# CHAPTER 4

## The Modal Coordinate Transformation Method

## 4.1 THEORY

### Preamble

This chapter is devoted entirely to the modal coordinate transformation method (Kim and Kim, 1997; Desangehere and Snoeys, 1985). A detailed derivation of the modal coordinate transformation method is presented. The numerical studies investigate the application of the method on a two degree-of-freedom system and the factors influencing the condition number of the modal matrix. This chapter also describes applying the modal filter to force reconstruction, which is validated by a numerical simulation.

For a proportionally damped multi-degree-of-freedom system, with $N$ degrees of freedom, the governing equations of motion can be written in matrix form as:

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+[K]\{x(t)\}=\{f(t)\} \tag{4.1}$$

where

$[M]$, $[K]$ and $[C]$ are the $(N \times N)$ mass, stiffness and damping matrices, respectively,

$\{\ddot{x}(t)\}$, $\{\dot{x}(t)\}$ and $\{x(t)\}$ are the $(N \times 1)$ acceleration, velocity and displacement vectors, respectively, and

$\{f(t)\}$ is the $(N \times 1)$ applied force vector.

The eigenvalue problem for the conservative (undamped) structure is written as:

$$[K][\Phi]=[M][\Phi][\Lambda] \tag{4.2}$$

where

$[\Phi]=[\{\phi\}_1,\{\phi\}_2,\ldots,\{\phi\}_N]^T$ is the modal matrix, consisting of $N$ modal vectors,

$[\Lambda]=diag\{\lambda_1,\lambda_2,\ldots,\lambda_N\}$ is the modal stiffness matrix and $\lambda_r=\omega_r^2$ for $r=1,N$

Substitute the following coordinate transformation into equation (4.1)

$$\{x(t)\}=[\Phi]\{p(t)\} \tag{4.3}$$

and pre-multiplying the resulting equation by $[\Phi]^T$ yields

$$[\overline{M}]\{\ddot{p}(t)\}+[\overline{C}]\{\dot{p}(t)\}+[\overline{K}]\{p(t)\}=[\Phi]^T\{f(t)\} \tag{4.4}$$

where

$[\overline{M}]$, $[\overline{K}]$ and $[\overline{C}]$ are the modal mass, stiffness and damping matrices, respectively.

For a mass-normalised modal matrix it follows that

$$[\Phi]^T[M][\Phi]=[I] \quad \text{and} \quad [\Phi]^T[K][\Phi]=[\Lambda]$$
$$[\Phi]^T[C][\Phi]=[\beta] \tag{4.5}$$

where

$[\beta]=diag\{\beta_1,\beta_2,\dots,\beta_N\}$ is the modal damping matrix and $\beta_r=2\zeta_r\omega_r$,

$\zeta_r$ is the modal damping factor for the $r$-th mode.

It will be assumed that the following relation holds:

$$[C][M]^{-1}[K]=[K][M]^{-1}[C] \tag{4.6}$$

Satisfying this condition, the damped system will possess the same mode shapes as its undamped counterpart. Thus, the eigenvectors will be real and equation (4.4) will contain $N$ uncoupled system equations.

Transforming equation (4.4) to the frequency domain and using equation (4.5), it follows for steady state conditions that

$$\left[-\omega^2[I]+i\omega[\beta]+[\Lambda]\right]\{P(\omega)\}=[\Phi]^T\{F(\omega)\} \tag{4.7}$$

where

$\{P(\omega)\}$ denotes the Fourier Transform of the modal coordinates ($\{p\}$ refers to the modal coordinates as a function of time),

Rearranging equation (4.7) it follows

$$\{P(\omega)\}=\left[-\omega^2[I]+i\omega[\beta]+[\Lambda]\right]^{-1}[\Phi]^T\{F(\omega)\} \tag{4.8}$$

Introducing equation (4.8) back into equation (4.3) in the frequency domain results in:

$$\{X(\omega)\}=[\Phi]\left[-\omega^2[I]+i\omega[\beta]+[\Lambda]\right]^{-1}[\Phi]^T\{F(\omega)\} \tag{4.9}$$

Equation (4.9) can be represented in a more familiar form, as the frequency response function

$$\{X(\omega)\}=\sum_{r=1}^{N}\frac{\{\phi\}_r\{\hat{\phi}\}_r^T}{\omega_r^2-\omega^2+i2\zeta_r\omega_r\omega}\{F(\omega)\} \tag{4.10}$$

where

$\{\phi\}_r$ is the $(N\times 1)$ modal vector corresponding to the measurement points,

$\{\hat{\phi}\}_r$ is the $(N \times 1)$ modal vector corresponding to the excitation points.

In real-world applications the mode shapes are usually identified at $n$ $(n < N)$ sensor locations while only $p$ $(p < N)$ of all possible modes are included. If the number of forces, $m$ and positions are known *a priori*, equation (4.10) reduces to

$$\{X(\omega)\} = \sum_{r=1}^{p} \frac{\{\phi\}_r \{\hat{\phi}\}_r^T}{\omega_r^2 - \omega^2 + i2\zeta_r\omega_r\omega} \{F(\omega)\} \tag{4.11}$$

where

$\{\phi\}_r$ is the $(n \times 1)$ modal vector corresponding to the measurement points,

$\{\hat{\phi}\}_r$ is the $(m \times 1)$ modal vector corresponding to the excitation points.

Employing matrix notation, equation (4.11) can be rewritten as

$$\{X(\omega)\} = [\Phi][S(\omega)][\hat{\Phi}]^T \{F(\omega)\} \tag{4.12}$$

where

$[S(\omega)]$ is an $(p \times p)$ diagonal matrix having the following terms on the diagonal:

$$s_r = \frac{1}{\omega_r^2 - \omega^2 + i2\zeta_r\omega_r\omega}$$

### 4.1.1 The Modal Coordinate Transformation Methodology:

The modal coordinate transformation method uses equation (4.12) to identify the input forces. This method comprises the following steps (Desanghere and Snoeys, 1985):

Step 1: Computation of the modal responses.

Successive integration of the measured acceleration signals to obtain the displacements. The displacements at the physical coordinates are transformed to the modal coordinates by multiplying the displacements with the pseudo-inverse of the modal matrix.

$$\{P(\omega)\} = [\Phi]^+ \{X(\omega)\} \tag{4.13}$$

where

$\{P(\omega)\}$ is a $(p \times 1)$ vector and denotes the Fourier Transform of the modal coordinates ($\{p\}$ refers to the modal coordinates as a function of time),

$[\Phi]^+$ denotes the pseudo-inverse of the $(n \times p)$ modal matrix, and

$\{X(\omega)\}$ is the $(n \times 1)$ physical response vector at the sensor locations.

<u>Step 2:</u> Computation of the modal forces.

$$\{F_m(\omega)\} = [S(\omega)]^{-1}\{P(\omega)\} \tag{4.14}$$

where

$\{F_m(\omega)\}$ is the $(p \times 1)$ modal force vector.

<u>Step 3:</u> Computation of the physical forces

$$\{\hat{F}(\omega)\} = [\Phi^T]^+\{F_m(\omega)\} \tag{4.15}$$

where

$\{F(\omega)\}$ is the $(m \times 1)$ input force vector, and

$[\Phi^T]^+$ indicates the pseudo-inverse of $[\Phi]^T$.

The least-squares estimation of the pseudo-inverse is used twice in the modal force identification. These pseudo-inverse matrices can be calculated using any of the methods previously described in Section 3.1.

The above procedure can be summarised in a single equation as:

$$\{\hat{F}(\omega)\} = [\Phi^T]^+[S(\omega)]^{-1}[\Phi]^+\{X(\omega)\} \tag{4.16}$$

### 4.1.2 Limitations Regarding the Modal Coordinate Transformation

Since the over-determined case is the most common in engineering problems, the same approach will be followed in force identification. This means that the pseudo-inverse is obtained by normal least-squares solutions while avoiding minimum norm solutions associated with the under-determined case. Consider the direct problem of equation (4.15) as

$$\{F_m(\omega)\} = [\Phi]^T\{F(\omega)\}$$

The number of responses, $n$, generally exceeds the number of modes, $p$, resulting in a rectangular modal matrix. Thus, solving the forces at the physical coordinates from knowledge of the modal forces, equation (4.16) is under-determined (i.e. more unknowns than data). If the number and positions of the forces are known *a priori*, the rows of $[\Phi]$ which correspond to these forces are grouped into the $(m \times p)$ matrix $[\Theta]$, while eliminating the positions from the force vector, $\{F(\omega)\}$, at which forces will be constrained to act, making it an $(m \times 1)$ vector. In this case, the solution for $\{F(\omega)\}$ becomes over-determined (i.e. more data than unknowns).

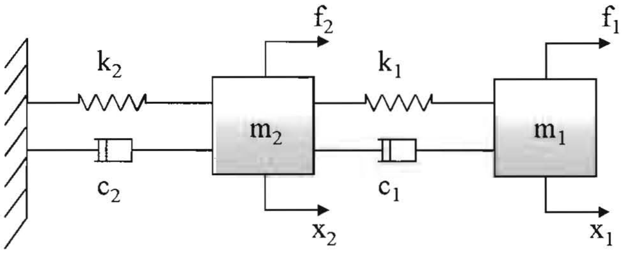
Thus, to obtain a least-squares solution, it is required that $n \geq p \geq m$ be satisfied, i.e. the number of response measurements should exceed or be equal to the number of modes, which in turn should be greater than or equal to the number of forces estimated.

## 4.2 TWO DEGREE-OF-FREEDOM SYSTEM

In this section the modal coordinate transformation technique is implemented on a simple 2 DOF lumped-mass system. Once again the system's response and modal parameters will be subjected to perturbation in order to simulate experimental measurements. It will be shown that the modal coordinate transformation technique does not suffer from the ill-conditioning of the force estimates at the system's resonance due to the matrix inversion, as was encountered for the frequency response function technique.

There are primarily two sources of error present in the modal coordinate transformation technique. These are the noise encountered in the structure's response measurements and the modal parameter extraction. Errors in the modal parameters will propagate to the identified forces through the double pseudo-inverse of the modal matrix. Kim and Kim (1997) studied the error propagation and found that for a lightly damped structure, errors in the damping values induce fewer error in the force identification than errors in the natural frequencies. Errors in the modal vectors were considered to have the most adverse effect on force predictions. Desanghere and Snoeys (1985) considered the modal coordinate transformation technique rather insensitive to measurement and curve-fitting perturbations. They claim that the precision of the identified forces were almost constant over the frequency range covered by the modes included in the analysis, regardless of the system's resonances.

Consider the following lumped-mass system.




| | | |
|---|---|---|
| $m_1 = 0.5$ kg | $k_1 = 1.4212 \times 10^5$ N/m | $\zeta_1 = 0.001$ |
| $m_2 = 1.0$ kg | $k_2 = 2k_1$ N/m | $\zeta_2 = 0.001$ |

*Figure 4.1 - 2 DOF lumped-mass system*

Once again, a harmonic forcing function is used to excite each of the masses.

$$f_1(t) = 150\cos(60\pi t) \qquad f_2(t) = 100\cos(60\pi t) \tag{4.17}$$

The response for each degree-of-freedom was solved analytically.
Perturbation of the natural frequencies, modal damping factors, mode-shapes and accelerations was conducted similar to that described in Section 3.2, for the frequency response function method. The same maximum error levels were used as before.

The contaminated modal parameters and responses were used to solve the forces.

## Results and Discussion

This technique was successfully implemented to identify the two harmonic forces applied to the 2 DOF lumped-mass system.

It is evident from Figures 4.2, 4.3 and 4.4 that the modal coordinate transformation technique does not suffer from the ill-conditioning of the force estimates at the system's resonance due to the inversion of the modal matrix. Figure 4.2 shows the force magnitudes as a function of the frequency.

Although it is general practice to take acceleration measurements, the formulation of the modal coordinate transformation method, as expressed by equation (4.16), uses displacements. The successive integration of the accelerations to displacements will amplify the errors in the low frequency range. This type of behaviour is evident in Figures 4.3 and 4.4 where the small numerical errors from the perturbation and the DFT are amplified prior to the excitation frequency.

The Moore-Penrose pseudo-inverse method was used to determine the pseudo-inverse of the modal matrix, since the columns of the modal matrix represent the individual mode-shapes and are orthogonal with respect to each other.

The FEN is depicted in Figure 4.5. The FEN is initially high in the low-frequency range, for the above-mentioned explanation, but decreases rapidly as it approaches the excitation frequency. At the excitation frequency the force estimates are in good agreement with the applied forces and the FEN reaches a minimum value. From here onward the FEN increases steadily, as the force estimates deviate from the applied forces.

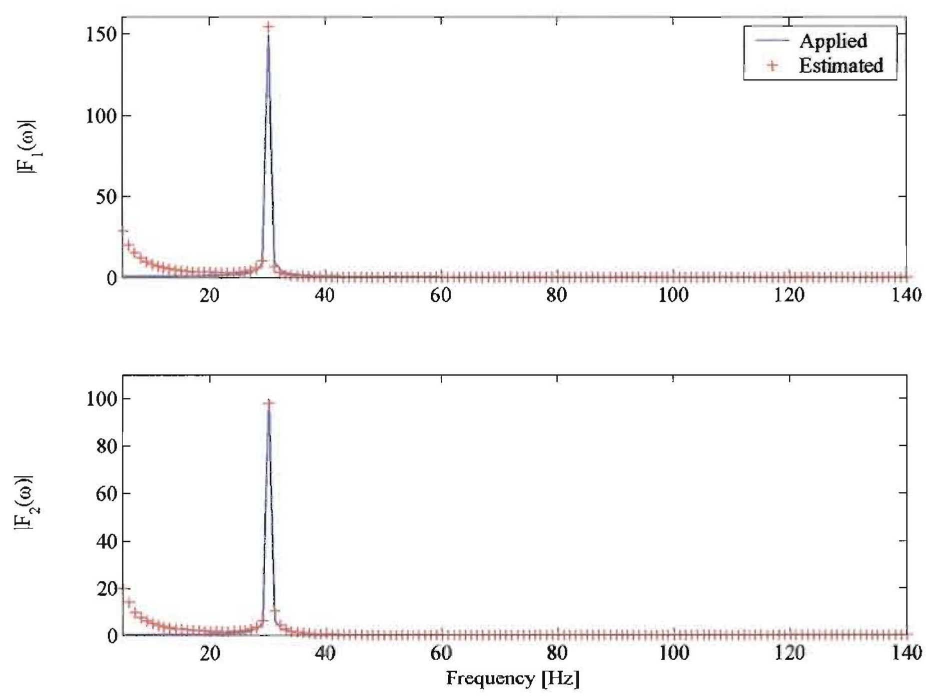



*Figure 4.2 – Applied and estimated force magnitudes for the 2 DOF lumped-mass system in the frequency domain*

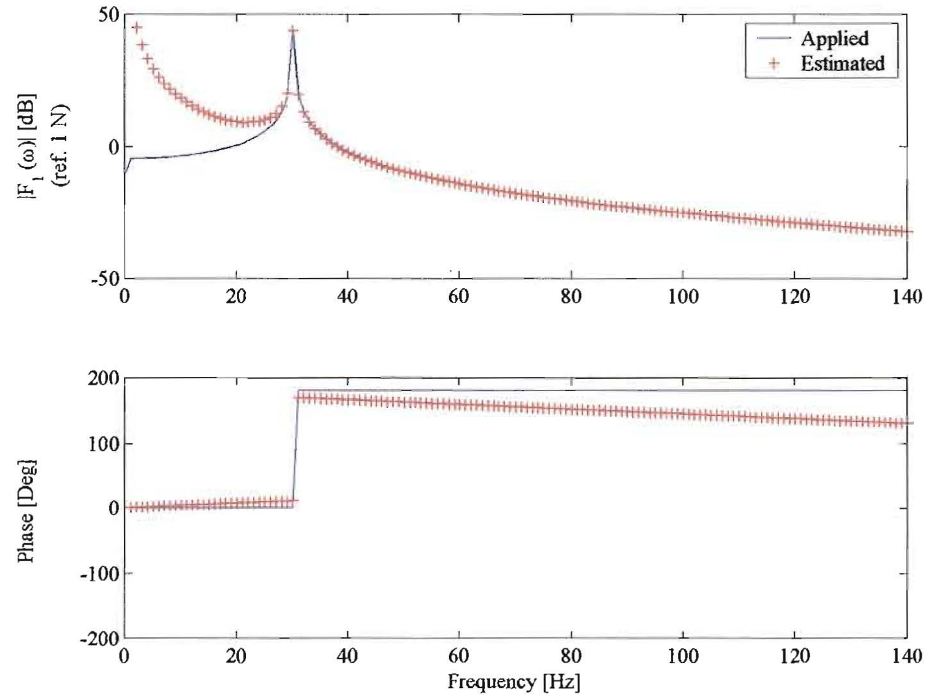



*Figure 4.3 – Applied and estimated force no.1 for the 2 DOF lumped-mass system in the frequency domain*

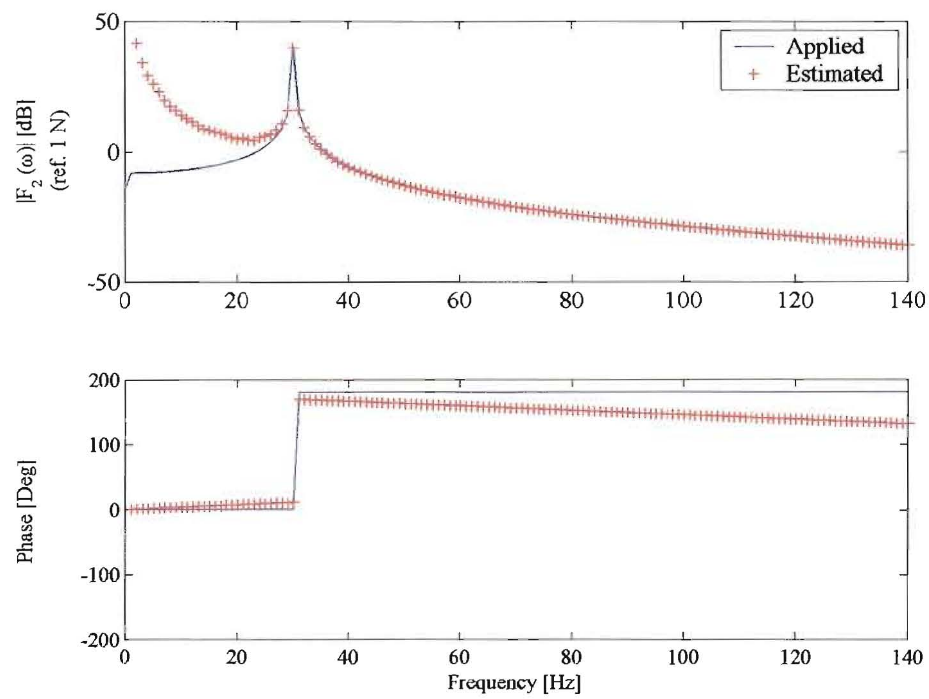



*Figure 4.4 – Applied and estimated force no.2 for the 2 DOF lumped-mass system in the frequency domain*

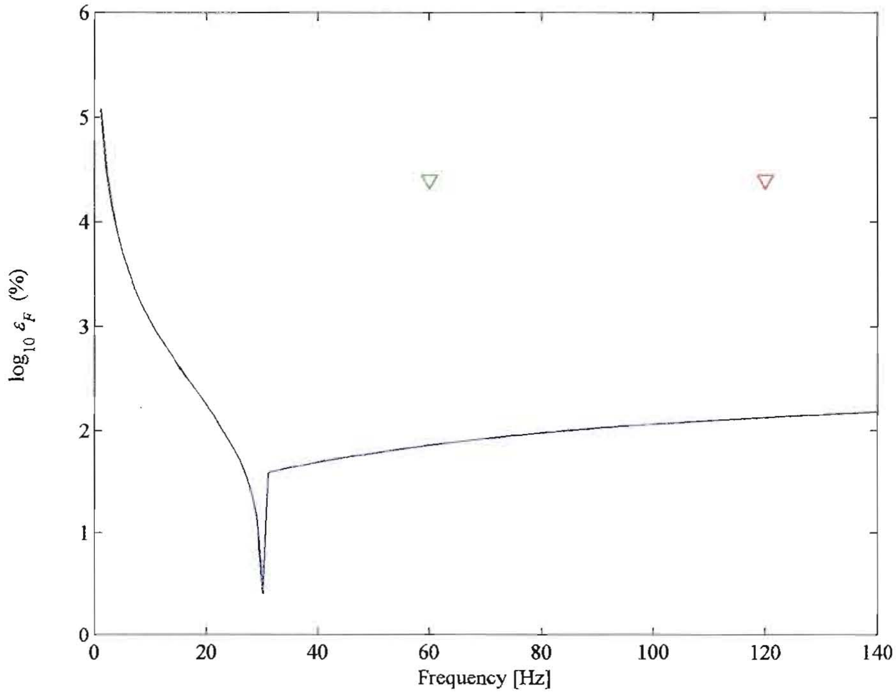



*Figure 4.5 – Force Error Norm (FEN) of the estimated forces,*
*∇ indicates the 2 DOF systems' resonant frequencies*

In view of the above numerical simulation we would like to comment on some of the advantages and disadvantages regarding the use of the modal coordinate transformation technique in general.

**Advantages of Modal Coordinate Transformation**

This technique requires that the pseudo-inverse be calculated only twice, regardless of the number of discrete frequencies included in the analysis. This reduces the computational time required to analyse large systems (Kim and Kim, 1997).

The points on a structure where the actual forces are applied may be inaccessible. Another point can then be artificially excited to extract the modal parameters while measuring the responses at the actual input locations. The force estimates at the inaccessible points can then be determined based on the reciprocity theorem (Kim and Kim, 1997).

**Disadvantages of Modal Coordinate Transformation**

The modal coordinate transformation technique requires a good set of modal parameters. This may prove hard to obtain for a complex structure.

Another disadvantage is that the estimated forces are limited to the frequency range based on the modes selected for the modal transformation (Shih *et al.*, 1989).

## 4.3 SIGNIFICANCE OF THE CONDITION NUMBER

As mentioned earlier, the definition of the condition number for a rectangular matrix can be stated as:

$$\kappa_2 = \frac{\sigma_{\max}([A])}{\sigma_{\min}([A])} \tag{4.18}$$

where

$\sigma([A])$ is the singular value of the matrix $[A]$.

From equations (4.13) and (4.15) it is apparent that the modal coordinate transformation method only requires the pseudo-inverse of the modal matrix, $[\Phi]$ and $[\Phi]^T$. The condition number of this technique will, thus be bounded by the product of the condition number of the individual pseudo-inverses.

$$\kappa_{\text{modal}} = \kappa_2([\Phi^T]) \cdot \kappa_2([\Phi]) \tag{4.19}$$

A large condition number will indicate an ill-conditioned system, i.e. a system that is prone to significant errors in the force estimates when inverted. The source of this ill-conditioning lies in the pseudo-inverse of $[\Phi]$ and $[\Phi]^T$. It is also important to note from equation (4.19), that the condition number remains constant for the entire frequency range of interest.

Hansen and Starkey (1990) extended the work of Starkey and Merrill (1989) by considering the condition of the modal coordinate transformation technique. They established a criterion for which some types of systems are ill-conditioned while others are not. Based on the fact that there is a close relationship between the singular values of a matrix $[A]$ and the eigenvalues of $([A]^T[A])$, they concluded that the condition number of the pseudo-inverse will be a function of the set of sensor locations, as well as of which modes are included.

The same FEM, as described in detail in Section 3.3, was used to investigate the factors that contribute to the condition number of the modal coordinate transformation technique.

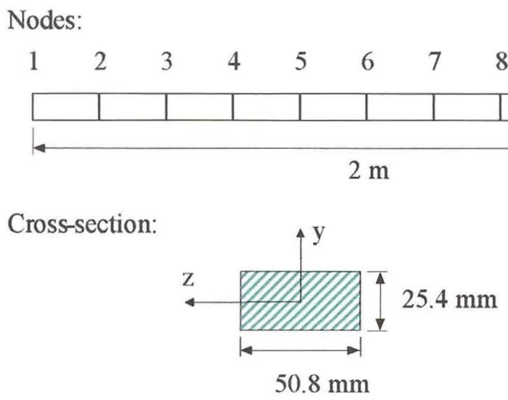



*Figure 4.6 - FEM of free-free beam and response locations*

### 4.3.1 Effect of the Response Selection

Firstly, we investigated how the different combinations of response locations influenced the condition number of the modal matrix. Three sets of response measurements were chosen, each consisting of four response locations. The first five bending modes were included in the analysis, while disregarding the rigid body modes. From the results in Table 4.1 we can conclude that the choice of response locations included in the modal matrix, can greatly affect the condition of the pseudo-inverse and subsequently influence the force predictions.

*Table 4.1 – Effect of different response sets on the condition number*

| Sets of response points | Condition Number |
|---|---|
| (8, 5, 6, 11) | 3.05 |
| (1, 7, 4, 10) | 18.88 |
| (1, 2, 3, 4) | 23.00 |

For a real-world structure involving a large number of sensors, selecting the appropriate response set that corresponds to the lowest condition number, from all possible sensor locations may prove to be a formidable task. In order to choose $r$ sensor locations out of a possible $n$, we would need to evaluate

$$\frac{n!}{r!(n-r)!} \tag{4.20}$$

combinations in search for the optimal modal matrix. Therefore, to choose 4 sensor locations out of a possible 11 for the above FEM example, 330 combinations must be evaluated, which is not practical.

Kammer (1990) introduced the effective independent algorithm that measures how each sensor location contributes to the rank of the modal matrix. Shelley *et al.* (1991) found this algorithm to be both effective and computationally efficient in selecting sensor locations in the application of modal filters.

Kammer's algorithm is supported in the *Structural Dynamics Toolbox*®(Balmès, 1997) and is accessed using the `fe_sens` function. This function sorts the selected sensor locations contained in the vector `sdof`, from most too least important. Applying this function to the FEM the following result was obtained:

```
sdof=[8; 5; 6; 11; 1; 7; 4; 10; 2; 3; 9]
```

The first four digits in `sdof` correspond to the first response set in Table 4.1. It can be seen that this response set produced the lowest condition for the particular choice of modes. Important to note is that Krammer's algorithm is dependent on the modes included in the analysis. Reducing the number of modes to include only the first four bending modes will result in a different set of sensor locations, as shown below:

```
sdof=[11; 7; 5; 1; 6; 4; 9; 3; 8; 2; 10]
```

Similar to the frequency response function technique, the inclusion of more response measurements will reduce the condition number of the modal matrix.

### 4.3.2 The Effect of the Number of Modes

The effect of the number of modes included in the modal matrix was considered next. The same combinations of response measurements were considered as before, but with different ranges of modes included. Table 4.2 shows the condition number for each of these cases.

*Table 4.2 – Effect of different modes on the condition number*

| Response points | Modes 1-5 | Modes 2-5 | Modes 1-4 | Modes 2-4 |
|---|---|---|---|---|
| (8, 5, 6, 11) | 3.05 | 4.85 | 22.37 | 3.49 |
| (1, 7, 4, 10) | 18.88 | 20.37 | 20.09 | 17.12 |
| (1, 2, 3, 4) | 23.00 | 46.79 | 98.12 | 14.49 |

The result shows that the modes included or excluded from the modal matrix can influence the condition number significantly. In general the condition number is smaller when more modes are included, than when fewer modes are used.

These results confirm Hansen's findings.

### 4.3.3 Conclusion

The modal coordinate transformation technique is generally well-conditioned, as long as a convenient set of sensor locations is chosen. Care should be taken during the selection of the response measurements and number of modes included to ensure the lowest possible condition number for the calculation of the pseudo-inverse.

## 4.4 MODAL FILTERS

### 4.4.1 Preamble

This section describes applying modal filters to the force identification process. First, modal filters are described and discussed. The modal filter is then applied to the modal coordinate transformation technique, which is validated by a numerical simulation.

The modal filter was originally introduced to deal with the spillover problem in the control of distributed-parameter systems (Meirovitch and Baruh., 1983), but has recently been extended to include other applications in vibration analysis. These include active vibration control (Meirovitch and Baruh, 1985), correlation between analytical and experimental models, vibration force identification (Zhang *et al.*, 1990) and on-line parameter estimation (Shelley *et al.*, 1992).

A modal filter works on the basis that it transforms the physical response coordinates to the modal coordinates and may be performed in either the time or frequency domain. For instance, by multiplying the measured response vector, $\{x\}$, by the modal filter matrix, $[\psi]^T$, the physical response vector is uncoupled into a vector of the single-degree-of-freedom modal coordinates responses, $\{p\}$.

$$\{p\}=[\psi]^T\{x\} \tag{4.21}$$

The first step in the force identification by means of the modal coordinate transformation method requires the computation of the modal responses. This is done by multiplying the measured displacement vector with the pseudo-inverse of the modal matrix.

$$\{P(\omega)\}=[\Phi]^{+}\{X(\omega)\} \tag{4.22}$$

In equation (4.22) the pseudo-inverse of the modal matrix performs in essence the role of the modal filter as described in equation (4.21). A problem with this approach is that complete and accurate estimations of all the modal vectors in the frequency range of interest are required. Furthermore, it is assumed that for an undamped or proportionally damped system the modal vectors are orthogonal with respect to the physical mass matrix, which is analytically imposed but may be violated during the parameter extraction process. If a particular modal vector contains errors this will propagate to all of the modal vectors through calculation of the pseudo-inverse.

The Reciprocal Modal Vector (RMV) can be employed as an alternative method for calculating the modal filter of a system. The reciprocal modal vectors are defined to be orthogonal to the modal vectors and are calculated from measured frequency response functions and modal parameters. The modal filter estimate for a given mode is not affected by errors associated with other modes.

### 4.4.2 Formulation

A brief description of the formulation of the reciprocal modal vector method follow (Zhang *et al.*, 1990). Specific details regarding the derivation of RMV and other modal filters can be found in Zhang *et al.* (1989); Shelley and Allemang (1992) and He and Imregun (1995).

The frequency response function was previously derived in terms of modal parameters as

$$\{X(\omega)\}=\sum_{r=1}^{N}\frac{\{\phi\}_r\{\hat{\phi}\}_r^T}{\omega_r^2-\omega^2+i2\zeta_r\omega_r\omega}\{F(\omega)\} \tag{4.23}$$

In the above equation the orthogonality criterion of the modal vectors is used to perform the transformation from the physical to the modal coordinates. Thus, for mass-normalised modal vectors it follows that

$$\{\phi\}_i^T[M]\{\phi\}_j=\delta_{ij} \tag{4.24}$$

where

$\delta_{ij}$ is the Kronecker delta function, and $\delta_{ij}$ is equal to zero for $i \neq j$ and is equal to 1 for $i=j$.

Since the physical mass matrix is usually not available in practice, this orthogonality criterion can be restated as:

$$\{\psi\}_i^T\{\phi\}_j = \delta_{ij} \tag{4.25}$$

where

$\{\psi\}_i^T$ is the reciprocal modal vector, corresponding to mode $i$.

The reciprocal modal vector is defined as the product of the transposed modal vector and the mass matrix:

$$\{\psi\}_i^T \equiv \{\phi\}_i^T[M] \tag{4.26}$$

Zhang *et al.* (1990) constructed the reciprocal modal vector of a particular mode, using mode vectors, eigenvalues and frequency response functions. If the $p$-th column of the frequency response function matrix may be expressed as

$$\{H_p(\omega)\} = \sum_{r=1}^{N}\left[\frac{\{\phi\}_r\phi_{pr}Q_r}{(i\omega-\lambda_r)} + \frac{\{\phi\}_r^*\phi_{pr}^*Q_r^*}{(i\omega-\lambda_r^*)}\right] \tag{4.27}$$

where

$\lambda_r$ and $\{\phi\}_r$ are the $r$-th complex eigenvalue and the associated modal vector,

$Q_r$ is the modal scaling factor, and

$*$ denotes the complex conjugate.

Premultiplying the reciprocal modal vector and rearranging the expression Zhang produced the following results:

$$\left\{\{H_p(\omega)\} - \frac{\{\phi\}_r^*\phi_{pr}^*Q_r^*}{(i\omega-\lambda_r^*)}\right\}^T\{\psi\}_r = \frac{\phi_{pr}Q_r}{(i\omega-\lambda_r)} \tag{4.28}$$

Equation (4.28) may be evaluated at a sufficient number of discrete frequencies to form an over-determined problem, which may be solved for $\{\psi\}_r$ in a least-squares manner. The resulting reciprocal modal vector will be orthogonal to the modal vectors within this frequency range.

The minimum number of sensors required to calculate the modal filter should be equal to or greater than the number of modes in the frequency range of interest.

The modal filter calculated from use of the reciprocal modal vector will replace the pseudo-inverse of the modal vector in the first step of the force identification process.

### 4.4.3 Seven Degree-of-Freedom System

A numerical simulation involving a seven DOF system, with non-proportional damping, is employed to illustrate the effectiveness of the proposed method. Although the formulation of the modal coordinate transformation method was developed based on the hypothesis of proportional damping, this simulation shows that for lightly damped structures the non-proportional damping model does not lead to significant errors in the force estimates.

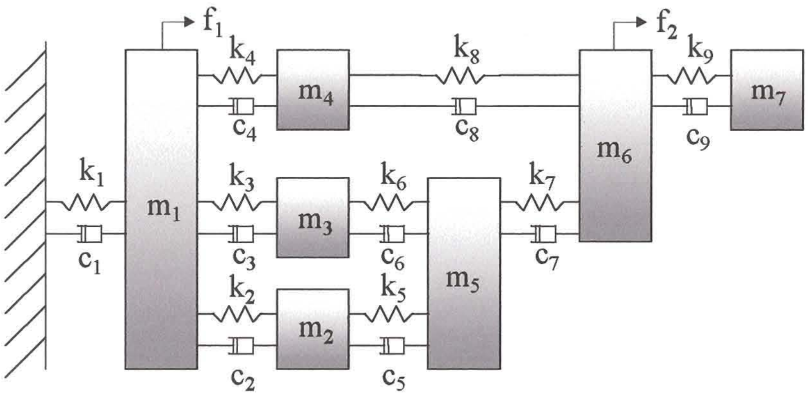



*Figure 4.7 - 7 DOF lumped-mass system*

*Table 4.3 Properties of the 7 DOF lumped-mass system*

| Mass [kg] | Stiffness [N/m] | Modal damping factors | Natural frequencies [Hz] |
|---|---|---|---|
| $m_1$=3.0 | $k_1$=5.0 x $10^6$ | 2.5 x $10^{-3}$ | 68.02 |
| $m_2$=2.0 | $k_2$=2.0 x $10^6$ | 1.0 x $10^{-3}$ | 135.56 |
| $m_3$=1.0 | $k_3$=1.0 x $10^6$ | 3.0 x $10^{-3}$ | 173.55 |
| $m_4$=2.0 | $k_4$=2.0 x $10^6$ | 2.0 x $10^{-3}$ | 212.86 |
| $m_5$=1.5 | $k_5$=1.0 x $10^6$ | 0.0 x $10^{-3}$ | 250.10 |
| $m_6$=2.0 | $k_6$=1.5 x $10^6$ | 5.0 x $10^{-3}$ | 316.76 |
| $m_7$=1.0 | $k_7$=2.0 x $10^6$ | 1.0 x $10^{-3}$ | 335.16 |
| - | $k_8$=2.0 x $10^6$ | - | - |

The system's natural frequencies and corresponding modal vectors were determined. The modal vectors were contaminated with uniformly distributed random errors with a maximum error level of 10 %, to simulate experimentally obtained data. This was done in accordance with the error model described in Section 3.2. These contaminated modal vectors will then be used to calculate the modal filter and the force estimates by means of the modal coordinate transformation method.

The modal filter was calculated as follows:

Columns 2 and 6 of the frequency response function matrix were constructed at 15 discrete frequencies, distributed equally throughout the frequency range and depicted as circles in Figure 4.8. These values, as well as the modal parameters of the mode of interest, were substituted into equation (4.28), from which the reciprocal modal vector was solved. This procedure was repeated for all seven modes. Each reciprocal modal vector represented a column of the modal filter matrix.

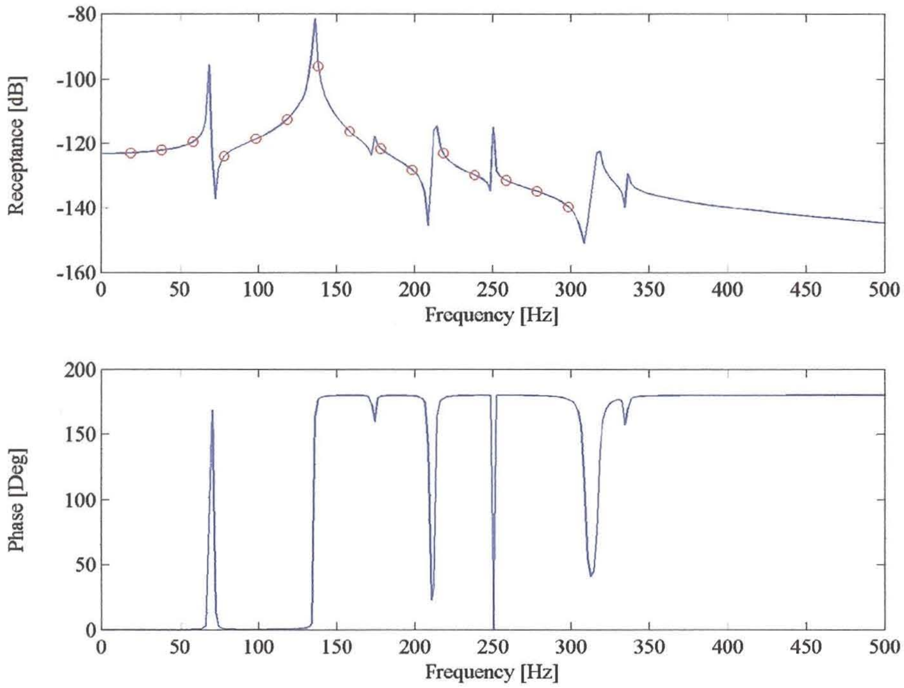



*Figure 4.8– Receptance, $\alpha_{12}(\omega)$ and discrete frequency points used in modal filter construction*

The validity of the modal filter can be checked based on the orthogonality criterion stated in equation (4.25). This equation can be rewritten in matrix notation as:

$$[\Phi]^T[\psi]=[I] \tag{4.29}$$

Substituting the previously obtained modal filter and modal matrix into the above equation, the following matrix was obtained:

$$[\Phi]^T[\psi]=\begin{bmatrix} 0.999 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.994 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.977 & 0.001 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.003 & -0.004 & 0.989 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.001 & -0.002 & 0.003 & 1.0 & 0.0 & 0.0 \\ -0.003 & 0.0 & 0.004 & -0.002 & 0.0 & 0.988 & 0.0 \\ 0.005 & 0.003 & 0.0 & 0.003 & 0.0 & 0.004 & 1.0 \end{bmatrix}$$

This matrix differs from the expected identity matrix due to the contaminated modal vectors included in the calculation of the reciprocal modal vectors and the modal truncation errors caused by the limited frequency range. However, it can be seen that the dominant values are still situated on the diagonal and are very close to unity.

The seven DOF-system was subjected to two harmonic forces with the same excitation frequency, acting on masses 1 and 6, respectively. The modal coordinate transformation method was used to calculate the force estimates, with the exception that the reciprocal modal vector was used for the modal coordinate transformation, rather than the pseudo-inverse of the modal matrix.

As can be seen from Figures 4.9 and 4.10, the estimated forces correspond well with the actual forces.

To conclude:

The modal filter, calculated from use of the reciprocal modal vector, might replace the pseudo-inverse of the modal vector in the first step of the force identification process.

Since the reciprocal modal vector method only requires the frequency response function matrix and modal parameters for only the mode of interest, the modal filter estimate for a given mode is not affected by errors associated with other modes.

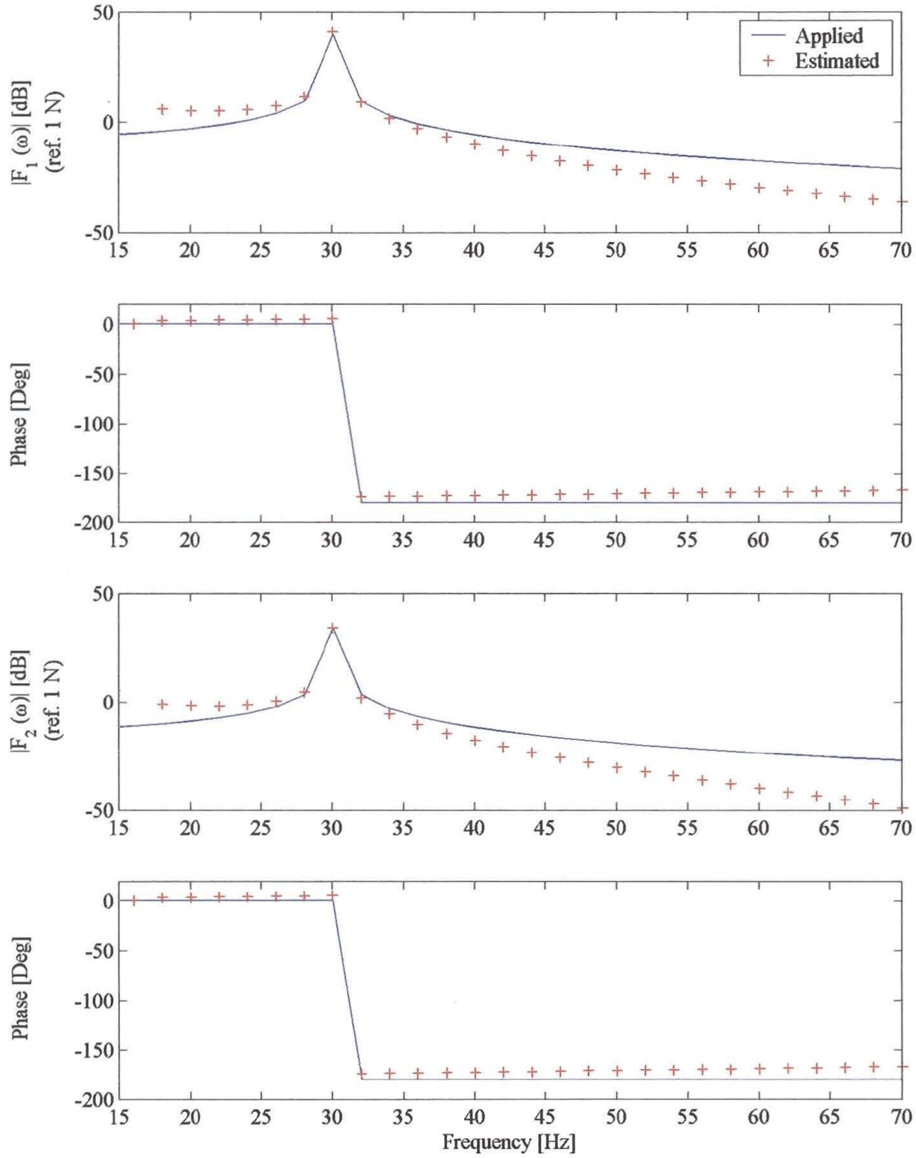



*Figure 4.9 – Applied and estimated forces in the frequency domain*

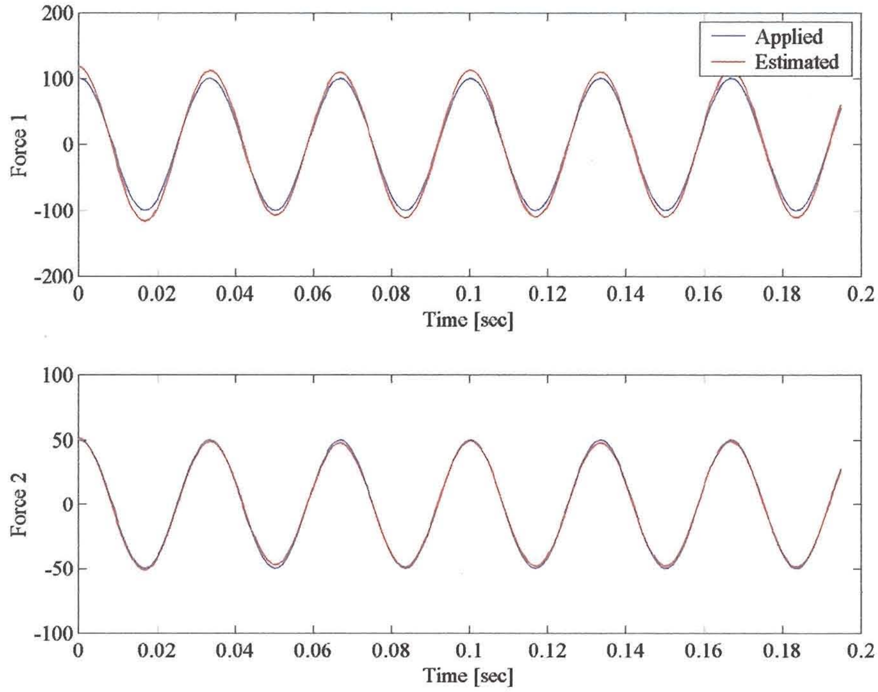



*Figure 4.10 – Applied and estimated forces in the time domain*