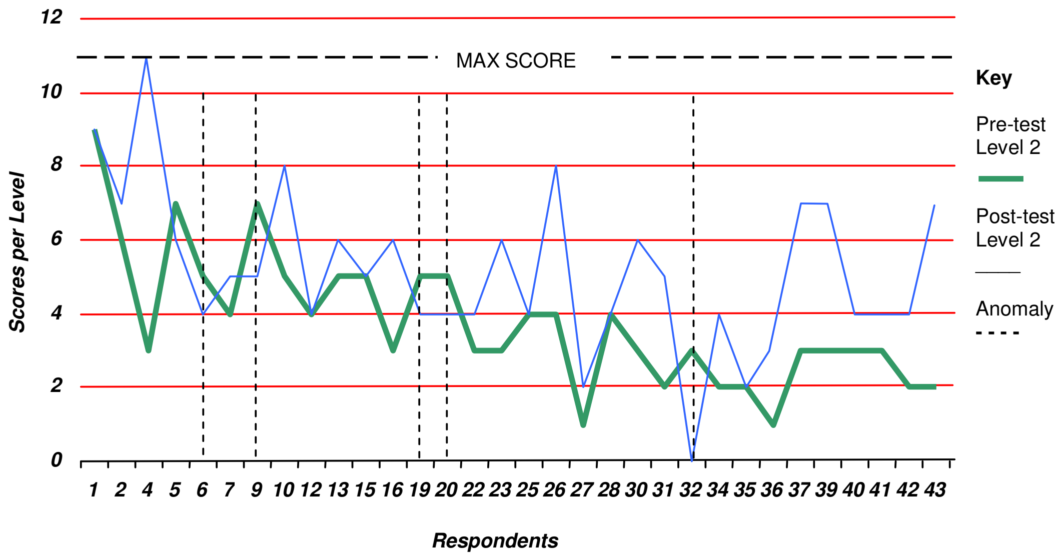


Figure 11. Participants' test scores on Level 2 in the pre-test and the post-test, n = 32.

Once again the picture presented of the respondents' results shows a marked improvement in the post-test. Amongst Respondents 35 to 43 in particular, there is a significant improvement after the intervention. Level 2 items reveal the same type of small anomalies found in the Level 1 responses, with the exception of Respondent 32.

Respondent 32:

	Ques 2	Ques 4	Ques 5	Ques 11	Ques 14	Ques 16	Ques 17	Ques 19	Ques 20	Ques 21	Ques 22	Total Level 2
Pre-test	0	1	0	0	0	0	0	1	1	0	0	3
Post-test	0	0	0	0	0	0	0	0	0	0	0	0

Investigation of the choices made by Respondent 32 yielded the following information. In Items 4, 19 and 20 this student did not make the same incorrect choices as in the pre-test, nor were the choices the same in Items 14, 16. However, in all the remaining items the same incorrect choices were made in both the pre- and post-tests. It may thus be safely assumed that this student experiences serious difficulties with work on this level.

Level 3

Performance on Level 3 in the pre- and post-tests reveal many similarities to the previous two charts, as can be seen in Figure 12.

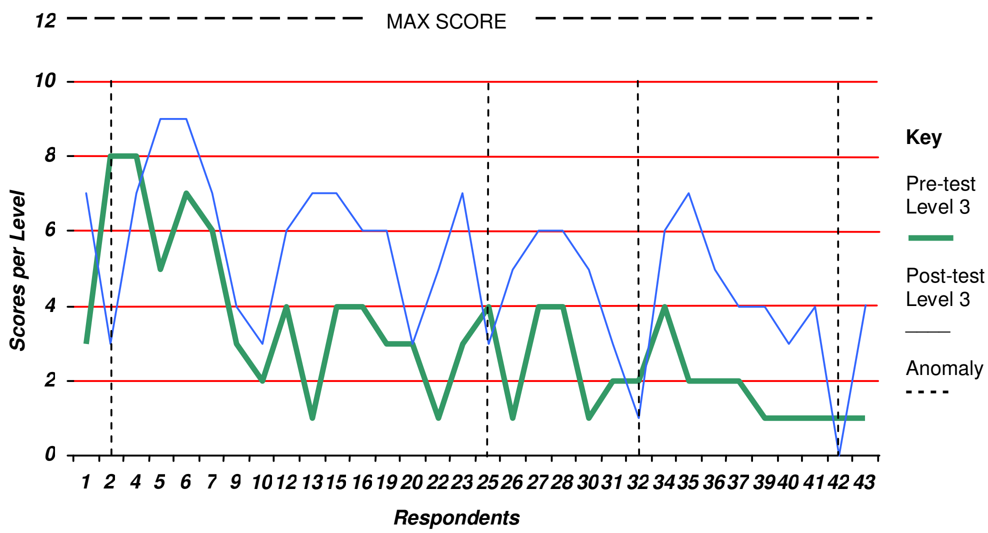


Figure 12. Respondent scores on Level 3 in the pre-test and the post-test, n = 32.

However, an interesting phenomenon occurs in terms of an inversion of results where Respondent 2, the second best performer in the pre-test on Level 3 places him/herself in the third quarter in the post-test. The question may well be asked, has this student's understanding decreased as a result of the intervention? Three other anomalies occur in this representation, all considerably smaller than in the case of Respondent 2. Nor do these anomalies represent the same respondents who represent anomalies in the previous charts.

Respondent 2:

	Ques 6	Ques 7	Ques 10	Ques 13	Ques 15	Ques 18	Ques 23	Ques 24	Ques 25	Ques 26	Ques 30	Ques 32	Total Level 3
Pre-test	1	0	1	1	1	1	0	0	1	0	1	1	8
Post-test	0	0	0	1	0	0	1	1	0	0	0	0	3

The responses observed here defy logic. Of the three responses that were correct in the post-test, only one had also been correct in the pre-test. However it cannot be safely stated that improved understanding was the reason for this, since every other question was answered incorrectly in the post-test. At the same time, it must be considered that this student's overall score for the pre-test dropped from 19 to 16 in the post-test, despite slight improvements in his/her score on Levels 1 and 2. It is possible that for this student at least, that there are extraneous factors that played a role during the writing of the post-test.

4.1.3.3 Analysis of relationship between pre- and post-test data

A *t*-test was done on the overall pre- and post-test data to examine the differences in achievement between the pre-test and post-test and the possible impact of instruction.

Table 12

T-test using pre- and post-test data

	Pre-test	*Post-test*
Mean	11.0625	16.34375
Observations	32	32
df	31	
t Stat	-5.023	
P(T<=t) two-tail	2.00E-05	

The calculated *t*-value on the data is -5.023 and the means for the two tests are significantly different. In other words, since the calculated P-value is less than 0.05, the conclusion is that the mean difference between the paired observations is statistically significant. In fact, there is a probability of approximately 2 in 100 000 that the observed difference between the means could be achieved by chance alone. This test therefore shows that there is a highly significant and large difference in achievement between the pre- and post-tests. This can be taken to indicate that the intervention which took place between the pre- and post-tests made a significant difference to the understanding of the students involved.

Comparison of specific items in the pre- and post-test results dealing with only rote learnt knowledge or formal deductive reasoning is of particular value. Items 29 and 31 each required a verbatim recall of the wording of a theorem. Both of these items were surprisingly well answered in the pre-test (seventeen and fifteen correct answers respectively), given the period of time (two years) since these statements were last learnt in matric. However, the riders (requiring formal deduction) related to each of these theorems, Items 30 and 32 respectively, were uniformly poorly answered. Eight pre-test respondents could answer Item 30 correctly, and five, Item 32. These statistics seem to indicate that rote learning of the theorems took place, generally unaccompanied by insight in terms of the application of those theorems. This is confirmed by the results for these items in the post-test. Items 29 and 31 received 30 and 25 correct answers respectively, while Items 30 and 32 had 22 and 23 correct answers respectively. This is a clear indication that, while the rote learning of the theorems remained intact, insight into how these theorems could be applied had been acquired during the course of the intervention. Once these students were properly taught about the basic tenets of proof construction and how it is applied with insight and logic, they were generally able to do such construction successfully.

4.1.4 THE EFFECTIVENESS OF EACH ALTERNATIVE.

The assessment instrument was designed so that 96% of the distracters offer an answer in which the geometric logic was meaningless. The remaining 4% were created to reveal a level of thinking below the level required for the correct answer. Analysis of each distracter would therefore not yield significant information with the exception of the anomalistic situation that arose regarding items 5, 8 and 12. In the pre-test these three items lay in the category of questions most poorly answered, but they are Level 1 questions. Items 5 and 12 even remained in that category in the post-test. Examining the incorrect choices that were made for these three items reveals the following:

Item 5: Distracter D was chosen by twelve and ten respondents in the pre- and post-test respectively, C by eleven and seven, and B by six and nine. Only three correct choices were made for this item in the pre-test and six in the post-test. Distracter D mentioned all the principle properties of a rhombus, while the correct answer stated the minimum possible properties that will always make a rhombus. Clearly, the majority of the students have not mastered the idea of minimum requirements with regard to shape analysis. Insights into minimum requirements for classification and class inclusion are closely linked.

Item 8: This item was anomalistic only in the pre-test. Distracter B was chosen by twelve students. C and D were evenly spread in the remaining incorrect choices. Eight correct choices were made for this item. All three distracters were, in fact, meaningless, yet the simplest of them which offered a technical impossibility, was most often selected. It is possible that language had a role to play in this selection.

Item 12: Distracter D was overwhelmingly the favourite choice, having been selected by 21 students in the pre-test and 16 in the post-test. Like D, A and C were all meaningless in terms of the geometric logic required, offering a random selection of the properties and vocabulary associated with types of triangles. Once again, it seems that language and a lack of geometric vocabulary were responsible for the poor performance of students with regard to this item.

4.2 THE PME STUDENTS' EXPERIENCE OF AND FEELINGS ABOUT EUCLIDIAN GEOMETRY

Information concerning the PME students' *experience* of learning Euclidian geometry at high school as well as how they *felt* about geometry was accessed during the two group interviews that were conducted. Since the sub-sample that was selected for interviewing was specifically chosen for its representivity, for the purpose of this research it is assumed that the experiences and feelings discussed in the interviews would be found in the sample population also. Specifically, answers to the questions, "What was the experience of the students at school in relation to the teaching and learning of Euclidian geometry?", "How was the acquisition of knowledge of theorems related to the acquisition of skills in solving riders?" and "How do PME students feel about Euclidian geometry?" were being sought. Thus the transcribed interview data was analysed in terms of these questions with the focus, in terms of coding themes, on words or sentences that expressed emotion generated by the study of Euclidian geometry.

4.2.1 PRIOR TO THE GEOMETRY MODULE

The students were firstly asked to recall their matric geometry results. Despite the fact that the question required only a numerical answer, every student's answer was dominated by expressions of emotion surrounding geometry. Enjoyment was associated with understanding, which they declared was not consistent or even frequent. Interviewee E (male, white and from an urban school) used such phrases as "pretty easy for me", "I enjoyed", "I understood" and "I could see the things more clearly", the latter of which he ascribed to having Technical Drawing as a subject. Despite the fact that he felt very positive about geometry, this candidate's performance in the pre-test gave him an overall 38%. Interviewee W (male, coloured and from a rural school) struggled with geometry, despite the fact that he enjoyed it. He scored 31% in the pre-test. Both Interviewees G (score of 50%, female, black and from an urban school) and D (score of 16%, male, black and from a rural school) said that they did quite well in high school geometry. Interviewee M (score of 63%, female, white and from a rural school) did not like geometry at all and claimed to have passed the second matric final exam (dealing with trigonometry, analytical geometry and Euclidian geometry) by concentrating on answering the first two sections, while leaving out many of the Euclidian geometry questions. Interestingly, their response to this first question reveals information which is totally belied by their scores in the pre-test.

4.2.1.1 What was the experience of the students at high school in relation to the teaching and learning of Euclidian geometry?

All the interviewees expressed the confusion and frustration they experienced being taught by educators who did not appear to have either mastered the subject or developed a positive attitude toward the subject. One interviewee described the process of learning geometry as follows: a teacher with apparently limited understanding leads to a learner who does not understand who becomes a learner who dislikes subject; then he gets a new teacher with good understanding and the learner understands and begins to like the subject. This reaction chain was confirmed by Interviewee W who found that geometry was like a punishment, until he began to understand what was going on. In the same way, Interviewee G declared that she hated geometry in Grades 8 and 9, until a teacher explained it step-by-step to her, and her attitude improved. All five interviewees were not convinced of their teachers' prowess in Euclidian geometry; they thought that their teachers explained poorly because they were themselves unsure of the reasoning behind the applications of the theory. Interviewee E stated the following (translated from Afrikaans):

> And they [the teachers] didn't really have a clue about what they were doing in their own geometry. For three years, I had the same teacher who would often look at the board, then quickly run back to her file to see what's in it, then back to the board. She couldn't explain just out of her own head … then she'd say, don't worry, you won't get this problem in the exam. I think that's where my problem came in. By the time we got to grades 11 and 12 lots of people had problems with geometry and then they just couldn't be fixed any more.

Although the situation described by Interviewee E took place in an urban school, his description correlated very well with the experience of Interviewee D, coming from an under-resourced rural school. He explained,

> My teacher is not perfect with geometry. He just put a problem on the blackboard and say, you and I prove that problem. So we discovered a lot of problems, but at the end we make a discussion, our own discussion, and do the stuff and at the end we pass.

In his situation, learners who were determined to pass were dependent upon their own resources (other textbooks and group discussions) to achieve enough mastery of geometry not to fail the exam. Their teacher, in his view, knew as little as they did about geometry.

In the geometry classroom in an under-resourced urban school, as experienced by Interviewee G, there was a slightly different situation. The teacher, in her opinion, while she might have understood what she was doing, was not concerned about the total lack of comprehension amongst most of the learners. G described the situation as follows:

> If one person understands in the class, then it's fine. Because there was this one girl who understood. And no .. none of us in the class understood. Our marks were pretty

> low; she had 90's and 80's. We were just low every time … Because, basically when we were in grade 11 half of our class was not even doing maths at the time. They all went to the history class.

Although she was in a well-resourced rural school, Interviewee M also experienced a lack of inclusion of the learners in the classroom. Her teacher would not consider any deviation from the method she chose for solving a rider (translated from Afrikaans):

> And what she often did when we asked her to explain, was to do it on the board, following her own path, although there are different paths. And if we asked her, 'Can it be like this?' then she would say to us, 'No, rather keep it the way I did it.' She would not explore the options.

All the interviewees thus expressed the opinion that the geometry teachers of their experience were either themselves lacking in understanding of the subject matter, or unable to teach it in such a way as to make sense of the theory in terms of its application. The interviewees were unable to discuss their high school experiences of learning geometry without expressing negative emotions about it.

4.2.1.2 How was the acquisition of knowledge of theorems related to the acquisition of skills in solving riders?

All the interviewees indicated that an insistence on learning theorems was not done with a view to their application on riders. Interviewee W explained the problem he experienced with riders, as opposed to just learning theorems (translated from Afrikaans): "The rider – you must look at it and see what information it's giving to you. Then you must find the answer yourself. That was a bit of a problem for me." He explained why he thought this was the case: "Because I think the teacher did not have a good attitude towards geometry, because he would just say to us, 'Here it is. Take this. Do this.'" As a result of not understanding, rote-learning was the order of the day. This student answered the theorem recall questions perfectly in the pre-test, but scored zero on their application. G said that at school she just applied herself to the theorems and omitted doing riders that required real insight. Student D even said that once he had come to believe that geometry was only about statements and reasons, and he was able to pass because statements and reasons could be learnt. However, he realised why his success was very limited: "You do not understand how to apply theorems, so you discover you will have more problems." Interviewee M confirmed the general tendency of concentrating on what could be rote-learnt (translated from Afrikaans):

> My teacher focused more on the theorem itself. And if we ran into problems, then we were always just given more problems out of the text book to do. And when we came back the next day she would put the problem on the overhead projector and then we all marked our work. So most of us just copied down the work because we could not do it in the first place. So we actually did not do much application.

The prevalence of an excessive emphasis on rote-learning in this sample's geometric schooling is demonstrated in the pre-test, where Items 29 and 31, dealing with simple recall of the wording of theorems, were extraordinarily well done, with 62,5% and 50% difficulty indices, whereas the application of each of those two theorems was very poorly done, with difficulty indices of 25% and 12,5% respectively. Interviewee M, who was the top scorer in the pre-test, explained that she began to experience problems in geometry when the real Euclidian proof construction began in Grade 11. She could even give examples of how riders generated confusion for her. This is borne out by her pre-test performance, in which all her marks were scored in questions where proof construction or theorem recollection was *not* required.

In grade 11 Interviewee G was taught by someone who, in her opinion, not only understood what he was doing, but used practical examples to explain the applications of the theory to the class. He believed in a hands-on approach, which, she said, made it better for her. In the pre-test, this student scored full marks on the first theorem recall question *and* its application, and zero on the next one. However, in the proof construction required in the last application she demonstrated the ability to think on Level 3, but without completing the rider successfully.

The interviewees all commented on the virtue of relating the theory to practical situations and said that if they could, that is one of the teaching methods they would enforce in their old classrooms. Interviewee G pointed out that dependence on the textbook was not conducive to understanding. By contrast, the way in which the geometry module they were currently doing was presented through problem-solving, made understanding accessible. She explained as follows: "Here [at university] you're just faced with the problem. In a way you think about it more critically than when it's being done for you in a text book and everything and with notes on top of that."

The fact that none of these five students did well in the application part of the pre-test, clearly corroborates the information gathered from them in discussion that, while they might have a fair knowledge of the theorems themselves, the teaching they had received at high school did not lead them to insight and understanding of what the theorems were about.

4.2.1.3 How do PME students feel about Euclidian geometry?

All the negative emotions mentioned by these students were placed in a context of not understanding the work. In fact, they stated unequivocally that the geometry class in general was a place of negative feeling and demotivation. Generally, in their experience, application to real-life problem-solving situations was rare. Instead, theorems were learnt and riders were required to be done for homework, which would then be corrected the next day with little attention or time given to the fact that the learners did not understand what they were doing.

Interviewee E analysed the reasons for the negativity towards geometry as experienced by himself and his classmates. He felt that there were three causes: firstly, some teachers in his experience did not encourage the asking of questions because, he thought, they did not know enough and were afraid of being stumped; secondly, others answered questions in such a way as to increase the learners' confusion; and thirdly, he thought that an inadequate foundation had been laid in terms of geometry at primary school level, so that there was no anticipated enjoyment of it when geometry was commenced in earnest at high school. Without such a grounding in which geometry was associated with pleasurable understanding, "you cannot suddenly acquire a positive attitude towards it in grades 11 and 12. That's where it's really supposed to be enjoyable, but now everybody hates geometry because nobody understands how things come about."

In contrast, their experience of the geometry module they had just begun had already changed their view of geometry: they were more positive, enjoyed the subject and understood the fundamental precepts that underlie geometric reasoning. This they ascribed to the confidence and ability of their lecturer and the practical, problem-solving, visual approach he adopted in his class.

4.2.2 AFTER THE GEOMETRY MODULE

The subsequent interview confirmed what had been established during the first interview: the grounding in the origin of geometric axioms which they received during the course of the module not only gave them pleasure in being able to "see" how things worked, for some, for the first time, but gave them insight and understanding into how and why proof works. The interview began with the students receiving their marked pre-tests and post-tests, and being given time to peruse both.

4.2.2.1 The experience of the students in the teaching and learning of geometry

The first question they were asked was, in view of the fact that each one of them had achieved a higher score in the post-test, to what would they ascribe the improvement? Both Interviewees E and W said that they were very positive about the subject because the question, "why?" had been answered for them repeatedly during the course of the module. E stated that his score had improved because his understanding had improved. W added that (translated from Afrikaans), "There are different ways of approaching a thing, but at school they never taught us this." Interviewee E made specific reference to the fact that he often had ideas of his own when solving a problem and he had the liberty to explain his ideas, knowing that the teaching methodology in this geometry module allowed for innovation. If his ideas were not geometrically sound, he was assisted in correcting them:

> In many things I had my own ideas… but I could ask [the lecturer] directly, "Listen, this method of mine, does it work or not?" If I thought I had a better way of proving something, I could ask him. If it did not work he would lead me with hints to the correct solution.

The fact that the lecturer gave opportunity for doing every sum and problem in class, and gave no homework at all, said Interviewee M, was what increased her understanding. She felt that being stuck and being able to ask for help immediately, prevented the formation of blockages regarding difficult aspects of the work.

All five interviewees made statements regarding the difference between the way they were taught geometry at school and the way the geometry module was presented at UP. They said that their questions were answered in an understandable way and that they were shown that riders can be solved in several different ways, not just the one selected by the teacher or the textbook. They all spoke highly of a hands-on methodology in which the practical aspect of geometry was emphasized. Interviewee G said she would like to have her own future classes involved in hands-on activities, "like touching things, like making things, building things". It is clear by the improved internal consistency coefficient of the post-test that where there were gaps in understanding, some possibly not even consciously perceived, such gaps were filled by doing applications of concepts through problem-solving in the presence of the lecturer.

4.2.2.2 The acquisition of theorem knowledge in relation to the acquisition of rider-solving skills

Interviewee D, whose improvement in the post-test was more significant than that of his four fellow-interviewees (from 16% overall score in the pre-test to 53% in the post-test), ascribed this remarkable increase to the fact that previously he had learnt the theorems, but could never apply any of them, but that he had now been shown how to apply his knowledge. This new insight played a role particularly in the second part of the test which consisted of theorems and riders in supply-type questions:

> I see my improvement is on Section Two. Because the first one, on section two, I hadn't understood anything there [in the pre-test]. So I just leave the space. So right now, because I know the concept and how to do and so, I answered it.

Interviewee G explained the improvement in her performance in the post-test by stating that the lecturer, when working out problems with the class, had not just made statements that had to be accepted, but had gone to considerable lengths to demonstrate why the statements were true. She explained,

> Ok, I found that I'm improving my…my mark changed because of the way the concepts are explained because [the lecturer] is not gonna say to you, "Ok, fine, an exterior angle whatever is equal to the opposite whatever" - he's gonna *show* it to you, basically that's what he does. And I think that's why I improved, basically.

She indicated that teaching in this module had taken place with a view to generating understanding as opposed to rote-learning. Knowledge of properties and axioms was directed toward problem solving: "My ideas changed because, in a way, it's [the module] changed my way of thinking. Cause you….cause in school you just thought geometry was just about lines and everything but now, it means you think systematically." She emphasized the virtue of being given a visual understanding of the concepts instead of only a theoretical one, explaining that visualizing shapes and their properties became easier once these had been demonstrated physically:

> I never understood that concept until the lecturer actually drew it in a test. So, basically, yea, I think … he helped us again with the visualising. Like, when somebody tells you something in words, like when we did the exercises… you know how to…to visualise it… Yeah, if you can't see something, you can't do it. That's just the way it is.

In corroboration of this statement, Interviewee E said he had now seen how the abstract could be explained using practical and visual techniques. He stated that his own ideas about how to teach geometry had been impacted by the module: "Now I can describe things, and then I can take it from there and present it to them visually so that they can understand."

The students interviewed all attested to their perception that the aim of the module was not to acquire theoretical knowledge in isolation, but to foster a conceptual understanding through practical activities which would facilitate the solving of such problems as riders. They uniformly expressed the opinion that their understanding had improved because they were given clear reasons for procedures they were unsure of, and were shown that the theorems in fact became tools for solving riders once the concepts behind the theorems were fully grasped.

4.2.2.3 How do the PME students feel about Euclidian geometry?

Where there were difficulties, the students were able to receive assistance immediately since all work was done in class, and so frustration and anxiety were not given a chance to set in. The feelings expressed by the students concerning geometry had changed radically from the first interview to the second, and they themselves ascribed this to the difference in the quality and methodology of the tuition they received, comparing their secondary school experience with their university module. Interviewee M explained the change in her attitude towards geometry as follows:

> My attitude is definitely positive…At school you had a teacher who was negative, so you just took on her attitude. Her attitude was that algebra was important, but geometry was just *there*, but now at varsity they have taught us geometry and the lecturer's positive attitude has rubbed off on us.

Interviewee W said he had been shown a new approach to proof construction, which caused him to do this section in the post-test with a more positive attitude than he had ever experienced before. This student, who had not *attempted* either of the proof constructions in the pre-test, had completed both successfully in the post-test. He said that his newly acquired positive attitude towards geometry had enabled him to *try* solving riders instead of *omitting* them as he had always done.

Each one of the interviewees testified to a change in their attitude towards geometry. They ascribed this change to three factors: they *enjoyed* what they were doing because they *understood* what they were doing; the lecturer inspired their confidence because of his thorough knowledge of his subject; and his positive attitude towards problem solving was contagious.

As was demonstrated by their results in the post-test, these five students were in fact capable of good, logical geometric reasoning, but had not previously been taught with insight into the acquisition of such skills. A rewarding "by-product" of the intervention was that they not only understood what was wrong with the way they had been taught at school, but how geometry should in fact be taught.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 SUMMARY OF THE FINDINGS

The way that geometry has been taught in South Africa has been largely textbook dependent as was confirmed by the students interviewed for this study and who were selected for that purpose on the grounds of their representivity. Generally, the textbooks present the theorems and their proofs as prologues to several exercises which then use those theorems in applied form, the emphasis therefore lying on deductive reasoning. Schoenfeld's research shows that "Most textbooks present "problems" that can be solved without thinking about the underlying mathematics, but by blindly applying the procedures that have just been studied" (Schoenfeld, 1988, p. 160). Very little by way of inductive reasoning is required, unless the individual educator introduces and guides such reasoning in his/her learning activities. There is very little doubt that most FET learners who intend becoming mathematics teachers are easily able to master the ability to learn and reproduce Euclidian theorems, having, in many cases, had them drilled into their heads through all-or-nothing theorem tests. Test results in this study show, however, that not long after matric they are unable to use any of the rote-learnt theorems in an application situation. In other words, the ability to reason deductively, if it was ever acquired, atrophies quickly. Freudenthal (1971) explains this phenomenon succinctly:

> We mathematicians retain the mathematics we learned, because it is our business. People usually forget what is not related to the world in which they live. For most people mathematics cannot be an aim in itself; if they have learnt it in an unrelated way, they will never be able to use it (p. 420).

For students who intend becoming teachers of mathematics, mathematics is "our business".

However, theorem proofs can be re-learnt. It is in the solution of riders, which often depends on proof construction that the problem arises. Battista and Clements (1995) found during the course of their research that "numerous attempts have been made to improve students' proof skills by teaching formal proof in different ways, albeit largely unsuccessful." (p. 49)

This study aimed to investigate the level of understanding of Euclidian geometry, in terms of theoretical knowledge as well as its problem-solving application, in pre-service mathematics education students at the University of Pretoria. In order to do so, a one group pre-test/ post-test procedure was conducted around an intensive geometry module, and a representational group of students was interviewed before and after the module to discuss their high school experiences of learning geometry and to analyse their attitudes towards the subject. The van Hiele Theory of Levels of Thought in Geometry was used as the theoretical framework for this study.

Larew (1999) states that nearly all research connecting the van Hiele theory to an explanation or description of how students learn geometry, used secondary school students or pre-service and in-service *elementary* teachers (p. 70). This study contributes to the body of research in this field in investigating what the position is of pre-service *senior secondary* school in this regard. This study also investigates the impact of a geometry module presented during the third year level of tertiary study and provides statistical proof that a tertiary education module aimed specifically at evolving PME students through the van Hiele levels is essential in order to break the cycle of PME students being released into the field without at least a Level 3 understanding of geometry, who are then unable to bring their learners, some of whom will become PME students, up to that level.

5.1.1 EFFICIENCY OF THE STUDENTS' CONTENT KNOWLEDGE APPLICATION IN THE SOLUTION OF RIDERS

The first research question namely, **How efficiently can FET phase mathematics pre-service education students apply their content knowledge of Euclidian geometry theorems to solve riders (including doing proofs)?** investigates, in terms of the theoretical framework, whether the students in this study function efficiently on Level 3 of the van Hiele model. Facility of functioning on the levels below Level 3, in view of the sequential nature of the model, is exposed by such an investigation.

This research shows that the students in question have unquestionably been found seriously lacking in terms of their content knowledge of Euclidian geometry as well as in the skills and insight necessary to solve problems, do riders and construct proofs. Their overall pre-test results show that the group as a whole did not even attain 50% on Level 1, and that adequate functioning on Levels 2 and 3 was even rarer. Judging by the percentages shown in the table below, more than half of this group of students is only efficiently functional on Level 0, having been unable to achieve sufficiently to be placed on the other three levels.

Table 13

Total percentage of correct answers and percentage of respondents per level in the pre-test for the whole group.

Pre-test	Van Hiele Level 1		Van Hiele Level 2		Van Hiele Level 3	
	% correct answers	% students	% correct answers	% students	% correct answers	% students
Results	42.5	37.5	37.5	9,3	25.75	3,1

International literature indicates that content knowledge in Euclidian geometry generally and mainly involves a rote learning of theorems, leading to what van de Walle (2004) describes as procedural knowledge and an instrumental understanding. Such learning is often not accompanied by insight or understanding into the fundamental precepts and root principles which govern the rules. Duval (2006) refers to the absence of "good conceptual knowledge" (p. 119). In the absence or paucity of such insight, application of these principles in the problem-solving area of riders is rendered extremely difficult, and is frequently entirely beyond the reach of many students. Schoenfeld (1986) emphasises the absence of a cognitive connection between what is learnt and how it is applied. The results of this study serve to confirm this view. Not only were the items in the assessment instrument requiring insight and application of knowledge uniformly poorly done, but the students themselves also confirmed in the interviews conducted with them that riders and proofs were where they encountered almost insurmountable problems in the geometry classroom. They commonly experienced what Weber (2001) refers to as an impasse, where they just simply came to a standstill in constructing a proof and could not continue.

Van der Sandt and Niewoudt (2005) found in their research with elementary school preset teachers that students had a better understanding of geometry after leaving matric than after their professional training. However, this study shows that, in the case of the senior secondary school preset teachers involved in this research, the understanding of geometry with which they leave matric is almost universally poor, despite the fact that their final matric marks may seem to indicate differently. Certainly, they are unable to teach geometry with such understanding. Were it not for the introduction by the University of Pretoria of a dedicated geometry module which takes the students through the van Hiele levels, these students would go into the field of mathematics teaching with what can only be described as a very low level of insight and knowledge in Euclidian geometry.

It stands to reason that no teacher of mathematics can teach on a level which lies beyond their own understanding. Muijs and Reynolds (2002) and Kotzé (2007) confirm the direct correlation between the educator's conceptual knowledge of mathematics and the quality of instruction which that educator is capable of delivering.

Having established by means of the pre-test that the PME students were not uniformly on the expected level (Level 3) which would enable them to teach the subject adequately once they entered the profession, it is safe to conclude that a module which uses geometric problem-solving techniques and which is designed to bring about improvement in terms of the van Hiele levels is absolutely essential. The students upon whom this research was conducted were in their third year of study. The mathematics courses they had completed prior to the geometry module had not increased their insight or competence in geometry to a level which would enable them to teach it.

Table 14

Total percentage of correct answers and percentage of respondents per level in the post-test for the whole group.

Post-test	Van Hiele Level 1		Van Hiele Level 2		Van Hiele Level 3	
	% correct answers	% students	% correct answers	% students	% correct answers	% students
Results	62.75	59.3	50.25	21.8	42.25	18.7

Comparing the table above with Table 13, it is clear that a vast improvement has taken place. The students have progressed from a score of 42.5% on Level 1 to 42.25% on Level 3. This improvement in achievement translates into a migration of the majority from sure competency on Level 0 to sure competency on Level 1, with a 12.5% increase of students on Level 2 and a 15.6% increase in students on Level 3, which may be considered significant.

While Level 3, the desired level of competence for a mathematics educator in a South African high school, had not been uniformly achieved by the end of the intervention, considerable progress had been made towards mastery at this level.

5.1.2 THE PME STUDENTS' HIGH SCHOOL EXPERIENCE OF EUCLIDIAN GEOMETRY IN THE VAN HIELE CONTEXT

With regard to the second question, **Was the preset FET trainees' experience of learning Euclidian geometry at high school conducive toward prompting their progression from one van Hiele level to the next**, one of the most strident pieces of information derived from the international literature is the fact that the way geometry is taught has not changed much during at least the last fifty years. Despite the revolutionary input made by the van Hieles in the late 1950's, classroom practice has not changed its emphasis from the learning of procedure to the understanding of relations and concepts. Although good teachers do not advocate blind memorisation, Schoenfeld (1988) found that what actually happened in the classroom in fact enforced the idea of rote learning. This research suggests that this also the case in the South African context. The results of the pre-test show clearly that the PME students, coming from a range of high schools, had not made significant and consistent progress beyond Level 0 of the van Hiele model during their high school years. What makes this finding most disturbing is the fact that these students did well enough in their matric final examinations to allow them entry into a university course which would train them to teach mathematics to senior high school learners. This begs the question: if it can be said of the mathematical "cream of the crop" that they operate comfortably and consistently only on Level 0, what can be said of those who did not do as well in matric?

a) What was the experience of the students at school in relation to the teaching and learning of Euclidian geometry?

According to Schoenfeld (1988), the subject matter was "presented, explained and rehearsed" (p. 159); grasping the concepts is more a matter of chance than design. Lessons are designed with success in the final assessment in mind. In order to facilitate acquisition of the material to be presented in final examinations, educators teach procedure: a step-by-step "recipe" for success; at least marks will be given for sound method. In this way, students may score good marks for the process they used in dealing with the rider, without understanding the concepts involved. When serious deductive reasoning is required, for example in the construction of complex proofs, such students are stranded. The interviews conducted with students in this research suggest that the situation Schoenfeld described in the United States in 1988, may very much still be the order of the day in South Africa.

Several reasons are proposed for understanding not being arrived at in the traditional geometry classroom: the van Hieles suggested that the teacher presented the subject matter on a higher level than the one on which the learners were at that time; Duval (2006) speaks of individual's skills in visual organisation, allowing recognition of figures within figures which represent concepts that would lead to solutions; Mogari found that in South Africa, very often the educators were themselves not in possession of adequate content knowledge and understanding. De Villiers (1997) found that sophisticated technological aids or the lack thereof were not the answer to the problematic nature of teaching and learning geometry. Respondents interviewed in this research were convinced that their teachers at high school did not know enough about geometry and themselves had a bad attitude towards geometry, as in fact was found by Mogari (2003).

b) How was the acquisition of knowledge of theorems related to the acquisition of skills in solving riders?

Students interviewed in this study testified to the fact that what Romburg and Carpenter (as cited in Schoenfeld, 1988, p 147) described as the absorption theory, was in fact the dominant methodology in their high school geometry classrooms. They stated that learners were given information and told to accept it as truth, without having to understand why they should do so. A slavish adherence to the textbook and its presentation of the subject matter allowed for little time being given to inductive thinking and constructive participation by the learners. Misconceptions, such as was demonstrated by of one of the interviewees who "discovered" that geometry was just about statements and reasons which could be memorised in order to pass, are rife. If understanding is elusive, rote learning is the only hope for such students. Examination of the pre-test results confirm that knowledge of theorems was very tenuously linked to problem-solving situations such as riders.

5.1.3 THE ATTITUDES OF SELECTED STUDENTS TO EUCLIDIAN GEOMETRY

The final research question, **What do these students feel about Euclidian geometry**, focused on attitudes. Studying mathematics generates emotion. The interviewees in this study, when asked about their matric results in geometry, first spoke about how they felt about the subject. Boaler (2000) confirmed that enjoyment and understanding were concomitant, but were rarely the case. More often annoyance, frustration, anxiety and fear accompanied the study of geometry. Pierre van Hiele (1986) spoke of an intuitive aspect to proof construction, which by its very definition means that there are some who do not grasp what lies behind the rules they are required to learn. An interviewee in this study put it simply: "I just did not get it". A distinctive characteristic of the fear and anxiety which underlie such a statement is that such emotions inhibit even further the acquisition of intuition and understanding, bringing students to a point where they refuse even to try. Another interviewee said of riders, "I just skipped them."

However, despite many students' dislike or even hatred of geometry, they persist in their study of mathematics because there is a perception that it facilitates better career choices (Boaler, 2000, and Mogari, 2003).

5.2 REFLECTIONS AND DISCUSSION

This research therefore serves to suggest that there are learners leaving secondary schools in South Africa to pursue a career in teaching mathematics do not have an in-depth understanding of geometric concepts. It would seem that the secondary schools attended by the students in this study do not function in terms of the teaching of geometry according to the van Hiele Theory of Levels of Geometric Thought, and teaching consequently does not appear to be conducive toward progress through such levels. Rote learning would still seem to be the order of the day, despite the fact that it does not necessarily generate understanding.

This "absorption theory" of teaching, observed more than twenty years ago by Schoenfeld, Romburg and Carpenter, amongst others in America, is still the foundation for the way geometry is taught in at the least the secondary schools attended by the interviewed for this research.: Interviewee E (male, white, educated in an urban school) stated (translated from Afrikaans),

> I think, to fix it, in a way, is to take many of those theorems and show us where they come from; walk that route with us. Many of those theorems, like the basic stuff regarding corresponding angles, alternate… that stuff must surely come from somewhere. That stuff was just given to us and we were told, "Here, accept this".

In corroboration of this, another interviewee (male, coloured, educated in an urban school) explained that his teacher only concentrated on the theorems, the work that could be rote-learnt:

> You had to know the theorems. You had to know them off by heart. He did not focus on their applications…I think that what I would change is the teacher's attitude towards geometry. Because I don't think the teacher had a positive attitude about geometry because he just said to us, "Here it is. Take this. Do this".

A third interviewee (male, black, educated in a rural school) added that his teacher seemed unable to explain since he was himself unsure of how the riders worked and what the theorems implied. This teacher's methodology thus consisted of presenting the work from the textbook to the class and making them does it:

> "So, and his introduction to geometry was we just read the problems and we just do the problems".

Lacking understanding, these students disliked and feared the subject. Instruction which does not include visualisation and explanation of the reasoning behind concepts and which moves directly from teaching of theorems to their application, makes the assumption that the formal deduction skills required for riders is in place. The van Hiele theory (1986) asserts that students at a lower level of thinking cannot be expected to understand instruction presented at a higher level of thinking: "This is the most important cause of bad results in the education of mathematics." (van Hiele, 1986, p. 66)

The van Hiele model, used as the conceptual framework for this study, has served to analyse the hierarchy of reasoning on which the PME students operate, as revealed in the results of the written assessments. The levels have been carefully defined by van Hiele and further refined in the work of Usiskin (1982) and so they could be strenuously applied to the test results. The van Hiele levels have also provided parameters for judgment when comparing the reasoning and understanding demonstrated in the pre- test with the post-test. Since the geometry module was specifically designed to take students through the van Hiele levels, using them as a conceptual framework for this study facilitated the assessment of the impact on the students of that module. However, this research confirmed that students can be on different levels simultaneously with regard to different content. It is also apparent that the classification by van Hiele of class inclusions as a Level 2 activity is questionable: it may be argued that the level of thinking required for this is more closely identifiable with Level 3 understanding then with Level 2. It is important to note, however, that the van Hiele model deals only with Euclidian geometry, a discipline which now constitutes only a small part of the new matric syllabus, introduced in 2008. Its relevance in the training of pre-service mathematics teachers remains valid, though, since these students will *still* be obliged to teach Euclidian geometry, although in lesser measure than before 2008.

The methodology used in the course of this study has proved successful in that valid and reliable results were produced which allowed detailed analysis of the assessments administered, and important conclusions. The decision to conduct interviews with a purposive sub-sample enabled the students to provide information regarding their experience of the high school geometry classroom, and also allowed for follow-up questioning by the researcher. Their statements provided corroboration of the original premise that initiated this study: that there exists a cycle in terms of geometry education in South Africa in which the ill-taught become the ill-teachers. Whilst these results are limited to this case, they provide a basis for further research which would be required in order to generalize findings across institutions. In retrospect, more data could have een collected regarding the background of the students under study to provide a deeper analysis of the assessment.

5.3 CONCLUSIONS AND RECOMMENDATIONS

1. The PME students in this study, prior to their completion of the geometry module, lacked the content knowledge, skills and insight in Euclidian geometry that is expected at matric level. The pre-test results revealed that half the group could only be classified as being on Level 0. This statistic implies that, at best, 50% of the group were competent in shape recognition, but did not demonstrate competence in property recognition nor in informal and formal deduction. Of the other half of the group, 75% could demonstrate competence only up to Level 1. This research suggests that it is an invalid assumption that, since the whole of this group met the requirements of the University to begin training as FET mathematics teachers in terms of their matric mathematics mark, their understanding of Euclidian geometry is on the desired matric level (Level 3). The students interviewed intimated that they either did very well in the other branches of mathematics required for matric which would then have compensated for an inadequate performance in Euclidian geometry, or they were able, through rote-learning, to achieve a mark for Euclidian geometry which did not reflect their lack of insight.

2. The pre-test results indicate and insufficient level of understanding of Euclidian geometry. The post-test results however, reveal that, while there are no longer any students on Level 0, 60% of the group have moved onto Level 1 as their maximum competence level. This implies that these students *were all* brought to greater insight by the teaching they received during the geometry module. Therefore the notion, expressed by the students who were interviewed as a common one among their peers, that they simply were not able to do geometry, is a false perception – clearly the quality of teaching they received made a difference.

3. The overall improvement in the group as revealed in the post-test results, consisted of an upward movement of only one level. This implies that the geometry module

offered did not bring about sufficient improvement for these students to be able to teach geometry adequately. This module requires modification with a view to increasing its depth in content knowledge and insight-generating activities, with possible small-group attention being given to students whose progress through the van Hiele levels is not fast or efficient enough. Further research on the nature and content of such a module is required. Based on the case in this study, it may be suggested that there is a need for teacher education institutions to implement a geometry module which is designed to improve insight not procedural knowledge. The pre-test results in this study reveal that a general mathematics course does *not* enhance understanding of geometry in terms of the van Hiele levels.

3. The students who were interviewed for this study uniformly expressed their dislike or fear of Euclidian geometry in general. The literature study confirms that the pleasure which learning should generate is absent when that which is being studied is not understood or questions that arise remain unanswered. The interviewees also described the positive change in their attitude towards the subject during the course of the module because of the way it was presented: all work was done in class under the supervision of the lecturer so that problems which might have arisen were dealt with immediately before they became exacerbated through time and increased workload.

4. The fact that the current (as from 2008) FET mathematics syllabus makes Euclidian geometry optional, is lamentable. Further research is required to ascertain whether the kind of insight and logical reasoning which is acquired through the study of Euclidian geometry is, in fact, acquired in the same measure through any of the other branches of school mathematics. Training of students for a career as mathematics educators which includes an in-depth van Hiele-based geometry module would facilitate the acquisition of insight and

relational understanding. Such acquisition would serve these students well in terms of how they think, whether future syllabi exclude the teaching of Euclidian geometry per se, or not.

5.4 LIMITATIONS OF THE STUDY

While this study is obviously limited in that it accesses a small sample of South African pre-service mathematics teachers at one South African university, its findings are nevertheless relevant as it provides some insight into a very important conceptual area of mathematics. There was a large difference in results between the pilot and the pre-test, as one would expect, given the purpose of the pilot. The pre- post-test design is not the strongest design possible, but it was the most feasible at the stage of the assessment. It must also be noted that the van Hiele theory works on a narrow view of what geometry is, and so an assessment instrument that measures only what the van Hiele model describes does not access other facets of geometry which are also valid indicators of mathematical thought applied to space. The lack of background information on all the students in the sample, in hindsight, was a limitation.

5.5 FINAL WORD

Geometry is worth being taught and learnt because it facilitates the evolvement of logical thinking and deductive skills in the minds of those who study it. While there are those who acquire formal deductive skills with less ease than others, this study proves that a broad range of students from a variety of cultural and secondary school backgrounds *do* respond positively to the type of teaching that is specifically designed to engender such thinking processes. Tertiary institutions which release mathematics education students into the field of mathematics teaching without first promoting their personal progress through the van Hiele levels of deductive thinking, are doing society a disfavour.

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LEGEND FOR ITEM ANALYSIS IN ADDENDA A AND B

Content:

Item = number of question in test

Level = van Hiele level

Freq correct = the number of times the correct answer was selected

Distracters = non-correct answers are either without mathematical meaning, or indicate understanding at a level lower than that required by the correct answer.

Freq chosen = the number of times that particular incorrect answer was selected

Purpose = type of content knowledge accessed by that particular item including:

- Property recognition
- Class inclusion
- Deduction
- Property classification
- Rote memory
- Proof construction

Format = type of question including:

- MCQ or Multiple Choice Question
- Supply type question

Difficulty %

Std error = Standard error

Discrim power = discriminating power

ADDENDUM A

Item analysis of pre-test

Content					Purpose	For-mat	Diffi-culty %	Std error	Discrim power
Item	Level	Freq correct	Distracters	Freq chosen					
1	1	10	A Meaningless	6	Property recog	MCQ	31.3	0.08	0.3
			B Meaningless	1					
			C Meaningless	15					
2	2	5	B Meaningless	0	Class inclusion	MCQ	25.0	0.07	0.1
			C Meaningless	0					
			D Meaningless	27					
3	1	24	A Meaningless	6	Property recog	MCQ	68.8	0.08	0.2
			B Meaningless	0					
			D Meaningless	2					
4	2	12	B Meaningless	13	Property recog	MCQ	31.3	0.09	0.3
			C Meaningless	4					
			D Meaningless	3					
5	2	1	B Meaningless	6	Class inclusion	MCQ	0.0	0.03	0.0
			C Meaningless	4					
			D Meaningless	3					
6	3	12	A Meaningless	7	Deduction	MCQ	37.5	0.09	0.3
			B Meaningless	10					
			D Meaningless	3					
7	3	12	A Meaningless	2	Deduction	MCQ	43.8	0.09	0.2
			B Meaningless	4					
			C Meaningless	14					
8	1	8	B Meaningless	12	Property recog	MCQ	12.5	0.08	0.0
			C Meaningless	5					
			D Meaningless	6					
9	1	19	A Meaningless	5	Property recog	MCQ	56.3	0.09	0.2
			B Meaningless	6					
			D Meaningless	2					
10	3	9	A Meaningless	17	Deduction	MCQ	37.5	0.08	0.1
			B Meaningless	3					
			C Meaningless	3					

Content					Purpose	For-mat	Diffi-culty %	Std error	Discrim power
Item	Level	Freq correct	Distracters	Freq chosen					
11	2	7	A Level 0	1	Property recog	MCQ	18.8	0.06	0.2
			C Meaningless	6					
			D Level 1	21					
12	1	4	A Meaningless	4	Property classif	MCQ	18.8	0.05	0.2
			C Meaningless	2					
			D Meaningless	22					
13	3	20	A Meaningless	9	Deduction	MCQ	75.0	0.09	0.3
			C Meaningless	2					
			D Meaningless	1					
14	2	18	B Meaningless	2	Deduction	MCQ	43.8	0.09	0.2
			C Meaningless	9					
			D Meaningless	3					
15	3	2	A Meaningless	7	Deduction	MCQ	6.3	0.04	0.1
			C Meaningless	5					
			D Meaningless	18					
16	2	1	A Meaningless	12	Property recog	MCQ	12.5	0.04	-0.1
			B Meaningless	4					
			D Meaningless	15					
17	2	19	A Level 1	1	Property recog	MCQ	75.0	0.09	0.0
			B Level 1	6					
			C Meaningless	6					
18	3	6	A Meaningless	3	Deduction	MCQ	25.0	0.07	0.3
			B Meaningless	18					
			C Meaningless	5					
19	2	15	A Meaningless	4	Class inclusion	MCQ	43.8	0.09	0.4
			B Meaningless	9					
			D Meaningless	4					
20	2	23	A Meaningless	5	Class inclusion	MCQ	81.3	0.08	0.1
			C Meaningless	3					
			D Meaningless	1					
21	2	5	B Meaningless	7	Class inclusion	MCQ	62.5	0.06	0.4
			C Meaningless	17					
			D Meaningless	3					

Content					Purpose	For-mat	Diffi-culty %	Std Error	Discrim power
Item	Level	Freq correct	Distracters	Freq chosen					
22	2	18	A Meaningless	6	Class inclusion	MCQ	62.5	0.09	0.4
			C Meaningless	3					
			D Meaningless	5					
23	3	5	B Meaningless	16	Deduction	MCQ	18.8	0.07	0.2
			C Meaningless	11					
			D Meaningless	0					
24	3	14	A Meaningless	7	Deduction	MCQ	43.8	0.09	0.2
			C Meaningless	10					
			D Meaningless	1					
25	3	3	A Meaningless	13	Deduction	MCQ	12.5	0.05	0.1
			B Meaningless	12					
			C Meaningless	4					
26	3	1	A Meaningless	25	Deduction	MCQ	0.0	0.03	0.0
			B Meaningless	5					
			C Meaningless	1					
27	1	23	A Meaningless	1	Property recog	MCQ	75.0	0.08	0.1
			B Meaningless	3					
			C Meaningless	2					
28	1	18	B Meaningless	7	Property recog	MCQ	50.0	0.09	0.3
			C Meaningless	3					
			D Meaningless	4					
29	1			16	Memory	Sup-ply	62.5	0.09	0.0
30	3			8	Proof	Sup-ply	25.0	0.08	0.3
31	1			13	Memory	Sup-ply	50.0	0.09	-0.1
32	3			6	Proof	Sup-ply	12.5	0.07	0.1
						Ave	36.5	0.07	0.2

ADDENDUM B

Item analysis of post-test

Content					Purpose	For-mat	Diff-iculty %	Std error	Discrim power
Item	Level	Freq correct	Distracters	Freq chosen					
1	1	16	A Meaningless	5	Property recog	MC Q	43.8	0.09	0.6
			B Meaningless	3					
			C Meaningless	8					
2	2	24	B Meaningless	0	Class inclusion	MC Q	62.5	0.08	0.3
			C Meaningless	0					
			D Meaningless	8					
3	1	31	A Meaningless	0	Property recog	MC Q	93.8	0.03	0.1
			B Meaningless	0					
			D Meaningless	1					
4	2	15	B Meaningless	8	Property recog	MC Q	50.0	0.09	0.8
			C Meaningless	6					
			D Meaningless	3					
5	2	6	B Meaningless	9	Class inclusion	MC Q	12.5	0.07	0.3
			C Meaningless	7					
			D Meaningless	10					
6	3	11	A Meaningless	8	Deduction	MC Q	50.0	0.09	0.5
			B Meaningless	8					
			D Meaningless	5					
7	3	17	A Meaningless	4	Deduction	MC Q	56.3	0.09	0.4
			B Meaningless	4					
			C Meaningless	7					
8	1	20	B Meaningless	10	Property recog	MC Q	62.5	0.09	0.5
			C Meaningless	1					
			D Meaningless	1					
9	1	24	A Meaningless	1	Property recog	MC Q	68.8	0.08	0.6
			B Meaningless	6					
			D Meaningless	1					
10	3	18	A Meaningless	7	Deduction	MC Q	62.5	0.09	0.5
			B Meaningless	5					
			C Meaningless	2					
11	2	7	A Level 0	1	Property recog	MC Q	18.8	0.07	0.1
			C Meaningless	3					
			D Level 1	21					

Content					Purpose	Format	Difficulty %	Std Error	Discrim power
Item	Level	Freq correct	Distracters	Freq chosen					
12	1	8	A Meaningless	6	Property classif	MCQ	12.5	0.08	0.3
			C Meaningless	2					
			D Meaningless	16					
13	3	16	A Meaningless	7	Deduction	MCQ	50.0	0.09	0.0
			C Meaningless	5					
			D Meaningless	4					
14	2	27	B Meaningless	4	Deduction	MCQ	68.8	0.07	0.1
			C Meaningless	1					
			D Meaningless	0					
15	3	1	A Meaningless	9	Deduction	MCQ	6.3	0.03	0.1
			C Meaningless	3					
			D Meaningless	19					
16	2	3	A Meaningless	11	Property recog	MCQ	6.3	0.05	0.1
			B Meaningless	7					
			D Meaningless	11					
17	2	16	A Level 1	4	Property recog	MCQ	62.5	0.09	0.5
			B Level 1	4					
			C Meaningless	8					
18	3	17	A Meaningless	2	Deduction	MCQ	50.0	0.09	0.8
			B Meaningless	11					
			C Meaningless	2					
19	2	15	A Meaningless	5	Class inclusion	MCQ	37.5	0.09	0.8
			B Meaningless	10					
			D Meaningless	2					
20	2	23	A Meaningless	8	Class inclusion	MCQ	75.0	0.07	-0.3
			C Meaningless	0					
			D Meaningless	1					
21	2	14	B Meaningless	12	Class inclusion	MCQ	50.0	0.09	0.5
			C Meaningless	6					
			D Meaningless	0					

Content					Purpose	For-mat	Diffi-culty %	Std error	Discrim power
Item	Level	Freq correct	Distracters	Freq chosen					
22	2	15	A Meaningless	4	Class inclusion	MCQ	37.5	0.09	0.3
			C Meaningless	2					
			D Meaningless	11					
23	3	14	B Meaningless	14	Deduction	MCQ	56.3	0.09	0.4
			C Meaningless	3					
			D Meaningless	1					
24	3	16	A Meaningless	2	Deduction	MCQ	43.8	0.09	0.6
			C Meaningless	12					
			D Meaningless	2					
25	3	5	A Meaningless	11	Deduction	MCQ	18.8	0.07	0.1
			B Meaningless	12					
			C Meaningless	4					
26	3	2	A Meaningless	13	Deduction	MCQ	12.5	0.04	0.3
			B Meaningless	6					
			C Meaningless	11					
27	1	19	A Meaningless	4	Property recog	MCQ	62.5	0.09	0.5
			B Meaningless	8					
			C Meaningless	1					
28	1	21	B Meaningless	6	Property recog	MCQ	62.5	0.09	0.5
			C Meaningless	3					
			D Meaningless	2					
29	1				Memory	Sup-ply	100.0	0.04	0.0
30	3				Proof	Sup-ply	75.0	0.08	0.0
31	1				Memory	Sup-ply	81.3	0.07	0.1
32	3				Proof	Sup-ply	75.0	0.08	0.3
						Ave	50.8	0.08	0.3

ADDENDUM C

Report of the First Interview

The interview was conducted on 19 March, 2007, five weeks after the writing of the pre-test, and four weeks into the geometry module. The interviewees did not have access to their pre-test results prior to the interview.

The first point of discussion in the pre-intervention interview required the interviewees to recall their level of performance in geometry during their matric finals. The way they felt about Euclidian geometry was spontaneously revealed by their answers. Interviewee E used such phrases as "pretty easy for me", "I enjoyed", "I understood" and "I could see the things more clearly", the latter of which he ascribed to having Technical Drawing as a subject. Despite the fact that he felt very positive about geometry, this candidate's performance in the pre-test gave him an overall 38%. Interviewee W struggled with geometry, despite the fact that he enjoyed it. He scored 31% in the pre-test. Both Interviewees G (score of 50%) and D (score of 16%) said that they did quite well in high school geometry. Interviewee M (score of 63%) did not like geometry at all and claimed to have passed the second matric final exam (dealing with trigonometry, analytical geometry and Euclidian geometry) by concentrating on answering the first two sections, while leaving out many of the Euclidian geometry questions. Interestingly, their response to this first question reveals information which is totally belied by their scores in the pre-test.

Key words from their answers to this question were: enjoy, problem, struggle, easy, good, dislike.

Upon then being asked specifically how they felt about geometry, Interviewee E stated that he greatly disliked geometry when it was introduced to him at school, and explained this feeling by saying his first geometry teacher did not explain very well and therefore he disliked the subject because he did not understand. The sequence of events as he experienced it is significant: teacher with apparently limited understanding > learner does not understand > learner dislikes subject; new teacher with good understanding > learner understands > learner likes subject. This reaction chain was confirmed by Interviewee W who found geometry was like a punishment, until he began to understand what was going on. In the same way, Interviewee G declared that she hated geometry in Grades 8 and 9, until a teacher explained it step-by-step to her, and her attitude improved. Interviewee D, by contrast, liked geometry because he found "it was all about statement and reasoning" which he was able to do, and this motivated him to work harder. However, in the pre-test he demonstrated no skill in this at all. He did, however, explain later that not knowing how to apply theorems led to problems in his life. Interviewee M explained that she began to experience problems in geometry when the real Euclidian proof construction began in Grade 11. She could even give examples of how riders generated confusion for her. This is borne out by her pre-test performance, in which all her marks were scored in questions where proof construction or theorem recollection was not required.

Key words: dislike, punishment, hated, statement and reason, motivated, confused, disinterest.

The next question required them to identify specifically where problems in doing geometry began, if, in fact they did experience such problems during theory high school career. While Interviewee E claimed not to have experienced problems in matric, Interviewee W had problems with riders: finding the information and trying to apply that information without assistance. Interviewee G simply skipped the application questions. Applications, particularly before having learnt his theorems, were where Interviewee D struggled. Interviewee M found the amount of information presented in a sketch together with the given written information and the required proof was altogether just confusing.

Key words: riders, apply, confusion, self. Four out of the five students declared that their problems arose when having to apply their knowledge to problem situations, when they were on their own with the problem.

The students were then asked to discuss their experience of the teaching of geometry at their school. Interviewee E did not refer at all to his first teacher whose understanding was not, according to him, good. Instead, he explained how the good teacher which he had subsequently, took great pains to explain carefully and logically, generating insight as she went. This student answered both the proof construction questions perfectly in the pre-test. Interviewee W remembered that his teacher had concentrated on rote learning of the theorems and had largely neglected dealing with application. This student answered the theorem recall questions perfectly in the pre-test, but scored zero on their application. Interviewee G explained how her teacher used practical methods of bringing home understanding and that this worked really well. He believed in a hands-on approach, which, she said, made it easy for her. In the pre-test, this student scored full marks on the first theorem recall question and its application, and zero on the next one. However, in the proof construction required in the last application she demonstrated the ability to think on Level 3, but without completing the rider successfully. Interviewee D confessed that his teacher was so bad at geometry that he could do none of the riders himself. Instead, he would put the rider on the board and then would try to help the learners through discussion to arrive at some sort of answer. D emphasized the many difficulties they experienced as a class, but through consulting amongst themselves the learners managed to gain enough knowledge to pass. This student was able to recall only one of the theorems in the test, and achieved zero for the applications. Interviewee M also said that her teacher concentrated exclusively on rote learning of the theorems and used the textbook exercises in a do-them-for-homework-write-down-the-correct-answers-the-next-day sort of process which, she claimed, led to no insight at all. This student achieved zero for the theorems and their riders in the pre-test.

Key words: thorough explanation, theorems, rote—learning, practical work, homework-correction routine.

Two of the interviewees, whose geometry improved toward the end of their schooling, lauded the techniques used by their teachers in detailed explanations and using practical applications. The other three complained of the typical pattern of rote-learnt theorems and exercises done at home followed by corrections presented by the teacher the next day.

When asked how they would improve on the teaching they received at school, given the insights they have acquired in to the teaching of mathematics since the commencement of their tertiary studies, this group of students revealed remarkable understanding. Interviewee E, referred to the teacher who, he felt, taught him so badly, seemed not to know what she was doing and kept running from the board to the textbook for confirmation of the correctness of her work. What she could not do, she claimed would not be in the exam. Thus the problem lay not in the teaching, but in what he perceived to be a lack of content knowledge in the teacher. He thought an improvement would be to trace the origins of statements and theorems, so that true understanding is created through answering the “why” question in learners’ heads, instead of what he experienced: never mind why this is so – just learn it! Interviewee W went one step further to say that he felt his teacher did not himself have a good attitude towards geometry, and transmitted this to his learners by making them do repeated exercises with little or no explanation leading to insight taking place. Interviewee G felt that marks would have improved and fewer learners would have abandoned the mathematics class had more practical applications of the geometry theorems been done. Interviewee D agreed, adding that geometry needed to be applied to real life, that learners would be more motivated and would understand why they were doing something if the theory were applied to the building of a bridge, for example. The teacher had a negative attitude, claimed Interviewee M. She would not allow any deviation from her way of thinking or her method of doing. Her manner created the impression that she disliked geometry and wanted only to get it over and done with.

Key words: teacher's content knowledge, teacher's attitude, learner participation, real world application.

The next question required the students to think about whether they felt any different about geometry now compared to how they felt at school. Seeing it from the teacher's point of view makes teaching mathematics very different, according to Interviewee E. It makes it possible, he explained, to see what mistakes both he and his teacher made. Interviewee W felt much better now about geometry than he did at school because of the way it is presented. State-of-the-art media are used, and the lecturer has a positive and confident attitude towards his subject. Interviewee G emphasized the practical aspect of tuition in the geometry module. The computer programmes used by the lecturer enable the students to see, often in three dimensions, how an application works, and this makes it easier for her to learn. She expressed her confidence in her own geometric ability, thanks to the method and quality of instruction. Interviewee D said he felt good about geometry now. This feeling was shared by Interviewee M who claimed that the practical application of geometric concepts created immediate comprehension for her. She also pointed to the usefulness of electronic media in the teaching of this subject. Interviewee G expanded this thought by adding that working out of a textbook with its usual format of an example followed by problems to do, was a poor way of learning;

that the method employed in this module involved very little theory, concentrating instead on application problems and their solutions: in fact, an inductive approach. She said this encouraged critical thinking of which learners at school were also capable, but never encouraged to use.

Key words: media, attitude of lecturer, content knowledge of lecturer, critical thinking, problem-solving approach.

The students were then asked about the input they would like to have into the teaching of geometry in South Africa, particularly in connection with the general atmosphere regarding geometry as they experienced it in their school classroom. Interviewee E declared that the atmosphere in his school class was unequivocally negative. He felt that this sort of response could be remedied by explaining why geometry worked the way it did in primary school already, so that learners were aware from the beginning of the origins of thinking which later would become axiomatic. He also felt that teachers should encourage the asking of questions – they generally do not do so, he felt, because they are unsure of the content which they are teaching – much as he has experienced himself on teaching practice. He felt that geometry was at its most enjoyable in Grades 11 and 12, but that by then the learners had come to hate it. His opinion was shared by Interviewee W who agreed that a better foundation for the learning of geometry should be laid in primary school, to avoid the negative attitude toward geometry which he also experienced in high school. Interviewee G felt that generally geometry is taught in isolation and that it would be much better to link it to other topics within the mathematics syllabus in any given year. It should also be made more fun, for example, by using colour in doing riders. Interviewee D re-emphasised the virtue of practical applications. Interviewee M pointed out that her entire class was negative about geometry at school. She felt that question and answer techniques needed to be improved: it is of no use, she claimed, to repeat an explanation exactly when a learner did not understand that word sequence the first time round. A different route needed to be followed.

Key words: negative attitude, explanation from basics up, not geometry in isolation, fun.

Asked whether having been taught about the van Hiele model in their mathematics methodology course enhanced their understanding of geometry, the students agreed, feeling that understanding the concept of levels and the fact that missing parts of one level led to problems on the next level, empowered them in their own learning. It also helped them to acquire insight into their own process of learning and the teaching that they will eventually be doing.

Key words: insight, perspective, steps.

ADDENDUM D

Report of the Second Interview

The post-module interview was conducted on the 21st of May, 2007, when the geometry module had been completed by the students in question. They had written the post-test one week prior to this interview. At the start of the interview, the students were each given their marked pre- and post-test to peruse and compare.

The first question they were asked was, in view of the fact that each one of them had achieved a higher score in the post-test, to what would they ascribe the improvement. Interviewee E explained that he fared better after being taught where formulae, rules and theorems come from. He is better able to understand when his question of “why?” is answered than when he was at school and just told to accept unquestioningly what he was given. Interviewee W agreed with this, and added that he was taught a new approach to proof construction in particular, which caused him to approach this section in the post-test with a more positive attitude than he had ever experienced before. This student, who had not attempted either of the proof constructions in the pre-test, had completed both successfully in the post-test. Interviewee G explained the improvement in her performance in the post-test by pointing out that the lecturer had not just made statements that had to be accepted in working out problems in class, but had gone to considerable lengths to demonstrate why the statements were true. Interviewee D, whose improvement in the post-test was more significant than those of his four fellow-interviewees (from 16% overall score in the pre-test to 53% in the post-test), ascribed this remarkable increase to the fact that previously he had known the theorems, but could never apply any of them, but that he had now been shown how to apply his knowledge. What had not made sense to him before, now made sense. The fact that the lecturer gave opportunity for the doing of every sum and problem in class, and gave no homework at all, said Interviewee M, was what increased her understanding. She felt that being stuck and being able to ask for help immediately, prevented the formation of blockages regarding aspects of the work.

Key words: teaching methodology, positive attitude, explanation of theorems, practical application, classwork instead of homework.

This observation led to the rest of the group being asked their thoughts concerning classwork replacing homework. Interviewee E explained that he sometimes has his own ideas about how a problem should be solved, but often his ideas did not quite work. Doing the exercises in class meant that he could check with the lecturer as he went along, so acquiring a correct method of solving relatively painlessly. Interviewee W added that is he struggled with something at home, he was apt to abandon it very quickly, thinking that he would find out later how it was done. However, finding out how to do the work while doing it, was a much better way. The problem with homework, claimed Interviewee G, was that it always has a lower priority than an assignment which needs handing in, for example. In the experience of Interviewee D having a problem directly answered as one struggles with its solution makes the solution easier to retain for exam purposes.

Key words: immediate explanation, better understanding

The next question focused more closely on the difference between the two test results. The students were required to explain what brought about the improvement that took place in a specific area, or in fact, in general. In matric, Interviewee E had experienced gaps in his knowledge with regard to some of the work which was dealt with during the course of the module. However, these gaps were now filled, hence the improvement in his marks.

Interviewee W explained that the improvement he achieved lay particularly in section two of the test which dealt with proof construction. This he ascribed to a more positive attitude and increased confidence in his own ability. Interviewee G emphasized again the benefits of the practical approach adopted by the lecturer: visualizing shapes and their properties became easier once these had been demonstrated physically. Interviewee D ascribed his success in the post-test to understanding what had not known before, and further, knowing how to apply his knowledge. Interviewee M added that the lecturer used techniques involving colour and the issuing of hints and tips in the solution of problems which made the work interesting and memorable.

Key words: gaps filled, positive attitude, visualization, application.

The students were then asked to state in one sentence what has changed in their attitude, if it has changed, and why such a change has taken place. Interviewee E stated firmly that his attitude had become more positive because his understanding had improved. The attitude of Interviewee W also became more positive, because, he said, if there was more than one way of solving a problem, all the ways were explained, particularly using real life examples; this increased his understanding. In corroboration of this Interviewee G said that her way of thinking had changed, because “in school you just thought geometry, geometry was just about lines and everything but now, it means you think systematically”. Interviewee D ascribed the change in his attitude to the difference in the quality of teaching he experienced between school and this module. Interviewee M echoed this thought in saying that the lecturer’s positive attitude was, in fact, contagious. At school, her teacher had emphasised the importance of algebra, at the cost of geometry.

Key words: positive, improved understanding, thinking systematically, lecturer's attitude.

How, they were asked, would their experiences in this module affect their own teaching of geometry? Interviewee E said he had seen how the abstract could be explained using practical and visual techniques. Interviewee W would also now strive to teach in a more practical way, allowing his learners to see and do, instead of just being given exercises to complete. Interviewee G agreed to this, adding that she would like to have her classes involved in hands-on activities, "like touching things, like making things, building things". Interviewee D also intended to employ such practical techniques in his teaching. Interviewee M spoke of both a practical, hands-on sort of methodology in teaching geometry, accompanied by a positive attitude toward the subject matter.

Key words: visualize the abstract, practical work.

ADDENDUM E

ADDENDUM F

UNIVERSITEIT VAN PRETORIA

UNIVERSITY OF PRETORIA

Euclidian Geometry

Euklidiese Meetkunde

60 MINUTE / MINUTES

EKSAMINATOR / EXAMINER:

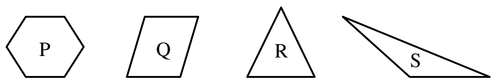
S van Putten

NAME/ NAAM:	**Studentenommer / Student number:**

Instruksies	**Instructions**
1. Beantwoord all vrae op die vraestel self. 2. Skryf netjies en ordelik sodat die antwoord in die gegewe spasie pas.	1. Answer all questions on the question paper itself. 2. Write in a neat and orderly way so that answers fit in the given spaces.

Section I/ Afdeling I

The following items are multiple choice questions. Circle the answer of your choice.
Die volgende items is meervoudige keuse vrae. Omkring die antwoord van u keuse.

1. 

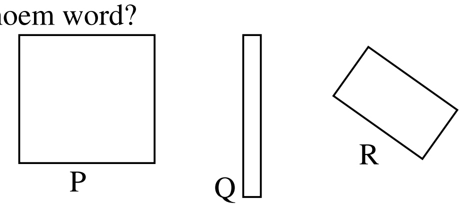
a. R is not a quadrilateral because none of its angles are obtuse. /
R is nie ‘n vierhoek nie omdat dit geen stomphoek het nie.

b. Q is not a quadrilateral because its diagonals are not equal in length. /
Q is nie ‘n vierhoek nie omdat die diagonale nie gelyke lengtes het nie.

c. P is not a quadrilateral because it can be divided into 6 equal triangles. /
P is nie ‘n vierhoek nie omdat dit in 6 gelyke driehoeke opgedeel kan word.

d. S is not a rhombus because its angles add up to 180°. /
S is nie ‘n ruit nie omdat die som van die hoeke 180° is.

2. Which of the figures below can be called rectangles? / Watter van die figure hieronder kan reghoeke genoem word?

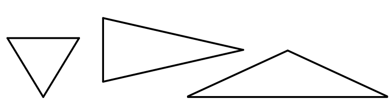


A) All can / Almal kan

B) Q only / Slegs Q

C) P and Q only / Slegs P en Q

D) Q and R only / Slegs Q en R

3. 

Here are three isosceles triangles. In every isosceles triangle it is true that … / Hier is drie gelykbenige driehoeke. In elke gelykbenige driehoek is dit waar dat

A) All three sides must be equal in length. / Al drie sye moet dieselfde lengte wees.

B) One side must be twice the length of another. / Een sy moet twee maal die lengte van ‘n ander wees.

C) Two angles must have the same measure. / Twee hoeke moet dieselfde grootte wees.

D) All three angles will be the same size. / Al drie hoeke sal dieselfde grootte wees.

4. Here are four definitions, each describing a certain polygon. / Hieronder volg vier definisies wat elk ‘n sekere veelhoek beskryf.

P: This is a quadrilateral with one pair of parallel sides. / Hierdie is ‘n vierhoek met een paar ewewydige sye.
Q: This is a quadrilateral in which the sum of any two consecutive interior angles is 180°. / Hierdie is ‘n vierhoek waarvan die som van enige twee opeenvolgende binnehoeke 180° is.
R: This is a quadrilateral in which the diagonals bisect each other. / Hierdie is ‘n vierhoek waarvan die diagonale mekaar halveer.
S: This is a quadrilateral in which the vertices are concyclic. / Hierdie is ‘n vierhoek waarvan die hoeke konsiklies is.

A) Definition Q always includes Definition P. / Definisie Q sluit altyd Definisie P in.

B) Q and S are the same quadrilateral./ Q en S is dieselfde vierhoek.

C) P and S are always the same quadrilateral. / P en S is altyd dieselfde vierhoek.

D) Definition S always includes Definition R. / Definisie S sluit altyd Definisie R in.

5. The minimum requirements that will always make a rhombus are: / Die minimum vereistes wat altyd ‘n ruit sal maak is:

A) Adjacent sides equal. / Aangrensende sye gelyk.

B) Diagonals bisect each other perpendicularly; adjacent sides equal. / Diagonale sny mekaar loodreg; aangrensende sye gelyk.

C) Both pairs of opposite sides equal and parallel; diagonals bisect each other. / Beide pare teenoorstaande sye gelyk en ewewydig; diagonale halveer mekaar.

D) Adjacent sides equal and non-perpendicular to each other; both pairs of opposite sides parallel. / Aangrensende sye gelyk en nie loodreg op mekaar nie; beide pare teenoorstaande sye ewewydig.

6. If two circles are drawn so that they touch internally, it is not necessarily true that … / As twee sirkels so geteken word dat hulle intern raak, is dit nie noodwendig waar dat …

A) A common tangent can be drawn through the point of contact. / ‘n Gemeenskaplike raaklyn deur die raakpunt geteken kan word nie.

B) The one circle is smaller than the other. / Die een sirkel kleiner is as die ander een nie.

C) One circle’s radius is the other circle’s diameter. / Die een sirkel se radius die ander sirkel se middellyn is nie.

D) The radii drawn from the centre of each circle to the point of contact will coincide. / Die radiusse geteken vanaf die middelpunt van elke sirkel na die raakpunt sal saamval nie.

7. Which of the following shapes can never be drawn as a cyclic quadrilateral? / Watter van die volgende vorms kan nooit as 'n koordevierhoek geteken word nie?

A) A square. / 'n Vierkant.

B) A rectangle. / 'n Reghoek.

C) A trapezium. / 'n Trapesium.

D) An obtuse parallelogram. / 'n Stomphoekige parallelogram.

8. If an altitude is dropped onto the hypotenuse of a right-angled triangle, then … / As 'n hoogtelyn getrek word na die skuinssy van 'n reghoekige driehoek, dan …

A) Two smaller similar triangles are created, similar to the original. / Word twee kleiner driehoeke gevorm, gelykvormig aan mekaar en aan die oorspronklike driehoek.

B) The altitude will bisect the hypotenuse. / Sal die hoogtelyn die skuinssy halveer.

C) The area of the original triangle is halved. / Word die oppervlakte van die oorspronklike driehoek gehalveer.

D) The sum of the squares of the hypotenuses of the two new small triangles is equal to the square of the altitude. / Is die som van die vierkante van die skuinssye van die twee kleiner driehoeke gelyk aan die vierkant van die hoogtelyn.

9. In a certain triangle, a line is drawn between two sides of the triangle, parallel to the third side. / In 'n sekere driehoek word 'n lyn getrek tussen twee van die sye van die driehoek, ewewydig aan die derde sy.

A) This line will always be equal to half the third side in length. / Hierdie lyn sal altyd gelyk wees aan helfde van die lengte van die derde sy.

B) This line will always join the midpoints of the two sides. / Hierdie lyn sal altyd die middelpunte van die twee sye verbind.

C) This line will always divide the two sides in equal proportion. / Hierdie lyn sal altyd die twee sye in dieselfde verhouding verdeel.

D) This line will always pass through the incentre (the centre of the incircle) of the triangle. / Hierdie lyn sal altyd deur die middelpunt van die ingeskrewe sirkel gaan.

10. Two tangents are drawn onto a circle from a particular point P. A line is drawn from P to the centre of the circle O. Radii connect the centre O with the points of contact of the tangents, T and S. It cannot be proved that … /
Twee raaklyne word getrek aan ‘n sirkel vanaf ‘n sekere punt P. ‘n Lyn word ook getrek vanaf P na die middelpunt O van die sirkel. Radiusse verbind middelpunt O met die kontakpunte S en T van die raaklyne. Dit kan nie bewys word dat …

A) OSPT is a cyclic quad. / OSPT ‘n koordevierhoek is nie.

B) The two triangles, POT and POS are congruent. / Die twee driehoeke, POT en POS kongruent is nie.

C) $O\hat{P}T = O\hat{P}S$. / $O\hat{P}T = O\hat{P}S$ nie.

D) Triangle OPT and triangle OPS are both isosceles. / Driehoek OPT en driehoek OPS beide gelykbenig is nie.

11. A certain quadrilateral PQRS is a rectangle because … / ‘n Sekere vierhoek PQRS is ‘n reghoek omdat …

A) It looks like a door. / Dit soos ‘n deur lyk.

B) All corners are 90°. / All hoeke 90° is.

C) Both pairs of opposite sides are parallel. / Beide pare teenoorstaande sye ewewydig is.

D) Combination of B) and C). / Kombinasie van B) en C).

12. Triangles can be classified into the following types: / Driehoeke kan in die volgende kategorieë geklassifiseer word:

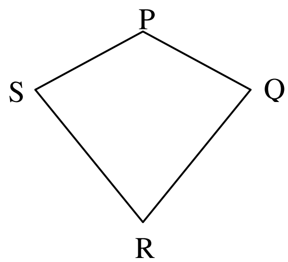
A) All sides equal, isosceles, all angles equal. / Alle sye gelyk, gelykbenig, alle hoeke gelyk.

B) Obtuse, right-angled, acute. / Stomphoekig, reghoekig, skerphoekig.

C) Obtuse, scalene, isosceles. / Stomphoekig, ongelyksydig, gelykbenig.

D) An infinite combination of A) and B). / ‘n Oneindige kombinasie van A) en B)

13.



A quadrilateral PQRS is given in which PQ and PS are equal, and RQ and RS are equal. In order to prove that $\hat{S} = \hat{Q}$, it would be necessary firstly to … / ‘n Vierhoek PQRS word gegee waar PQ en PS gelyk is, en RQ en RS gelyk is. Om te bewys dat $\hat{S} = \hat{Q}$, is dit eerstens nodig om …

A) Construct diagonals PR and QS. / Diagonale PR en QS te konstrueer.

B) Construct diagonal PR. / Diagonaal PR te konstrueer.

C) Acquire more information about quadrilateral PQRS. / Meer inligting oor vierhoek PQRS te verkry.

D) None of the above. / Geen een van bogenoemde antwoorde.

14. If the opposite interior angles of a quadrilateral are supplementary, then … / As die teenoorstaande binnehoeke van ‘n vierhoek supplementêr is, dan …

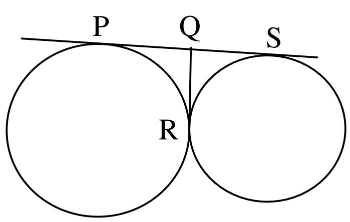
A) The quadrilateral will always be cyclic. / Sal die vierhoek altyd ‘n koordevierhoek wees.

B) The quadrilateral could be cyclic. / Kan die vierhoek moontlik ‘n koordevierhoek wees.

C) The quadrilateral is a parallelogram. / Is die vierhoek ‘n parallelogram.

D) The angles are all 90°. / Sal al die hoeke 90° wees.

15.



If two unequal circles touch externally and a common tangent is drawn from the contact point R to intersect another common tangent at Q, as in the sketch, it can then be proved that … / As twee ongelyke sirkels mekaar ekstern raak en ‘n gemeenskaplike raaklyn word getrek vanaf die raakpunt R om ‘n ander gemeenskaplike raaklyn in Q te sny, soos in die skets aangedui, dan kan dit bewys word dat …

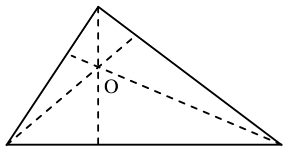
A) $RQ \perp PS$

B) $PQ = QS$

C) $\Delta\ PQR \equiv \Delta\ SQR$

D) All of the above. / Al die bogenoemde.

16.



If the three altitudes of a triangle are drawn as in the sketch, then … / As die drie hoogtelyne van 'n driehoek geteken word soos in die skets, dan …

A) Triangles are created which are similar to the original triangle. / Word driehoeke gevorm wat gelykvormig is aan die oorspronklike driehoek.

B) The area of each of the smaller triangles is $\frac{1}{6}$ of that of the original triangle. / Is die oppervlakte van elk van die kleiner driehoeke 'n $\frac{1}{6}$ van dié van die oorspronklike driehoek.

C) Three cyclic quads are created. / Word drie koordevierhoeke gevorm.

D) A circle can be drawn to touch all three sides of the original triangle, with O as its centre. / Kan 'n sirkel geteken word om aan al drie sye van die oorspronklike driehoek te raak, met O as die middelpunt.

17. If two triangles are equiangular, then … / As twee driehoeke gelykhoekig is, dan …

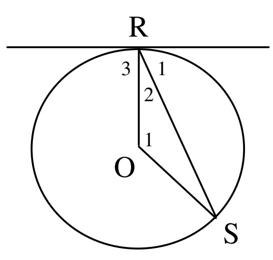
A) Their sides are in proportion. / Is hulle sye in verhouding.

B) They are similar. / Is hulle gelykvormig.

C) Each of their angles measure 60°. / Is die grootte van elke hoek 60°.

D) A) and B). / A) en B).

18. In order to calculate the size of $\hat{O}_1$ in the following diagram, it is sufficient to know the size of … / Om die grootte van $\hat{O}_1$ in die volgende skets te bereken, is die minimum inligting wat benodig word om die grootte te weet van …



A) $\hat{R}_1$ and $\hat{R}_2$ / $\hat{R}_1$ en $\hat{R}_2$

B) $\hat{S}$ and $\hat{R}_2$ / $\hat{S}$ en $\hat{R}_2$

C) $\hat{R}_3$

D) $\hat{S}$

19. Here are two statements: / Hier volg twee stellings:

1: Figure F is a rectangle. / Figuur F is 'n reghoek.

2: Figure F is a triangle. / Figuur F is 'n driehoek.

A) If 1 is true, then 2 is true. / As 1 waar is, dan is 2 ook waar.

B) If 1 is false, then 2 is true. / As 1 vals is, dan is 2 waar.

C) 1 and 2 cannot both be true. / 1 en 2 kan nie albei waar wees nie.

D) 1 and 2 cannot both be false. / 1 en 2 kan nie albei vals wees nie.

20. Here are two statements: / Hier volg twee stellings:

S: Δ ABC has three sides of the same length. / In Δ ABC is al drie sye gelyk.

T: In Δ ABC, $\hat{B}$ and $\hat{C}$ have the same measure. / In Δ ABC, is $\hat{B}$ en $\hat{C}$ ewegroot.

A) Statements S and T cannot both be true. / Stellings S en T kan nie albei waar wees nie.

B) If S is true, then T is true. / As S waar is, dan is T waar.

C) If T is true, then S is true. / As T waar is, dan is S waar.

D) If S is false, then T is false. / As S vals is, dan is T vals.

21. Which statement is true? / Watter stelling is waar?

A) All properties of rectangles are properties of all squares. / Alle eienskappe van reghoeke is eienskappe van all vierkante.

B) All properties of squares are properties of all rectangles. / Alle eienskappe van vierkante is eienskappe van all reghoeke.

C) All properties of rectangles are properties of all parallograms. / All eienskappe van reghoeke is eienskappe van all parallelogramme.

D) All properties of squares are properties of all parallelograms. / All eienskappe van vierkante is eienskappe van all parallelogramme.

22. What do all rectangles have that some parallelograms do not have? / Watter een van hierdie eienskappe het alle reghoeke wat sommige parallelogramme nie het nie?

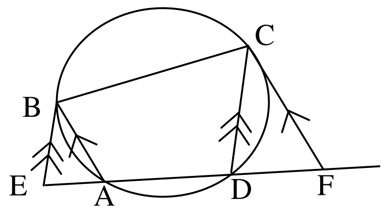
A) Opposite sides equal. / Teenoorstaande sye gelyk.

B) Diagonals equal. / Hoeklyne gelyk.

C) Opposite sides parallel. / Teenoorstaande sye ewewydig.

D) Opposite angles equal. / Teenoorstaande hoeke gelyk.

23.



If two trapeziums EBCD and ABCF are drawn on the same baseline EADF and have a circle drawn through points A, B, C, and D, then it can be proved that … / As twee trapesiums EBCD en ABCF op dieselfde basislyn geteken word, en 'n sirkel word getrek deur punte A, B, C, en D, dan kan dit bewys word dat …

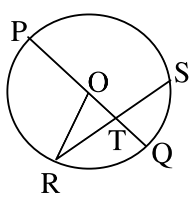
A) $\Delta BEA /// \Delta CFD$

B) ABCD is a cyclic quadrilateral. / ABCD 'n koordevierhoek is.

C) $\Delta BEA \equiv \Delta CFD$

D) EBCF is a cyclic quadrilateral. / EBCF 'n koordevierhoek is.

24.



PT = 8 cm
TQ = 2 cm
Chord RTS ⊥ diameter PQ. The length of RS is … /
Koord RTS ⊥ diameter PQ. Die lengte van RS is …

A) 5 cm

B) 8 cm

C) $\sqrt{10}$ cm

D) 9 cm

25. Here are two statements. / Hier volg twee stellings.

I: If a figure is a rectangle, then its diagonals bisect each other. / As 'n figuur 'n reghoek is, dan halveer sy hoeklyne mekaar.

II: If the diagonals of a figure bisect each other, the figure is a rectangle. / As die hoeklyne van 'n figuur mekaar halveer, dan is die figuur 'n reghoek.

A) To prove that I is true, it is sufficient to prove that II is true. / Om I waar te bewys, is dit genoeg om II waar te bewys.

B) To prove II is true, it is sufficient to prove that I is true. / Om II waar te bewys, is dit genoeg om I waar te bewys.

C) To prove II is true, it is sufficient to find one rectangle whose diagonals bisect each other. / Om II waar te bewys, is dit genoeg om een reghoek te kry waarvan die hoeklyne mekaar halveer.

D) To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other. / Om te bewys dat II vals is, is dit genoeg om een nie-reghoek te vind waarvan die hoeklyne mekaar halveer.

26. In geometry… / In meetkunde …

A) Every term can be defined and every true statement can be proved true. / Kan elke term gedefiniëer word en elke waar stelling waar bewys word.

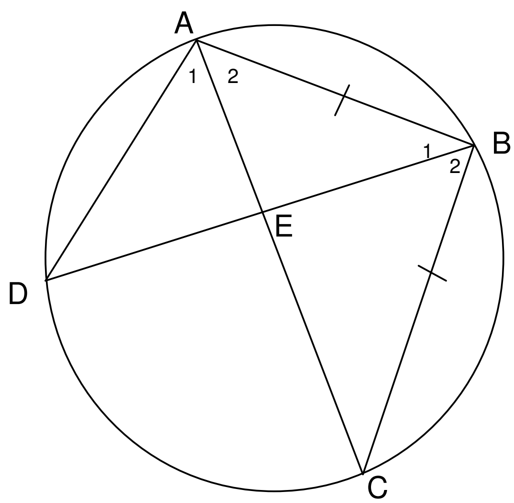
B) Every term can be defined, but it is necessary to assume that certain statements are true. / Kan elke term gedefiniëer word, maar dit is nodig om aan te neem dat sekere stellings waar is.

C) Some terms must be left undefined, but every true statement can be proved true. / Moet sommige terme ongedefiniëer gelaat word, maar elke waar stelling kan waar bewys word.

D) Some terms must be left undefined and it is necessary to have some statements which are assumed true. / Moet sommige terme ongedefiniëer gelaat word, en is dit nodig om sommige stellings te hê wat as waar aangeneem word.

Section II/ Afdeling II

27. a) Complete the following sentence:
Angles in the same segment of a circle ________________________________
Voltooi die volgende sin:
Hoeke in dieselfde sirkelsegment ____________________________________

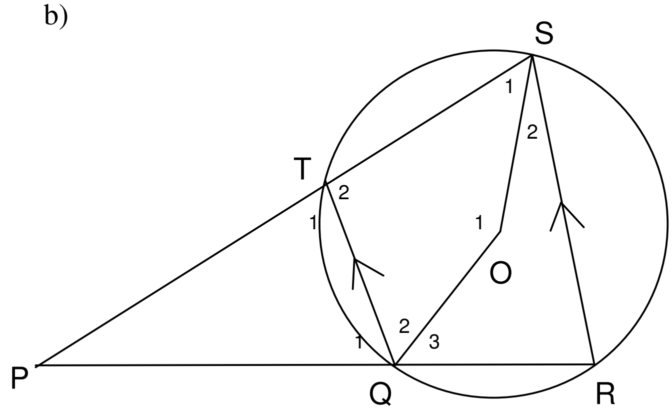
b) 

Given that chords AB and BC are equal, and that chords AC and BD intersect at E, prove that AB is a tangent to circle DEA.

Gegee dat AB en BC gelyke koorde is, en dat AC en BD mekaar in E sny, bewys dat AB 'n raaklyn is aan sirkel DEA.

28. a) Complete the following sentence:
The exterior angle of a cyclic quad is ______________________________________
Voltooi die volgende sin:
Die buitehoek van 'n koordevierhoek ____________________________________

b)



In the accompanying figure, TQRS is a cyclic quad with centre O. TQ // SR and ST and RQ produced meet in P.

In die meegaande figuur is TQRS 'n koordevierhoek met middelpunt O.
TQ // SR en ST en RQ verleng sny in P.

Prove that: / Bewys dat:

PT = PQ