# RNP AND LEWIS RNP FOR ALL 14 NATURAL TENSOR NORMS OF GROTHENDIECK

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Abstract. We classify all 14 natural tensor norms of Grothendieck according to the possession or the lack of the Radon-Nikodym property (RNP) and the Lewis Radon-Nikodym property.

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*Key words:* Tensor norm, Radon-Nikodym property, Lewis-Radon-Nikodym property.

**1. Introduction.** We will assume acquaintance with Grothendieck's Résumé [6] (cf. Diestel-Fourie-Swart [2] and Defant-Floret [1]).

A tensor norm (or $\otimes$-norm) $\alpha$ is said to have the *Radon-Nikodym property* (RNP) if $\mathcal{L}^\alpha(X, Y^*) = X^* \overset{\alpha}{\otimes} Y^*$ isometrically for all Banach spaces $X$ and $Y^*$, where $Y^*$ has the approximation property and the Radon-Nikodym property (see Maepa [9, p. 234]). We say that $\alpha$ has the *Lewis Radon-Nikodym property* (Lewis RNP) if $\mathcal{L}^\alpha(X, \ell^1) = X^* \overset{\alpha}{\otimes} \ell^1$ isometrically for every Banach space $X$ (see Lewis [7, p. 58]).

A $\otimes$-norm is said to be *natural* if it can be obtained from $\wedge$ by taking the dual, transpose, left or right injective or projective hull finitely often. The 14 natural $\otimes$-norms of Grothendieck appear in a table on page 163 of Diestel-Fourie-Swart [2] and each one in turn will be scrutinized for any possession of the RNP or the Lewis RNP or none of these.

Frequently useful in these deliberations are the following facts established in Maepa [9]:

Theorem 1. (Maepa [9, Theorem 3.1]) *Let $\alpha$ be a $\otimes$-norm. If $\alpha$ has the Lewis RNP, then $\alpha/$ has the RNP.*

Here $\alpha/$ is the tensor norm whose integral operators factor through an $L$-space $L$ into the bidual space with an $\alpha$-integral first factor and a bounded linear second factor.

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THEOREM 2. (Maepa [9, Theorem 3.2]) *If $\alpha$ has the RNP, then so does $\backslash\alpha$.*

Here $\backslash\alpha$ is a tensor norm whose integral operators factor through a $C$-space $C$ into the bidual space with a bounded linear first factor and an $\alpha$-integral second factor.

THEOREM 3. (Maepa [9, Theorem 3.3]) *If $\alpha$ is symmetric and has the RNP, then so does $/\alpha$.*

Here $/\alpha$ is a tensor norm such that for every $L$-space $L$ the composition of an $/\alpha$-integral operator with every operator from $L$ is $\alpha$-integral.

PROPOSITION 4. (Maepa [9, Proposition 3.4])

1. *If $\alpha$ has the Lewis RNP, then so does $\backslash\alpha$.*
2. *If $\alpha$ is symmetric and has the Lewis RNP, then so does $/\alpha$.*

**2. The 14 natural $\otimes$-norms.** The following result is basic to what is to follow:

THEOREM 5. *$\wedge$ has the RNP (and so has the Lewis RNP).*

*Proof.* Let $X$ and $Y$ be Banach spaces such that $Y^*$ has the approximation property and the Radon-Nikodym property. By Diestel-Uhl Jr [3, Theorem VIII.4.6], which also appears implicitly in Grothendieck [5], $\mathcal{L}^\wedge(Y, X^*) = \mathcal{L}_\wedge(Y, X^*)$ isometrically. By Diestel-Uhl Jr [3, Corollary VIII.2.11], $\mathcal{L}^\wedge(Y, X^*) = \mathcal{L}^\wedge(X^{**}, Y^*)$ isometrically. But $\mathcal{L}^\wedge(X, Y^*) \hookrightarrow \mathcal{L}^\wedge(X^{**}, Y^*)$ is an isometric embedding, thanks to Diestel-Uhl Jr [3, Corollaries VIII.2.12 and 13]. Therefore

$$\begin{aligned} \mathcal{L}^\wedge(X, Y^*) &\hookrightarrow \mathcal{L}^\wedge(X^{**}, Y^*) \\ &= \mathcal{L}^\wedge(Y, X^*) \\ &= \mathcal{L}_\wedge(Y, X^*) \\ &= X^* \overset{\wedge}{\otimes} Y^*, \end{aligned}$$

since $Y^*$ has the approximation (cf. Diestel-Fourie-Swart [3, Proposition 1.5.5]) and $\wedge$ is symmetric. It follows that the embedding $\mathcal{L}^\wedge(X, Y^*) \hookrightarrow X^* \overset{\wedge}{\otimes} Y^*$ is isometric. Thus $X^* \overset{\wedge}{\otimes} Y^* = \mathcal{L}^\wedge(X, Y^*)$ isometrically by $Y^*$'s approximation property, and so, the RNP of $\wedge$ has been confirmed. □

THEOREM 6. $\mathrm{C}^* = /\wedge$ *has the RNP.*

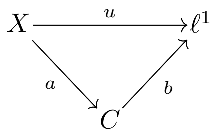
*Proof.* This follows from Theorems 5 and 3 (in this order). □

THEOREM 7. $\gamma^* = \backslash\mathrm{C}^*$ *has the RNP.*

*Proof.* This follows from Theorems 6 and 2. $\square$

THEOREM 8. $\mathrm{C} = \backslash\!\vee$ *has the Lewis RNP.*

*Proof.* For $u : X \longrightarrow \ell^1$ to be $\backslash\!\vee$-integral, $u$ must factorize through a $C$-space



where $a$ and $b$ are bounded linear operators. Now, $b : C \longrightarrow \ell^1$ must be weakly compact because $\ell^1$ is weakly sequentially complete. By Schur's theorem $b$ is compact. So $b$ is the limit of a sequence $(b_n)$ of finite rank bounded linear operators $b_n : C \longrightarrow \ell^1$ – everything in sight has the metric approximation property – with $\|b_n\| \leq \|b\|$. It follows that $u$ is the $\backslash\!\vee$-integral-norm-limit of the sequence $u_n = b_n a$ of finite rank operators. $\square$

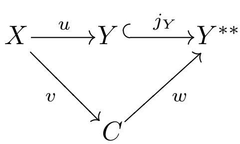
THEOREM 9. $\mathrm{H}^* = \backslash \vee /$ *has the RNP.*

*Proof.* This follows from Theorem 1. $\square$

*Note.* For any tensor norm $\alpha$, all $\alpha$-nuclear operators are compact.

THEOREM 10. $\mathrm{C}$ *fails the RNP.*

*Proof.* Recall that an operator $u : X \longrightarrow Y$ is C-integral with $\|u\|_{\mathrm{C}} \leq 1$ if and only if there are a $C$-space $C$ and bounded linear operators $v : X \longrightarrow C$ and $w : C \longrightarrow Y^{**}$, each of norm $\leq 1$, such that the following diagram commutes:

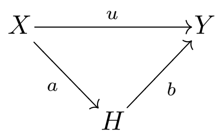


The C-integral norm of $u$ is $\|u\|_{\mathrm{C}} = \inf\{\|v\|\|w\| : \text{the above factorization holds}\}$. Thus to show that C fails the RNP it suffices to find a C-integral operator that is *not* compact. Banach (cf. Grothendieck [4]) showed that there is an isometric quotient operator $q : C[0,1] \twoheadrightarrow \ell^2$ of $C[0,1]$ onto $\ell^2$; namely, the operator that takes $f \in C[0,1]$ to the sequence $(\int_0^1 f(t) r_n(t) dt)_n$ where $(r_n)$ is the sequence of Rademacher functions. $\square$

THEOREM 11. $\mathrm{H} = /\wedge\backslash$ *has the Lewis RNP, hence* H/ *has the RNP.*

*Proof.* That H has the Lewis RNP is a consequence of Proposition 5.6 of [9] and the consequence for H/ is again an application of Theorem 1. □

Recall that $u : X \longrightarrow Y$ is H-integral if $u$ admits a factorization




where $a, b$ are bounded linear operators; the H-integral norm of $u$ is $\|u\|_{\mathrm{H}} = \inf\{\|a\|\|b\| :$ such that the above factorization holds$\}$.

THEOREM 12. H *fails the RNP.*

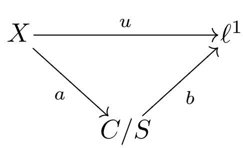
*Proof.* H fails the RNP because $id_{\ell^2}$ is H-integral but not compact. □

THEOREM 13. $\backslash$H *has the Lewis RNP but not the RNP.*

*Proof.* This is a consequence of Theorem 11 and Proposition 4(1). Again, Banach's quotient operator $q : C[0,1] \twoheadrightarrow \ell^2$ is $\backslash$H-integral but still is not compact. □

THEOREM 14. $\gamma$ *has the Lewis RNP but not the RNP.*

*Proof.* If $u : X \longrightarrow \ell^1$ is $\gamma$-integral, then $u$ factorizes as follows:

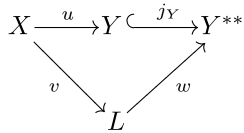



where $a, b$ are bounded linear operators and $C/S$ is a quotient of a $C$-space. Let $q : C \twoheadrightarrow C/S$ be the isometric quotient operator. Then $b(B_{C/S}) = bq(B_C)$. But $wq : C \longrightarrow \ell^1$ is compact, so $bq(B_C)$ is relatively compact, that is, $b(B_{C/S})$ is relatively compact and $b : C/S \longrightarrow \ell^1$ is a compact operator. Again, $\ell^1$ has the metric approximation property, so there is a sequence $(b_n)$ of finite rank operators $b_n : C/S \longrightarrow \ell^1$ with $\|b_n\| \leq \|b\|$ so $b =$ operator $- \lim_n b_n$. If $u_n = b_n a$, then $u_n$ is a sequence of finite rank operators $u_n : X \longrightarrow \ell^1$ so that $\|u - u_n\|_\gamma \to 0$.

Now, $\gamma$ fails the RNP because the Banach quotient operator is $\gamma$-integral but not compact. □

THEOREM 15. $\mathrm{L} = \vee/$ *fails the Lewis RNP.*

*Proof.* Recall that $u : X \longrightarrow Y$ is L-integral if $u$ admits a factorization




where $v, w$ are bounded linear operators; the L-integral norm of $u$ is $\|u\|_{\mathrm{L}} = \inf\{\|v\|\|w\| : \text{such that the above factorization holds}\}$. There is a quotient operator $q : L^1[0,1] \twoheadrightarrow \ell^1$ of $L^1[0,1]$ onto $\ell^1$; $Q$ is L-integral but not compact. □

THEOREM 16. $\lambda = \mathrm{L}\backslash$ *fails the Lewis RNP.*

*Proof.* Recall that $u : X \longrightarrow Y$ is $\lambda$-integral with $\|u\|_\lambda \leq 1$ if and only if for every $C$-space $C$ and every operator $i : Y \longrightarrow C$ with $\|i\| \leq 1$, the operator $iu : X \longrightarrow C$ is L-integral with $\|iu\|_{\mathrm{L}} \leq 1$. The identity operator $id_{\ell^1}$ on $\ell^1$ is $\lambda$-integral but not compact. □

THEOREM 17. $\vee$ *fails the Lewis RNP.*

*Proof.* This is a consequence of Proposition 4.5 of [9]. □

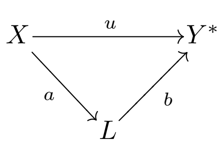
THEOREM 18. $\mathrm{L}^* = \wedge\backslash$ *fails the RNP.*

*Proof.* After all, Grothendieck's Fundamental Inequality ensures that every operator from $\ell^1$ to $\ell^2$ is $\wedge\backslash$-integral; $\ell^1 \hookrightarrow \ell^2$ is not compact and so is not $\wedge\backslash$-nuclear. □

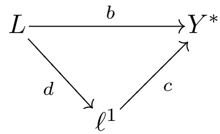
OPEN PROBLEM. We do not know if $\mathrm{L}^* = \wedge\backslash$ has the Lewis RNP.

THEOREM 19. $\lambda^*$ *has the RNP if and only if it has the Lewis RNP.*

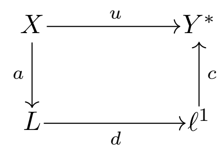
*Proof.* We need only show that if $\lambda^*$ has the Lewis RNP, then it has the RNP. This in mind, suppose that $Y^*$ has the RNP and the approximation property and let $u : X \longrightarrow Y^*$ be $\lambda^*$-integral. Then $u$ factorizes as follows (cf. Diestel-Fourie-Swart [2, Corollary 2.4.14 (b)]):



where $a$ is $\wedge\backslash$-integral (aka, absolutely summing), with $\|u\|_{\lambda^*} = \inf\{\|a\|_{\wedge\backslash}\|b\| :$ such that the above factorization holds$\}$. But the Lewis-Stegall Theorem (Diestel-Uhl Jr [3, Theorem III.1.8] and Lewis-Stegall [8]) ensures that $b : L \longrightarrow Y^*$ factorizes as follows:



where $\|b\| = \inf\{\|c\|\|d\| :$ such that the above factorization holds$\}$. Of course, we now have the following pictorial description of $u$:



Since $\lambda^*$ has the Lewis RNP, $da : X \longrightarrow \ell^1$ is $\lambda^*$-nuclear and so $cda = u$ is too. The $\lambda^*$-nuclear norm of $u$ is easily seen to be $\leq \|a\|_{\wedge\backslash}\|d\|\|c\|$. □

Next, we close with the

Open problem. Does $\lambda^*$ have the Lewis RNP? Alternatively, is it so that if $u : X \longrightarrow L^1$ is absolutely summing and $v : L^1 \longrightarrow \ell^1$ is a bounded linear operator, then $vu$ is quasi-nuclear?

For any normed spaces $X$ and $Y$, a bounded linear operator $u : X \longrightarrow Y$ is said to be *quasi-nuclear* if there exists a sequence $(x_i^*)$ in $X^*$ with $\sum_{i=1}^{\infty} \|x_i^*\| < \infty$ such that for every $x \in X$ it holds that $\|ux\| \leq \sum_{i=1}^{\infty} | < x, x_i^* > |$. Quasi-nuclear operators are treated in some detail in Pietsch ([10] and [11]). It is worth noting that $vu$ would be nuclear, and so quasi-nuclear (cf. [10, Satz 2.6] and [11, Proposition 3.2.5]), if either $v$ were also absolutely summing [11, Theorem 3.3.5] or there were a normed space $G$ containing $\ell^1$ such that $vu$ is nuclear as a mapping from $X$ into $G$ (cf. [10, Theorem 2.1] and [11, Proposition 3.2.7]).

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