Cayley graphs of given degree and diameters 3, 4 and 5

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Abstract
Let \( C_{d,k} \) be the largest number of vertices in a Cayley graph of degree \( d \) and diameter \( k \). We show that \( C_{d,3} \geq \frac{3}{16}(d - 3)^3 \) and \( C_{d,5} \geq 25(d^2 - 7)^5 \) for any \( d \geq 8 \), and \( C_{d,4} \geq 32(d^2 - 8)^4 \) for any \( d \geq 10 \). For sufficiently large \( d \) our graphs are the largest known Cayley graphs of degree \( d \) and diameters 3, 4 and 5.

Keywords: Cayley graph, degree, diameter

A Cayley graph \( C(G, X) \) is specified by a group \( G \) and a unit-free generating set \( X \) for this group such that \( X = X^{-1} \). The vertices of \( C(G, X) \) are the elements of \( G \) and there is an edge between two vertices \( u \) and \( v \) in \( C(G, X) \) if and only if there is a generator \( a \in X \) such that \( v = ua \).

The degree-diameter problem for Cayley graphs is to determine the largest number of vertices in Cayley graphs of given degree and diameter. Let \( C_{d,k} \) be the largest order of a Cayley graph of degree \( d \) and diameter \( k \). The number of vertices in a graph of maximum degree \( d \) and diameter \( k \) can not exceed the Moore bound \( M_{d,k} = 1 + d + d(d - 1) + \ldots + d(d - 1)^{k-1} \). In [1] and [2] Bannai and Ito improved the upper bound and showed that for any \( d, k \geq 3 \) there are no graphs of order greater than \( M_{d,k} - 2 \), therefore \( C_{d,k} \leq M_{d,k} - 2 \) for such \( d \) and \( k \). Since the Moore graphs of diameter 2 and degree 3 or 7, and the potential Moore graph(s) of diameter 2 and degree 57 are non-Cayley (see [3]), Cayley graphs of order equal to the Moore bound exist only in the trivial cases when \( d = 2 \) or \( k = 1 \).

We focus on constructions of Cayley graphs of small diameter. The case \( k = 2 \) and \( d \to \infty \) has been widely studied. The best known construction of diameter 2 was recently found by Šiagiová and Širáň [8] who showed that \( C_{d,2} \geq d^2 - 6\sqrt{2}d^2 \) for an infinite set of degrees \( d \). Macbeth et. al. [7] presented large Cayley graphs giving the bound \( C_{d,k} \geq k(d - 3)^k \) for any diameter \( k \geq 3 \) and degree \( d \geq 5 \). Let us also mention the Faber-Moore-Chen graphs [4] of odd degree \( d \geq 5 \), diameter \( k \), such that \( 3 \leq k \leq \frac{d-1}{2} \), and order
These graphs are vertex-transitive and in [7] it is proved that for any \( k \geq 4 \) and sufficiently large \( d \) the Faber-Moore-Chen graphs are not Cayley. Large Cayley graphs of given degree \( d \) and diameter \( k \), where both \( d \) and \( k \) are small, were obtained by use of computers, see [5] and [6].

For diameters 3, 4 and 5 we present Cayley graphs which yield the bounds

\[
C_{d,3} \geq \frac{3}{16}(d-3)^3 \quad \text{and} \quad C_{d,5} \geq 25\left(\frac{d-2}{4}\right)^5
\]

for any \( d \geq 8 \), and

\[
C_{d,4} \geq 32\left(\frac{d-8}{5}\right)^4
\]

for any \( d \geq 10 \), improving thus the corresponding bounds of [7] for large \( d \). Particularly for diameter 3 we improve the lower bound considerably. It can be easily checked that the graphs of Faber, Moore and Chen are smaller than our graphs for diameter 3 and large degree, and they are larger than our graphs for diameters 4 and 5. However, for \( k = 4 \) and \( d \geq 21 \), and for \( k = 5 \) and \( d \geq 23 \), the Faber-Moore-Chen graphs are non-Cayley. To the best of our knowledge, for sufficiently large \( d \) there is no construction of Cayley graphs of degree \( d \) and diameter 3, 4 or 5 of order greater than the order of our graphs.

Now we describe the groups \( G \) which we use to produce large Cayley graphs. Let \( H \) be a group of order \( m \geq 2 \) with unit element \( e \). We denote by \( H^k \) the product \( H \times H \times \ldots \times H \), where \( H \) appears \( k \) times. Let \( \alpha \) be the automorphism of the group \( H^k \) which shifts coordinates by one to the right, that is, \( \alpha(x_1, x_2, \ldots, x_k) = (x_{k}, x_1, x_2, \ldots, x_{k-1}) \). The cyclic group of order \( p \) will be denoted by \( Z_p \).

We study the semidirect products \( G = H^k \rtimes Z_p \), where \( p \) is a multiple of \( k \), with multiplication given by

\[
(x, y)(x', y') = (x \alpha^y(x'), y + y'),
\]

where \( \alpha^y \) is the composition of \( \alpha \) with itself \( y \) times, \( x, x' \in H^k \) and \( y, y' \in Z_p \). Elements of \( G \) will be written in the form \( (x_1, x_2, \ldots, x_k; y) \), where \( x_1, x_2, \ldots, x_k \in H \) and \( y \in Z_p \).

We consider generating sets \( X \) which consist of classes of elements of the form \( (x_1, x_2, \ldots, x_k; y) \) where \( x_i, 1 \leq i \leq k \), is either \( e \) or \( g \) for any \( g \in H \). In the case of diameters 3 and 5 we found generating sets for large Cayley graphs using four such classes, whereas for diameter 4 we needed five classes. In our search for relatively small generating sets (to result in large Cayley graphs in terms of their degree) it proved efficient to consider generating sets containing \( (k+1) \)-tuples as above with at most two non-identity entries among the first \( k \) coordinates; increasing this number did not yield better graphs.
We now state and prove our main result.

**Theorem 1.**
(i) \(C_{d,3} \geq \frac{3}{16}d^3\) for \(d \geq 8\) such that \(d\) is a multiple of 4.
(ii) \(C_{d,4} \geq 32\left(\frac{d}{4}\right)^4\) for \(d \geq 10\) such that \(d\) is a multiple of 5.
(iii) \(C_{d,5} \geq 25\left(\frac{d}{4}\right)^5\) for \(d \geq 8\) such that \(d\) is a multiple of 4.

**Proof.** We use the group \(G\) with multiplication (1) defined earlier.
(i) Let \(G = H^3 \times \mathbb{Z}_{12}\) and \(X = \{a_g, a_g', b_h, \bar{b}_{h'} \mid \text{for any } g, g', h, h' \in H\}\) where 
\[a_g = (g, g, e; 1), \quad a_g' = (g', e, g'; -1), \quad b_h = (h, e, e; 8)\text{ and } \bar{b}_{h'} = (e, h', e; 4).\]
Since \(a_g^{-1} = a_g'\) and \(b_h^{-1} = \bar{b}_{h'}\), we have \(X = X^{-1}\). The Cayley graph \(C(G, X)\) is of degree \(d = |X| = 4m, m \geq 2\) and order \(|G| = 12m^3 = 12\left(\frac{d}{4}\right)^3 = \frac{3}{16}d^3\).

We show that the diameter of \(C(G, X)\) is at most 3, which is equivalent to showing that each element of \(G\) can be obtained as a product of at most 3 generators of \(X\). For any \(x_1, x_2, x_3 \in H\) we have
\[(x_1, x_2, x_3; 0) = (x_1, e, e; 8)(x_3, e, e; 8)(x_2, e, e; 8),\]
\[(x_1, x_2, x_3; 1) = (x_1x_3^{-1}, e, e; 8)(x_3, x_3, e; 1)(e, x_2, e; 4),\]
\[(x_1, x_2, x_3; 2) = (x_1, e, x_1; -1)(x_2, e, x_2; -1)(e, x_2^{-1}x_1^{-1}x_3, e; 4),\]
\[(x_1, x_2, x_3; 3) = (x_1x_2^{-1}, e, e; 8)(x_3, e, e; 8)(x_2, e, x_2; -1),\]
\[(x_1, x_2, x_3; 4) = (x_3, e, x_3; -1)(e, x_3^{-1}x_1x_2^{-1}, e; 4)(x_2, x_2, e; 1),\]
\[(x_1, x_2, x_3; 5) = (x_1, e, e; 8)(x_3x_2^{-1}, e, e; 8)(x_2, x_2, e; 1),\]
\[(x_1, x_2, x_3; 6) = (x_2x_3^{-1}, x_2x_3^{-1}, e; 1)(x_3, x_3, e; 1)(e, x_3x_2^{-1}x_1, e; 4).\]
It is easy to see that if \((x_1, x_2, x_3; y) = abc\), where \(a, b, c \in X\) and \(0 \leq y \leq 6\), then
\[x_1^{-1}x_2^{-1}x_3^{-1}x_1^{-1}(\text{mod 3})+1; x_2^{-1}x_3^{-1}(\text{mod 3})+1; x_3^{-1}(\text{mod 3})+1; -y) = c^{-1}b^{-1}a^{-1}.\]
Note that the diameter of \(C(G, X)\) cannot be smaller than 3, because the order is greater than the Moore bound for diameter 2.

(ii) Let \(G' = H^4 \times \mathbb{Z}_{32}\) and \(X' = \{a_g, a_g', b_h, \bar{b}_{h'} \mid \text{for any } g, g', h, h', j \in H\}\), where 
\[a_g = (g, e, e, e; 1), \quad a_g' = (e, e, e, g'; -1), \quad b_h = (e, h, e, e; 7)\text{ and } \bar{b}_{h'} = (e, e, h', e; 16).\]
Clearly, \(X = X^{-1}\) since \(a_g^{-1} = a_g'\), \(b_h^{-1} = \bar{b}_{h'}\) and \(c_j^{-1} = c_{j-1}\). The Cayley graph \(C(G', X')\) has degree \(d = |X'| = 5m\) and order \(|G'| = 32m^4 = 32\left(\frac{d}{5}\right)^4\).

We show that every element of \(G'\) can be expressed as a product of 4 generators of \(X'\). For any \(x_1, x_2, x_3, x_4 \in H\) we have
\[(x_1, x_2, x_3, x_4; 0) = b_{x_2} a_{x_4 x_3^{-1} x_2} \bar{b}_{x_2^{-1} x_3} \bar{a}_{x_1},\]
\[(x_1, x_2, x_3, x_4; 1) = \bar{b}_{x_4 x_3^{-1}} c_{x_1 x_3^{-1} x_3} \bar{b}_{x_3} \bar{a}_{x_2},\]
\[(x_1, x_2, x_3, x_4; 2) = a_{x_1} c_{x_3} a_{x_2} c_{x_4},\]
\[(x_1, x_2, x_3, x_4; 3) = a_{x_1 x_2^{-1} x_3^{-1}} c_{x_3} \bar{b}_{x_4} \bar{b}_{x_2},\]
\[(x_1, x_2, x_3, x_4; 4) = a_{x_1} a_{x_2} a_{x_3} a_{x_4},\]
\[(x_1, x_2, x_3, x_4; 5) = c_{x_2 x_4^{-1}} a_{x_3^{-1} x_2} b_{x_2} b_{x_4}^{-1} b_{x_4},\]
\[(x_1, x_2, x_3, x_4; 6) = a_{x_4 x_3^{-1}} a_{x_3} b_{x_2} a_{x_2},\]
\[(x_1, x_2, x_3, x_4; 7) = c_{x_2} b_{x_3} a_{x_1} a_{x_1^{-1} x_4},\]
\[(x_1, x_2, x_3, x_4; 8) = c_{x_2} a_{x_1} c_{x_3 x_4^{-1}} b_{x_4},\]
\[(x_1, x_2, x_3, x_4; 9) = c_{x_2 x_3^{-1}} a_{x_4} b_{x_3} a_{x_1},\]
\[(x_1, x_2, x_3, x_4; 10) = b_{x_3} a_{x_2} a_{x_1} a_{x_3^{-1} x_2},\]
\[(x_1, x_2, x_3, x_4; 11) = c_{x_2} a_{x_1 x_4^{-1}} \bar{b}_{x_4} a_{x_3},\]
\[(x_1, x_2, x_3, x_4; 12) = b_{x_2} a_{x_3^{-1} x_3} b_{x_3} \bar{a}_{x_4^{-1} x_1},\]
\[(x_1, x_2, x_3, x_4; 13) = \bar{a}_{x_2} c_{x_1} a_{x_3} a_{x_2},\]
\[(x_1, x_2, x_3, x_4; 14) = a_{x_1 x_2^{-1}} b_{x_3} a_{x_3^{-1} x_4} b_{x_2},\]
\[(x_1, x_2, x_3, x_4; 15) = c_{x_2 x_3^{-1} x_2} \bar{b}_{x_4} a_{x_1} b_{x_3}^{-1} a_{x_3},\]
\[(x_1, x_2, x_3, x_4; 16) = a_{x_1 x_3^{-1}} a_{x_2} b_{x_4 x_3^{-1}} b_{x_3}.\]

Elements of $G'$ with the last coordinate $y$, where $17 \leq y \leq 31$, can be obtained as inverses of the above ones, therefore the diameter of $C(G', X')$ is at most 4. It is easy to show that, for example, no element of $G'$ with the last coordinate 4 can be obtained as a product of at most 3 elements of $X'$, hence the diameter of $C(G', X')$ cannot be smaller than 4.

(iii) Let $G'' = H^5 \times \mathbb{Z}_5$ and $X'' = \{a_g, \bar{a}_{g'}, b_h, \bar{b}_{h'} \mid \text{for any } g, g', h, h' \in H\}$ where $a_g = (g, e, e, e, e; 1), \bar{a}_{g'} = (e, e, e, g'; -1), b_h = (h, e, e, h, e; -4)$ and $\bar{b}_{h'} = (e, e, h', e, h; 4)$. The Cayley graph $C(G'', X'')$ is of degree $d = |X''| = 4m$ and order $|G''| = 25m^5 = 25(\frac{d}{4})^5$.

In order to prove that the diameter of $C(G'', X'')$ is at most 5, it suffices to show that any element $(x_1, x_2, x_3, x_4, x_5; y)$ of $G''$, where $0 \leq y \leq 12$, can be obtained as a product of 5 generators of $X''$. It can be checked that
\[(x_1, x_2, x_3, x_4, x_5; 0) = a_{x_2} a_{x_3} a_{x_3^{-1}} a_{x_4} b_{x_5},\]
\[(x_1, x_2, x_3, x_4, x_5; 1) = \bar{b}_{x_3} a_{x_2 x_3^{-1}} b_{x_2} a_{x_3^{-1} x_5} a_{x_1},\]
\[(x_1, x_2, x_3, x_4, x_5; 2) = b_{x_2 x_3^{-1}} a_{x_2 x_3^{-1}} a_{x_5} b_{x_2} \bar{b}_{x_3},\]
\[(x_1, x_2, x_3, x_4, x_5; 3) = a_{x_2} b_{x_5} a_{x_3} a_{x_2 x_3^{-1} x_5} b_{x_5}^{-1} x_2,\]
\[(x_1, x_2, x_3, x_4, x_5; 4) = a_{x_2} a_{x_3}^{-1} a_{x_5}^{-1} x_1 a_{x_2^{-1} x_4}.\]
(x₁, x₂, x₃, x₄, x₅; 5) = aₓ₁aₓ₂aₓ₃aₓ₄aₓ₅,
(x₁, x₂, x₃, x₄, x₅; 6) = bₓ₁bₓ₂aₓ₂⁻¹x₄x₁⁻¹aₓ₃⁻¹x₅bₓ₁,
(x₁, x₂, x₃, x₄, x₅; 7) = aₓ₁x₂x₄⁻¹bₓ₄x₂⁻¹aₓ₅bₓ₂aₓ₃,
(x₁, x₂, x₃, x₄, x₅; 8) = aₓ₅x₂⁻¹x₄⁻¹x₁x₃⁻¹bₓ₃x₁x₄⁻¹bₓ₄bₓ₂bₙ₄⁻¹x₅bₓ₁,
(x₁, x₂, x₃, x₄, x₅; 9) = aₓ₅bₓ₂bₓ₁aₓ₁⁻¹x₃aₓ₄⁻¹x₄,
(x₁, x₂, x₃, x₄, x₅; 10) = bₓ₅bₓ₄bₓ₅⁻¹x₃aₓ₄⁻¹x₂aₓ₃⁻¹x₅bₓ₁,
(x₁, x₂, x₃, x₄, x₅; 11) = bₓ₅bₓ₂aₓ₂⁻¹x₄aₓ₃⁻¹x₅aₓ₁,
(x₁, x₂, x₃, x₄, x₅; 12) = aₓ₁x₂x₄⁻¹bₓ₄x₂⁻¹bₓ₅bₓ₂aₓ₅⁻¹x₅bₓ₁.

Hence, C₄,₅ ≥ 25(4⁻⁵) for any d ≥ 8 such that d is a multiple of 4. □

By adding new elements to the generating sets, we get Cayley graphs of any degree d ≥ 10 if k = 4, and d ≥ 8 if k = 3 or 5.

**Theorem 2.**

(i) C₄,₃ ≥ ¾(d − 3)³ for any d ≥ 8.

(ii) C₄,₄ ≥ 32(d−8)⁴ for any d ≥ 10.

(iii) C₄,₅ ≥ 25(d−2)⁵ for any d ≥ 8.

**Proof.** We use the notation of the proof of Theorem 1.

(i) By Theorem 1, C₄,₃ ≥ ¾d³ for d = 4m, where m ≥ 2. Let u be an element of G such that u ∉ X and u ≠ u⁻¹. Let X₁ = X ∪ {v}, X₂ = X ∪ {u, u⁻¹} and X₃ = X ∪ {u, u⁻¹, v} where v = (e, e, e, 6). Then the Cayley graph C(G, X₁) has degree d = |X₁| = 4m + 1, diameter at most k = 3 and order |G| = 12m³ = ¾(d − i)³, where i = 1, 2, 3. Hence C₄,₃ ≥ ¾(d − 3)³ for any d ≥ 8.

(ii) We know that C₄,₄ ≥ 32(4⁻⁸)⁴ for d = 5m, m ≥ 2. Let uᵢ, i = 1, 2, 3, 4, be non-involuntary elements of G' such that uᵢ ∉ X'. Let Xᵢ' = X' ∪ {uᵢ, uᵢ⁻¹, 1, ..., uᵢ⁻¹} where i = 1, 2, 3, 4. The Cayley graph C(G', Xᵢ') has degree d = |Xᵢ'| = 5m + 2i and order |G'| = 32m⁴ = 32(4⁻²i)⁴. It remains to show that C₄,₄ ≥ 32(4⁻⁸)⁴ for d = 11 and 13. However, if m = 2, then G' has an involution which is not in X'. By using this involution, we can obtain Cayley graphs of order 32(10⁻⁸)⁴ = 2⁹ and degree 11 or 13.

(iii) Let m ≥ 2 be even. Then the order of H is even and G'' must contain an involution, say v, other than the identity. Let X'' = X'' ∪ {v} and X''ₙ = X'' ∪ {v, uᵢ, ..., uᵢ, uᵢ⁻¹, ..., uᵢ⁻¹} for i = 1, 2, 3, where uᵢ ≠ uᵢ⁻¹, uᵢ ∉ X''. The graph C(G'', X''ₙ), i = 0, 1, 2, 3, is of degree |X''ₙ| = 4m + 2i + 1, diameter at most k = 5 and order |G''| = 25m⁵ = 25(4⁻²𝑖⁻¹)⁵.
Since for any $m \geq 2$ we can obtain a Cayley graph of degree $4m + 2$ and order $25(d-2)^5$ by adjoining $u_1$ and $u^{-1}_1$ to $X''$, we have $C_{d,5} \geq 25(d-7)^5$ for any $d \geq 8$ as desired. \qed

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**References**


