$\mathcal{A}$-transvections and Witt's theorem in symplectic $\mathcal{A}$-modules

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Abstract. Building on prior joint work by Mallios and Ntumba, we study *transvections* (J. Dieudonné), a theme already important from the classical theory, in the realm of *Abstract Geometric Algebra*, referring herewith to *symplectic $\mathcal{A}$-modules.* A characterization of *$\mathcal{A}$-transvections*, in terms of *$\mathcal{A}$-hyperplanes* (Theorem 1.4), is given together with the associated *matrix* definition (Corollary 1.5). By taking the domain of coefficients $\mathcal{A}$ to be a PID-*algebra sheaf*, we also consider the analogue of a form of the classical *Witt's extension theorem*, concerning $\mathcal{A}$-symplectomorphisms defined on appropriate *Lagrangian sub-$\mathcal{A}$-modules* (Theorem 2.3 and 2.4).

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Introduction

Here is a further attempt at investigating classical notions/results such as *transvections*, *Witt's theorem for symplectic vector spaces*, the *characterization of singular symplectomorphisms of symplectic vector spaces of finite* (*even*) *dimension* within the context of Abstract Differential Geometry (à la Mallios), [10, 11]. This endeavor, as already signalled in [14], is for the purpose of rewriting and/or recapturing a great deal of classical symplectic (differential) geometry without any use (*at all*!) of any notion of "*differentiability*" (differentiability is here understood in the sense of the standard *differential geometry* of C^∞-manifolds).

Now, we take the opportunity to review succinctly the basic notions of Abstract Geometric Algebra which we are concerned with in this paper. Most of the concepts in this paper are defined on the basis of the classical ones; see to this effect, Artin [2], Berndt [4], Crumeyrolle [6], Deheuvels [7], Lang [9]. Our main reference is Mallios [10].

We also recall some notions, which may be found in our recent papers such as [12], [13], [14], and [15]. Let $\mathcal{F}$ and $\mathcal{E}$ be $\mathcal{A}$-modules and $\phi : \mathcal{F} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ an $\mathcal{A}$-bilinear morphism. Then, we say that the triple $((\mathcal{F}, \mathcal{E}; \phi); \mathcal{A}) \equiv (\mathcal{F}, \mathcal{E}; \phi) \equiv (\mathcal{F}, \mathcal{E}; \mathcal{A})$ forms a *pairing of* $\mathcal{A}$*-modules* or an $\mathcal{A}$*-pairing.* The sub-$\mathcal{A}$-module $\mathcal{F}^\perp$ of $\mathcal{E}$ such that, for every open subset U of X, $\mathcal{F}^\perp(U)$ consists of all $r \in \mathcal{E}(U)$ with $\phi_V(\mathcal{F}(V), r|_V) = 0$ for any open $V \subseteq U$, is called the *right kernel* of the pairing $(\mathcal{F}, \mathcal{E}; \mathcal{A})$. In a similar way, one defines the *left kernel* of $(\mathcal{F}, \mathcal{E}; \mathcal{A})$ to be the sub-$\mathcal{A}$-module $\mathcal{E}^\perp$ of $\mathcal{F}$ such that, for any open subset U of X, $\mathcal{E}^\perp(U)$ is the set of all (local) sections $r \in \mathcal{F}(U)$ such that $\phi_V(r|_V, \mathcal{E}(V)) = 0$ for every open $V \subseteq U$.

If $(\mathcal{E}, \phi)$ is a self $\mathcal{A}$-pairing with ϕ symmetric or skew-symmetric, the kernel $\mathcal{E}^\perp$ is called the *radical sheaf* (or *sheaf of* $\mathcal{A}$*-radicals,* or simply $\mathcal{A}$*-radical*) of $\mathcal{E}$. If $\mathcal{F}$ is a sub-$\mathcal{A}$-module of $\mathcal{E}$, the radical of $\mathcal{F}$ consists of those sections of $\mathcal{F}^\perp$ that are also sections of $\mathcal{F}$. In other words, rad $\mathcal{F} = \mathcal{F} \cap \mathcal{F}^\perp$. In general, if $(\mathcal{F}, \mathcal{E}; \mathcal{A})$ is a pairing of free $\mathcal{A}$-modules, then rad $\mathcal{E} := \mathcal{E} \cap \mathcal{E}^\perp$, and similarly rad $\mathcal{F} := \mathcal{F} \cap \mathcal{F}^\top$. An $\mathcal{A}$-module $\mathcal{E}$ such that rad $\mathcal{E} \neq 0$ (resp. rad $\mathcal{E} = 0$) is called *isotropic* (resp. *non-isotropic*); $\mathcal{E}$ is *totally isotropic* if ϕ is identically zero. For any open $U \subseteq X$, a non-zero section $r \in \mathcal{E}(U)$ is called *isotropic* if $\phi_U(r, r) = 0$.

N.B. We assume throughout the paper, unless otherwise mentioned, that the pair $(X, \mathcal{A})$ is an *algebraized space*, where $\mathcal{A}$ is a *unital* $\mathbb{C}$*-algebra sheaf* such that *every nowhere-zero section of* $\mathcal{A}$ is *invertible.* (Consider for example sheaves of continuous, smooth and holomorphic functions.)

1. Symplectic $\mathcal{A}$-transvections

By analogy to the classical notion of hyperplane, we call $\mathcal{A}$*-hyperplanes* of a free $\mathcal{A}$-module $\mathcal{E}$ free sub-$\mathcal{A}$-modules of $\mathcal{E}$ of corank 1 (cf. [12]).

We notice that if $\mathcal{E}$ is a free $\mathcal{A}$-module and $\mathcal{F}$ an $\mathcal{A}$-hyperplane of $\mathcal{E}$, then every $\mathcal{A}$-endomorphism ϕ of $\mathcal{E}$ that leaves $\mathcal{F}$ stable induces on the *line* $\mathcal{A}$*-module* $\mathcal{E}/\mathcal{F}$ an $\mathcal{A}$*-homothecy,* which we denote by $\widetilde{\phi}$. More explicitly, if U is open in X and s a section of $\mathcal{E}/\mathcal{F}$ over U, then

$$\widetilde{\phi}(s) \equiv \widetilde{\phi}_U(s) = a_U s \equiv as$$

for some $a_U \equiv a \in \mathcal{A}(U)$. The coefficient sections a_U are such that $a_V = a_U|_V$ whenever V is contained in U. The global section $a_X \equiv a$ is called the *ratio* of the $\mathcal{A}$-homothecy $\widetilde{\phi}$.

Lemma 1.1. *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, and $\mathcal{F}$ a proper free sub-$\mathcal{A}$-module of $\mathcal{E}$. Then, the following assertions are equivalent.*

(1) *$\mathcal{F}$ is an $\mathcal{A}$-hyperplane of $\mathcal{E}$.*
(2) *For every (local) section $s \in \mathcal{E}(U)$ such that $s|_V \notin \mathcal{F}(V)$ for every open $V \subseteq U$,*

$$\mathcal{E}(U) = \mathcal{A}(U)s \oplus \mathcal{F}(U).$$

(3) *For every open* $U \subseteq X$*, there exists a section* $s \in \mathcal{E}(U)$ *with* $s|_V \notin \mathcal{F}(V)$*, where* V *is any open subset contained in* U*, such that*

$$\mathcal{E}(U) = \mathcal{A}(U)s \oplus \mathcal{F}(U).$$

(4) *The free sub-*$\mathcal{A}$*-module* $\mathcal{F}$ *is a maximal sub-*$\mathcal{A}$*-module in the inclusion-ordered set of proper free sub-*$\mathcal{A}$*-modules of* $\mathcal{E}$*.*

Proof. (1) $\Rightarrow$ (2): For every open $U \subseteq X$ and section $s \in \mathcal{E}(U)$ such that $s|_V \notin \mathcal{F}(V)$ for any open $V \subseteq U$, it is clear that $\mathcal{A}(U)s + \mathcal{F}(U)$ is a direct sum. On the other hand, the equivalence class containing s is a nowhere-zero section of $\mathcal{E}/\mathcal{F}$; it spans $\mathcal{E}(U)/\mathcal{F}(U)$ since $\mathcal{E}(U)/\mathcal{F}(U)$ has rank 1. It thus follows that $\mathcal{E}(U) = \mathcal{A}(U)s + \mathcal{F}(U)$.

(2) $\Rightarrow$ (3): Evident.

(3) $\Rightarrow$ (1): Since $\operatorname{rank}(\mathcal{E}/\mathcal{F})(U) = \operatorname{rank}(\mathcal{E}(U)/\mathcal{F}(U)) = \operatorname{rank}(\mathcal{A}(U)s) = 1$ for every open $U \subseteq X$ and $s \in \mathcal{E}(U)$ with $s|_V \notin \mathcal{F}(V)$, where V is any open subset contained in U.

(2) $\Rightarrow$ (4): Let $\mathcal{F}'$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ containing $\mathcal{F}$ and such that $\operatorname{rank} \mathcal{F}' > \operatorname{rank} \mathcal{F}$. For every open U there exists a section $s \in \mathcal{F}'(U)$ such that $s|_V \notin \mathcal{F}(V)$ for every open $V \subseteq U$. By (2), for every open $U \subseteq X$, $\mathcal{E}(U) = \mathcal{A}(U)s \oplus \mathcal{F}(U)$; but $\mathcal{A}(U)s \oplus \mathcal{F}(U)$ is contained in $\mathcal{F}'(U)$, therefore $\mathcal{F}' = \mathcal{E}$.

(4) $\Rightarrow$ (2): Let U be an open set in X. There exists a section $s \in \mathcal{E}(U)$ with $s|_V \notin \mathcal{F}(V)$ for any open $V \subseteq U$; then $\mathcal{A}(U)s \oplus \mathcal{F}(U)$ contains strictly $\mathcal{F}(U)$, thus $\mathcal{A}(U)s \oplus \mathcal{F}(U) = \mathcal{E}(U)$, since $\mathcal{F}$ is maximal. □

Lemma 1.1 will be referred to in the proof of Theorem 1.4, which characterizes the kind of $\mathcal{A}$-*transvections* dealt with in the course of this paper. For the classical notion of transvection, see [2], [5, p. 152, Proposition 12.9], [6], [7, p. 419 ff], [8], [9, p. 542- 544]. To this end, we require some preparations.

Definition 1.2. (Mallios) Let $\mathcal{E}$ be an $\mathcal{A}$-module. An element $\phi \in \mathcal{E}nd\ \mathcal{E} \equiv \mathcal{E}nd_{\mathcal{A}}\mathcal{E} := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$ is called a **homothecy** of **ratio** $\alpha \in \mathcal{A}$ if

$$\phi = \alpha \cdot \mathrm{I}.$$

It goes without saying that I, in Definition 1.2 above, stands for the identity element of the $\mathcal{A}$-*algebra sheaf* $\mathcal{E}nd\ \mathcal{E}$. That is,

$$\mathrm{I} \equiv \mathrm{I}_{\mathcal{E}} := \mathrm{Id}_{\mathcal{E}} \in \mathcal{E}nd\ \mathcal{E}.$$

We notice that

$$\mathcal{A} \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \equiv \mathcal{E}nd\ \mathcal{A},$$

hence,

$$\mathcal{A}^{\bullet} \simeq (\mathcal{E}nd\ \mathcal{A})^{\bullet} \equiv \mathcal{A}ut\ \mathcal{A}.$$

Therefore, a *homothecy of* $\mathcal{E}$ *of ratio* $\alpha \in \mathcal{A}^{\bullet}$ *is* an $\mathcal{A}$-*isomorphism of* $\mathcal{E}$, viz. an *element of* $\mathcal{A}ut\ \mathcal{E}$.

Now, suppose that $\mathcal{F}$ is a sub-$\mathcal{A}$-module of an $\mathcal{A}$-module $\mathcal{E}$ such that

$$\mathcal{E}/\mathcal{F} \simeq \mathcal{A}. \tag{1.1}$$

($\mathcal{F}$ is an $\mathcal{A}$-*hyperplane* of $\mathcal{E}$). Moreover, let $\phi \in \mathcal{E}nd\ \mathcal{E}$ such that

$$\phi(\mathcal{F}) \subseteq \mathcal{F}; \tag{1.2}$$

then, ϕ gives rise to an element, say $\widetilde{\phi}$, of $\mathcal{E}nd(\mathcal{E}/\mathcal{F})$; viz.

$$\widetilde{\phi} \in \mathcal{E}nd(\mathcal{E}/\mathcal{F}),$$

such that, in view of (1.2),

$$\widetilde{\phi} \circ q = q \circ \phi,$$

with $q : \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{F}$, the canonical $\mathcal{A}$-epimorphism. However, due to (1.1), one gets

$$\widetilde{\phi} \in \mathcal{E}nd(\mathcal{E}/\mathcal{F}) \simeq \mathcal{E}nd\ \mathcal{A} \simeq \mathcal{A}, \tag{1.3}$$

viz. one obtains

$$\widetilde{\phi} = \alpha \in \mathcal{A} \simeq \mathcal{E}nd\ \mathcal{A},$$

or even

$$\widetilde{\phi} = \alpha \equiv \alpha \cdot \mathrm{I},$$

with α the *ratio* of $\widetilde{\phi}$. Thus, ϕ *induces a homothecy of* $\mathcal{E}/\mathcal{F}(\simeq \mathcal{A})$ *of ratio* α. In particular, if $\mathcal{F}$ is a free sub-$\mathcal{A}$-module of a free $\mathcal{A}$-module $\mathcal{E}$, by the *rank* (*dimension*) *formula* (cf. [17]), viz.

$$\mathrm{rank}(\mathrm{Im}\widetilde{\phi}) + \mathrm{rank}(\ker \widetilde{\phi}) = \mathrm{rank}(\mathcal{E}/\mathcal{F}) = \mathrm{rank}\ \mathcal{A} = 1,$$

one sees that α is either zero or nowhere zero.

Definition 1.3. (Mallios) Let $\mathcal{E}$ be an $\mathcal{A}$-module. An element $\phi \in \mathcal{E}nd\ \mathcal{E}$ is called an **$\mathcal{A}$-transvection** if the following conditions hold true:

(*i*) There exists an $\mathcal{H} \subseteq \mathcal{E}$, sub-$\mathcal{A}$-module, with

$$\mathcal{E}/\mathcal{H} \simeq \mathcal{A}.$$

(*ii*) $\phi|_{\mathcal{H}} = \mathrm{I}$.
(*iii*) $\mathrm{im}(\phi - \mathrm{I}) \subseteq \mathcal{H}$.

According to Definition 1.3, it is clear that an element $\phi \in \mathcal{E}nd\,\mathcal{E}$, where $\mathcal{E}$ is an $\mathcal{A}$-module, *is an* $\mathcal{A}$*-transvection if and only if it is locally so.*

In the light of [2, p. 160, Definition 4.1], Definition 1.3 can be rephrased as follows. An $\mathcal{A}$-*transvection* (with respect to an $\mathcal{A}$-*hyperplane* $\mathcal{H}$, par abus de language) of an $\mathcal{A}$-module $\mathcal{E}$ is an $\mathcal{A}$-endomorphism of $\mathcal{E}$, *which keeps every section of* $\mathcal{H}$ *fixed and moves any other section* $s \in \mathcal{E}(U)$ *by some section of* $\mathcal{H}(U)$, namely $\phi(s) - s \in \mathcal{H}(U)$.

Theorem 1.4. *Let* $\mathcal{E}$ *be a free* $\mathcal{A}$*-module,* $\mathcal{H}$ *an* $\mathcal{A}$*-hyperplane of* $\mathcal{E}$*,* ϕ *an* $\mathcal{A}$*-endomorphism of* $\mathcal{E}$ *that fixes every section of* $\mathcal{H}$*, and* $\widetilde{\phi}$ *the* $\mathcal{A}$*-homothecy, of ratio* α*, induced by* ϕ *on the line* $\mathcal{A}$*-module* $\mathcal{E}/\mathcal{H}$*. Then,*

(1) *If* α *is nowhere* 1*, there exists a unique line* $\mathcal{A}$*-module* $\mathcal{L} \subseteq \mathcal{E}$ *such that* $\mathcal{E} = \mathcal{H} \oplus \mathcal{L}$ *and* $\mathcal{L}$ *is stable by* ϕ*, i.e.* $\phi(\mathcal{L}) \cong \mathcal{L}$.

(2) *If* $\alpha = 1$, *then for every* $\mathcal{A}$*-morphism* $\theta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ *with* $\ker\theta \cong \mathcal{H}$, *there exists, for every open subset* $U \subseteq X$, *a unique section* $r \in \mathcal{H}(U)$ *such that*

$$\phi(s) = s + \theta(s)r \tag{1.4}$$

for every $s \in \mathcal{E}(U)$.

Proof. **Assertion** (1). *Uniqueness.* Let $\mathcal{L}$ be a line $\mathcal{A}$-module satisfying the hypotheses of the assertion, and s a nowhere-zero global section of $\mathcal{L}$ (such a section s does exist because $\mathcal{L} \cong \mathcal{A}$ and $\mathcal{A}$ is unital). Therefore, there exists $b \in \mathcal{A}(X)$ such that $\phi(s) = \beta s$. Next, assume that q is the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto $\mathcal{E}/\mathcal{H}$. It is clear that $\widetilde{\phi}_X(q_X(s)) = \beta q_X(s) \equiv \beta q(s)$; thus $\widetilde{\phi}_X$ is a homothecy of ratio $\alpha = b$, hence, by hypothesis, β is nowhere 1. Now, let u be an element of $\mathcal{E}(X)$ such that $u \notin \mathcal{H}(X)$; then there exists a non-zero $\lambda \in \mathcal{A}(X)$ and an element $t \in \mathcal{H}(X)$ such that

$$u = \lambda s + t.$$

It follows that

$$\phi(u) = \lambda\beta s + t.$$

Of course, $\phi(u)$ and u are colinear if and only if $t = 0$. Thus, we have proved that every section $u \in \mathcal{E}(X)$ which is colinear with its image $\phi(u)$ belongs to $\mathcal{L}(X)$. A similar argument holds should we consider the decomposition $\mathcal{E}(U) = \mathcal{H}(U) \oplus \mathcal{L}(U)$, where U is any other open subset U of X. Hence, $\mathcal{L}$ is the unique complement of $\mathcal{H}$ in $\mathcal{E}$, up to $\mathcal{A}$-isomorphism, and stable by ϕ.

Existence. Since α is nowhere 1 on X, there exists a nowhere-zero section $s \in \mathcal{E}(X)$ such that

$$\widetilde{\phi}_U(q_U(s|_U)) := \widetilde{\phi}_U(q_U(s_U)) \neq q_U(s_U) =: q_U(s|_U)$$

for any open $U \subseteq X$. As $\widetilde{\phi}\circ q = q\circ\phi$, it follows that $r_U := \phi_U(s_U) - s_U$ does not belong to $\mathcal{H}(U)$, for any open $U \subseteq X$. The line $\mathcal{A}$-module $\mathcal{L} := [r_U]_{X\supseteq U,\ open}$ clearly complements $\mathcal{H}$. It remains to show that $\mathcal{L}$ is stable by ϕ: To this end, we first observe that every s_U does not belong to the corresponding $\mathcal{H}(U)$, and, by Lemma 1.1, $\mathcal{E}(U) \cong \mathcal{A}(U)s_U \oplus \mathcal{H}(U)$. So, since $r_U \notin \mathcal{H}(U)$ for every open $U \subseteq X$, there exists for every r_U sections $\alpha_U \in \mathcal{A}(U)$ and $t_U \in \mathcal{H}(U)$ such that

$$r_U = \alpha_U s_U + t_U. \tag{1.5}$$

We deduce from (1.5) that

$$\phi_U(r_U) = (\alpha_U + 1)r_U,$$

and the proof is complete.

Assertion 2. *Uniqueness.* Let us fix an open set U in X. The uniqueness of r such that (1.4) holds is immediate, as $\theta_U(s) \equiv \theta(s) \neq 0$ for some $s \in \mathcal{E}(U)$. Relation (1.4) also shows that if $s \in \mathcal{E}(U)$ and $\theta(s)$ is nowhere zero, then necessarily

$$r = (\theta(s))^{-1}(\phi(s) - s).$$

Existence. Suppose given a section $s_0 \in \mathcal{E}(U)$ such that $s_0|_V \notin \mathcal{H}(V)$ for any open $V \subseteq U$. Let us consider the section $r = (\theta(s_0))^{-1}(\phi(s_0) - s_0)$. Clearly, $r \in \mathcal{H}(U)$; indeed

$$(q \circ \phi)(s_0) - q(s_0) = (\widetilde{\phi} \circ q)(s_0) - q(s_0) = 0.$$

The two $\mathcal{A}(U)$-morphisms $s \longmapsto \phi(s)$ and $s \longmapsto s + \theta(s)r$ are equal, since they take on, on one hand, the same value at s_0, and, on the other hand, the same value at every $s \in \mathcal{H}(U)$. □

In the course of this paper, we are interested in $\mathcal{A}$-transvections of free $\mathcal{A}$-modules of finite rank $\mathcal{E}$ such that locally for Condition (iii) of Definition 1.3, one has one and only one section $s_0^U \in \mathcal{H}(U)$ such that

$$\phi_U(s) := s + \theta_U(s)s_0^U,$$

for every $s \in \mathcal{E}(U)$, and where $\theta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ is such that $\ker\theta$ *is $\mathcal{A}$-isomorphic to $\mathcal{H}$*. Such $\mathcal{A}$-transvections shall be called **$\mathcal{A}$-transvections** of **classical type**.

So, assume $\mathcal{E}$ is free of rank n and $(e_1, \ldots, e_n)$ a basis for $\mathcal{E}(U)$, where U is a fixed open subset of X, such that $(e_1, \ldots, e_{n-1})$ is a basis for $\mathcal{H}(U)$. The matrix representing ϕ_U is given by

$$(\phi_U^{ij}) := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \lambda s_0^1 \\ 0 & 1 & \cdots & 0 & 0 & \lambda s_0^2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \lambda s_0^{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \in M_n(\mathcal{A}(U)) \simeq M_n(\mathcal{A})(U),$$

where $\lambda := \theta_U(e_n) \in \mathcal{A}(U)$ and $s_0^U \equiv s_0 := s_0^1 e_1 + \cdots + s_0^{n-1} e_{n-1}$. If we consider the determinant $\mathcal{A}$-morphism $\partial et : M_n(\mathcal{A}) \longrightarrow \mathcal{A}$ (cf. [10, p. 294]), it follows that

$$\overline{\partial et_U}(\phi_U^{ij}) \equiv \partial et_U(\phi_U^{ij}) =: \det{}_U(\phi_U^{ij}) = 1$$

(we have assumed that $\overline{\partial et} : \Gamma(M_n(\mathcal{A})) \longrightarrow \Gamma(\mathcal{A})$ is the $\Gamma(\mathcal{A})$-morphism of complete presheaves of sections of sheaves $M_n(\mathcal{A})$ and $\mathcal{A}$ that corresponds to ∂et); hence, $\mathcal{A}$-transvections are invertible.

Keeping with the notations above, the inverse of an $\mathcal{A}$-transvection ϕ is the $\mathcal{A}$-transvection ϕ^{-1} such that

$$\phi_U^{-1}(s) := s - \theta_U(s)s_0^U$$

for every open $U \subseteq X$ and section $s \in \mathcal{E}(U)$.

Fix an open subset U of X and let $\phi \in \mathcal{E}nd_{\mathcal{A}}\mathcal{E}$ be an $\mathcal{A}$-transvection. If $s_0 \equiv s_0^U \in \mathcal{E}(U)$ is nowhere zero, we may assume it to be one of the basis elements of $\mathcal{H}(U)$; therefore the matrix of ϕ_U will just be the *identity natrix with one non-zero element off the main diagonal.* Conversely, any $\mathcal{A}$-endomorphism ϕ of a free $\mathcal{A}$-module of finite rank $\mathcal{E}$ such that, for every open $U \subseteq X$, the matrix representing ϕ_U with respect to some basis is the identity matrix with one non-zero entry off the main diagonal is an $\mathcal{A}$-transvection.

We formalize the above argument in the following

Corollary 1.5. *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of rank n, $\mathcal{H}$ an $\mathcal{A}$-hyperplane of $\mathcal{E}$, and ϕ an $\mathcal{A}$-transvection of classical type. Then, for every open $U \subseteq X$, there exists a basis of $\mathcal{E}(U)$ such that the matrix (ϕ_U) of ϕ_U in this basis is of the form*

$$(\phi_U) = I_n + \lambda M^{ij}, \quad i \neq j, \tag{1.6}$$

where $\lambda \in \mathcal{A}(U)$ and $(M^{ij})_{1\leq i,j\leq n}$ represents a canonical basis of $M_n(\mathcal{A}(U))$. Matrix (1.6) *is called an* $\mathcal{A}(U)$**-transvection matrix.**

For the need of Proposition 1.8 below, we make the following important observation (cf. Lemma 1.10), concerning symplectic $\mathcal{A}$-modules of finite rank, the proof of which is based on the concept:

Definition 1.6. Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, ϕ and ϕ' non-degenerate $\mathcal{A}$-bilinear forms on $\mathcal{E}$ and $\mathcal{E}'$, respectively. Moreover, let ψ be an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{E}'$. An $\mathcal{A}$-morphism $\theta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}', \mathcal{E})$ such that

$$\phi' \circ (\psi, \mathrm{Id}) = \phi \circ (\mathrm{Id}, \theta). \tag{1.7}$$

is called an **adjoint** of ψ, and is denoted ψ^*.

Section-wise, Equation (1.7) means that for every open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$, $t \in \mathcal{E}'(U)$,

$$\phi'(\psi(s), t) \equiv \phi'_U(\psi_U(s), t) = \phi_U(s, \theta_U(t)) \equiv \phi(s, \theta(t)).$$

Keeping with the notations of Definition 1.6 above, we have

Proposition 1.7. *θ is unique whenever it exists.*

Proof. Suppose that θ_1 and θ_2 are adjoint of ψ, so given any open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$, $t \in \mathcal{E}'(U)$,

$$\phi^L_U(\theta_{1,U}(t))(s) = \phi^L_U(\theta_{2,U}(t))(s),$$

where $\phi^L \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}'^*)$ is given by

$$\phi^L_U(u)(v) \equiv (\phi^L)_U(u)(v) := \phi_V(u|_V, v)$$

for sections $u \in \mathcal{E}(U)$ and $v \in \mathcal{E}'(V)$. Since s is arbitrary in $\mathcal{E}(U)$,

$$\phi^L_U(\theta_{1,U}(t)) = \phi^L_U(\theta_{2,U}(t)).$$

But ϕ^L is injective, therefore

$$\theta_{1,U} = \theta_{2,U}.$$

Finally, since U is arbitrary, $\theta_1 = \theta_2$. □

Let us now enquire on the existence of the adjoint of an $\mathcal{A}$-morphism $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$, where $\mathcal{E}$ and $\mathcal{E}'$ are $\mathcal{A}$-modules equipped with $\mathcal{A}$-bilinear forms ϕ and ϕ', respectively.

Proposition 1.8. *Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, equipped with non-degenerate $\mathcal{A}$-bilinear forms ϕ and ϕ', respectively. If $\mathcal{E}$ is* free *and* of finite rank, *then for every $\mathcal{A}$-morphism $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$ there exists an adjoint, denoted ψ^*, which is given by*

$$\psi^* = (\phi^L)^{-1} \circ {}^t\psi \circ {\phi'}^L,$$

where ${}^t\psi : (\mathcal{E}')^* \longrightarrow \mathcal{E}^*$ is the transpose of ψ.

Proof. Let U be an open subset of X, $s \in \mathcal{E}(U)$ and $t \in \mathcal{E}'(U)$. Using the right insertion $\mathcal{A}$-morphism ${\phi'}^L$, one has

$$\phi'_U(\psi_U(s), t) = {\phi'}^L_U(t)(\psi_U(s)) = ({}^t\psi)_U({\phi'}^L_U(t))(s). \tag{1.8}$$

Since $\mathcal{E}$ has finite rank and ϕ is non-degenerate, ϕ^L is an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{E}^*$; so ${}^t\psi \circ {\phi'}^L$ may be written

$${}^t\psi \circ {\phi'}^L = \phi^L \circ ((\phi^L)^{-1} \circ {}^t\psi \circ {\phi'}^L).$$

It follows from (1.8) that

$$\begin{aligned} \phi'_U(\psi_U(s), t) &= [\phi^L_U(((\phi^L_U)^{-1} \circ ({}^t\psi)_U \circ {\phi'}^L_U)(t))](s) \\ &= \phi_U(s, ((\phi^L_U)^{-1} \circ ({}^t\psi)_U \circ {\phi'}^L_U)(t)), \end{aligned}$$

which ends the proof. □

Corollary 1.9. *Adjoints commute with restrictions.*

Proof. Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, ϕ and ϕ' non-degenerate $\mathcal{A}$-bilinear forms on $\mathcal{E}$ and $\mathcal{E}'$, respectively. Assume that $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$. Let U be an open subset of X, and $s,\ t$ be sections of $\mathcal{E}$ and $\mathcal{E}'$ on U, respectively. By Definition 1.6, we have

$$\phi'_U(\psi_U(s), t) = \phi_U(s, (\psi^*)_U(t)).$$

On the other hand, since ϕ_U and ϕ'_U are non-degenerate and

$$\psi_U \in \mathrm{H}om_{\mathcal{A}(U)}(\mathcal{E}(U), \mathcal{E}'(U)),$$

then by virtue of [5, pp. 385, 386], we have

$$\phi'_U(\psi_U(s), t) = \phi_U(s, (\psi_U)^*(t)).$$

On account of uniqueness of adjoints, we have

$$(\psi^*)_U = (\psi_U)^*,$$

as desired. □

Lemma 1.10. *Let $(\mathcal{E}, \omega)$ be a symplectic $\mathcal{A}$-module of finite rank, and f an $\mathcal{A}$-endomorphism of $\mathcal{E}$. Then, if f satisfies two of the three following conditions, it satisfies all of them three, and* $\mathrm{Id} + f$ *is called a* **singular $\mathcal{A}$-symplectomorphism of** $(\mathcal{E}, \omega)$*:*

1. $\mathrm{Id} + f$ *is an $\mathcal{A}$-automorphism of $\mathcal{E}$;*
2. *f is ω-skewsymmetric, i.e., for any open $U \subseteq X$ and sections $s, t \in \mathcal{E}(U)$,*

$$\omega_U(f_U(s), t) + \omega_U(s, f_U(t)) = 0;$$

(3) *Im* $f \equiv f(\mathcal{E})$ *is totally isotropic, i.e.,*

$$\omega|_{f(\mathcal{E})} = 0.$$

Proof. Using the equality

$$\omega_U(s + f_U(s), t + f_U(t)) - \omega_U(s,t) = \omega_U((f_U + f_U^*)(s), t) + \omega_U(f_U(s), f_U(t)),$$

where U is any open subset of X, s and t sections of $\mathcal{E}$ over U, one easily checks the implications: (1), (2) $\Rightarrow$ (3); (1), (3) $\Rightarrow$ (2); and (2), (3) $\Rightarrow$ (1). □

On account of Lemma 1.10, we have the following. Let $(\mathcal{E}, \omega)$ be a *symplectic orthogonally convenient* $\mathcal{A}$*-module of finite rank,* and $\phi \in \mathcal{E}nd\ \mathcal{E}$ a (symplectic) $\mathcal{A}$-transvection of $(\mathcal{E}, \omega)$. Suppose that

$$\phi = I + \psi,$$

where $I = \mathrm{Id}_{\mathcal{E}}$ and $\psi \in \mathcal{E}nd\ \mathcal{E}$. Then, necessarily, if Im ψ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, then rank $\psi :=$ rank Im $\psi = 1$, i.e., ϕ is a nowhere-identity $\mathcal{A}$-transvection. This necessary condition for nowhere-identity $\mathcal{A}$-transvections is not sufficient, for if $\mathcal{H}$ is the sub-$\mathcal{A}$-module of $\mathcal{E}$ defining ϕ, one must have

$$\psi(\mathcal{E}/\mathcal{H}) = 0,$$

i.e.

$$\psi^2 = 0.$$

Using Lemma 1.10, we thus obtain

Corollary 1.11. *Let* $(\mathcal{E}, \omega)$ *be a symplectic orthogonally convenient* $\mathcal{A}$*-module of finite rank. There is a bijection between* $\mathcal{A}$*-symplectomorphisms of the form* $I + \psi$ *such that Im* ψ *is a free sub-*$\mathcal{A}$*-module of* $\mathcal{E}$ *and the nowhere-identity symplectic* $\mathcal{A}$*-transvections.*

We shall now describe precisely such $\mathcal{A}$-symplectomorphisms. We recall the following definition and subsequent remarks, due to A. Mallios: A given algebra sheaf $\mathcal{A}$ is said to be a **PID algebra sheaf** *if, given a free* $\mathcal{A}$*-module and a sub-*$\mathcal{A}$*-module* $\mathcal{F} \subseteq \mathcal{E}$*, one has that* $\mathcal{F}$ *is section-wise free* (*i.e. for every open* $U \subseteq X$, $\mathcal{A}(U) \equiv \Gamma(U, \mathcal{A})$ *is a PID algebra*). An important *example* here is, of course, the *sheaf of* $\mathbb{C}$*-valued polynomials* (in one variable, with respect to the *constant sheaf* $\mathcal{C}_X$ [10, p. 17ff]). See also [14, p. 192, Definition 12].

So, let us fix an open subset U of X and suppose that rank $\psi_U = 1$: it follows that $\psi_U(s) = \alpha_U(s)s_0^U$, for every $s \in \mathcal{E}(U)$ and where α_U is an $\mathcal{A}(U)$-morphism $\mathcal{E}(U) \longrightarrow \mathcal{A}(U)$. It is clear that $\alpha \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$. The necessary and sufficient condition, according to Lemma 1.10, for $\mathrm{Id}_{\mathcal{E}(U)} + \psi_U$ to be symplectic is that

$$\omega_U(\psi_U(s), t) + \omega_U(s, \psi_U(t)) = \omega_U(s_0^U, \alpha_U(s)t - \alpha_U(t)s) = 0, \tag{1.9}$$

for all $s, t \in \mathcal{E}(U)$. Since ω is nondegenerate, we may associate with α_U a section $r \in \mathcal{E}(U)$ such that

$$\alpha_U(s) = \omega_U(r, s), \quad s \in \mathcal{E}(U);$$

therefore (1.9) becomes

$$\omega_U(s_0^U, \omega_U(r,s)t - \omega_U(r,t)s) = 0. \tag{1.10}$$

If $\mathcal{A}$ is a PID algebra sheaf, it follows that the symplectic free $\mathcal{A}$-module $(\mathcal{E}, \omega)$ is a *generalized* (*or locally*) *orthogonally convenient*, cf. [17, Definition 3.2]; hence, by [17, Corollary 4.1], if $\mathcal{G}$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, then

$$\text{rank } \mathcal{G} + \text{rank } \mathcal{G}^{\perp_\omega} = \text{rank } \mathcal{E}.$$

Consequently, $r^{\perp_{\omega_U}}$ is a hyperplane of $\mathcal{E}(U)$; whence it follows that if we take t in $r^{\perp_{\omega_U}}$, (1.10) reduces to

$$\omega_U(s_0^U, t) = 0.$$

Applying [17, Corollary 4.1] here as well and since ω is non-degenerate, one has

$$s_0^U \in (r^{\perp_{\omega_U}})^{\perp_{\omega_U}};$$

therefore there exists $\lambda_U \in \mathcal{A}(U)$ such that

$$r = \lambda_U s_0^U.$$

Thus, the symplectic $\mathcal{A}(U)$-automorphism $\phi_U = \mathrm{Id}_{\mathcal{E}(U)} + \psi_U$ is of the form:

$$\phi_U(s) = s + \lambda_U \omega_U(s_0^U, s) s_0^U$$

for every $s \in \mathcal{E}(U)$.

So, we have proved the following

Theorem 1.12. *Let $\mathcal{A}$ be a PID algebra sheaf, $(\mathcal{E}, \omega)$ a sympletic $\mathcal{A}$-module of finite rank, and ϕ a symplectic $\mathcal{A}$-transvection of $\mathcal{E}$. Then, for every open subset U of X,*

$$\phi_U(s) = s + \lambda_U \omega_U(s_0^U, s) s_0^U$$

for every $s \in \mathcal{E}(U)$.

2. Witt's theorem and symplectic orthogonally convenient $\mathcal{A}$-modules

As suggested in the title of this section, our first aim is to find an analogue of Witt's theorem (cf. [19, pp. 46-48]) for symplectic $\mathcal{A}$-modules. For this purpose, we refer the reader to [14] and [15] for useful details regarding symplectic $\mathcal{A}$-modules and symplectic bases (of sections). Sheaves of symplectic groups arise in a natural way when one considers $\mathcal{A}$-isomorphisms between symplectic $\mathcal{A}$-modules which respect the symplectic structures involved, see [14]. For some other versions of Witt's theorem, see [13] and [16]. Finally, the section ends with a characterization of singular $\mathcal{A}$-symplectomorphisms of symplectic orthogonally convenient $\mathcal{A}$-modules of finite rank. Orthogonally convenient $\mathcal{A}$-modules were introduced in [17].

For the classical Witt's theorem, see [1, pp. 368-387], [2, pp. 121, 122], [4, p. 21], [19], [6, pp. 11, 12], [7, pp. 148- 152], [9, pp. 591, 592], [18, p. 9]. But, first we need the following definition (cf. [16]).

Definition 2.1. A pairing $(\mathcal{F}, \mathcal{E}; \mathcal{A})$ of free $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{E}$ into the $\mathbb{C}$-algebra sheaf $\mathcal{A}$ is called an **orthogonally convenient pairing** if given free sub-$\mathcal{A}$-modules $\mathcal{F}_0$ and $\mathcal{E}_0$ of $\mathcal{F}$ and $\mathcal{E}$, respectively, their orthogonal $\mathcal{F}_0^\perp$ and $\mathcal{E}_0^\perp$ are free sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively.

Definition 2.2. Let $(\mathcal{E}, \omega)$ be a symplectic orthogonally convenient $\mathcal{A}$-module of finite rank.

(*i*) A free sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$ with $\omega|_{\mathcal{F}}$ non-degenerate is called a **symplectic orthogonally convenient sub-$\mathcal{A}$-module** of $\mathcal{E}$.
(*ii*) A free sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$ with $\mathcal{F}^\perp$ isotropic is called **coisotropic.**
(*iii*) A free sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$ which is both isotropic and coisotropic is called a **Lagrangian sub-$\mathcal{A}$-module.**

From [17, Corollary 4.1], if $\mathcal{F}$ is Lagrangian, then

$$\text{rank } \mathcal{F} = \text{rank } \mathcal{F}^\perp.$$

Theorem 2.3. *Let $\mathcal{A}$ be a PID algebra sheaf, $\mathcal{E}$ a symplectic free $\mathcal{A}$-module of rank $2n$ (ω is the symplectic structure on $\mathcal{E}$), $\mathcal{F}$ a Lagrangian (free) sub-$\mathcal{A}$-module of $\mathcal{E}$ and $\mathcal{G}$ any sub-$\mathcal{A}$-module of $\mathcal{E}$ such that $\mathcal{F}$ and $\mathcal{G}$ are supplementary. Then, using $\mathcal{G}$ we can construct a Lagrangian sub-$\mathcal{A}$-module $\mathcal{H}$ of $\mathcal{E}$ such that $\mathcal{E} \simeq \mathcal{F} \oplus \mathcal{H}$.*

Proof. The restriction ω' of ω to $\mathcal{F} \oplus \mathcal{G} \subseteq \mathcal{E} \oplus \mathcal{E}$ is also non-degenerate. In fact, let $\mathcal{F}_{\omega'}^\perp$ and $\mathcal{G}_{\omega'}^\perp$ denote the kernels of $\mathcal{F}$ and $\mathcal{G}$ respectively. More precisely, for every open $U \subseteq X$,

$$\mathcal{F}_{\omega'}^\perp(U) = \{r \in \mathcal{G}(U) |\ \omega'(\mathcal{F}(V), r|_V) = 0 \text{ for any open } V \subseteq U\}$$

and similarly

$$\mathcal{G}_{\omega'}^\perp(U) = \{r \in \mathcal{F}(U) |\ \omega'(r|_V, \mathcal{G}(V)) = 0 \text{ for any open } V \subseteq U\}.$$

Analogously we denote by $\mathcal{F}_\omega^\perp$ and $\mathcal{G}_\omega^\perp$ the kernels of $\mathcal{F}$ and $\mathcal{G}$ respectively with respect to the $\mathcal{A}$-bilinear morphism $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$, i.e. for every open $U \subseteq X$,

$$\mathcal{F}_\omega^\perp(U) = \{r \in \mathcal{E}(U) |\ \omega(\mathcal{F}(V), r|_V) = 0 \text{ for any open } V \subseteq U\}$$

and

$$\mathcal{G}_\omega^\perp(U) = \{r \in \mathcal{E}(U) |\ \omega(\mathcal{G}(V), r|_V) = 0 \text{ for any open } V \subseteq U\}.$$

It is obvious that $\mathcal{F}_\omega^\perp = \mathcal{F}_\omega^\top$ and $\mathcal{G}_\omega^\perp = \mathcal{G}_\omega^\top$. By hypothesis, we are given that $\mathcal{F} = \mathcal{F}_\omega^\perp$. Clearly, for every open $U \subseteq X$, $\mathcal{F}_{\omega'}^\perp(U) \subseteq \mathcal{F}_\omega^\perp(U)$ and $\mathcal{G}_{\omega'}^\perp(U) \subseteq \mathcal{G}_\omega^\perp(U)$. But since $\mathcal{F}_\omega^\perp(U) = \mathcal{F}(U)$ and $\mathcal{F}(U) \cap \mathcal{G}(U) = 0$, $\mathcal{F}_{\omega'}^\perp(U) = 0$. Thus, $\mathcal{F}_{\omega'}^\perp = 0$. On the other hand, let $r \in \mathcal{G}_{\omega'}^\perp(U) \subseteq \mathcal{F}(U) \cap \mathcal{G}_\omega^\perp(U)$. As $\mathcal{E}(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$, we deduce that $r \in \text{rad } \mathcal{E}(U) = 0$, therefore $r = 0$. Hence, $\mathcal{G}_{\omega'}^\perp = 0$. Since $\omega' : \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{A}$ is non-degenerate, the $\mathcal{A}$-morphism $\widetilde{\omega'} : \mathcal{F} \longrightarrow \mathcal{G}^*$ such that for every open $U \subseteq X$, and sections $r \in \mathcal{F}(U)$ and $s \in \mathcal{G}(U)$, $\widetilde{\omega'}(r)(s) := \omega'(r, s)$ is bijective.

Let us construct the sought Lagrangian complement $\mathcal{H}$ of $\mathcal{F}$ in $\mathcal{E}$. For every open $U \subseteq X$, we let

$$\mathcal{H}(U) := \{r + \phi(r) |\ r \in \mathcal{G}(U)\},$$

where $\phi : \mathcal{G} \longrightarrow \mathcal{F}$ is some $\mathcal{A}$-morphism. It is clear that $\mathcal{H}$ is a sub-$\mathcal{A}$-module of $\mathcal{E}$. For $\mathcal{H}$ to be Lagrangian, it takes the following: For every open $U \subseteq X$ and sections $r, s \in \mathcal{G}(U)$

$$\omega(r + \phi(r), s + \phi(s)) = 0$$

i.e.

$$\omega(r, s) = \widetilde{\omega'}(\phi(s))(r) - \widetilde{\omega'}(\phi(r))(s). \tag{2.1}$$

Let $\phi' := \widetilde{\omega'} \circ \phi : \mathcal{G} \longrightarrow \mathcal{G}^*$, so that (2.1) becomes

$$\omega(r, s) = \phi'(s)(r) - \phi'(r)(s). \tag{2.2}$$

Clearly, by taking $\phi'(r) = -\frac{1}{2}\omega(r, -)$ for every $r \in \mathcal{G}(U)$, (2.2) is satisfied. By setting $\phi := (\widetilde{\omega'})^{-1} \circ \phi'$, we contend that the claim holds. In fact, fix an open subset U of X, and suppose that $(r_1, \ldots, r_n)$ is a basis of $\mathcal{G}(U)$. If $a_1, \ldots, a_n \in \mathcal{A}(U)$ such that

$$a_1(r_1 + \phi(r_1)) + \ldots + a_n(r_n + \phi(r_n)) = 0,$$

one has that

$$\underbrace{a_1 r_1 + \ldots + a_n r_n}_{\in \mathcal{G}(U)} = \underbrace{-\phi(a_1 r_1 + \ldots + a_n r_n)}_{\in \mathcal{F}(U)}.$$

Since $\mathcal{F}(U) \cap \mathcal{G}(U) = 0$, it follows that

$$\phi(a_1 r_1 + \ldots + a_n r_n) = 0.$$

As the chosen ϕ' is injective and $\widetilde{\omega'}$ is an $\mathcal{A}$-isomorphism, ϕ is injective; thence

$$a_1 r_1 + \ldots + a_n r_n = 0;$$

so that $a_1 = \cdots = a_n = 0$. Now, let us show that $\mathcal{F}(U) \cap \mathcal{H}(U) = 0$. For this purpose, suppose that $r \in \mathcal{F}(U) \cap \mathcal{H}(U)$. Then for some $s \in \mathcal{G}(U)$

$$r = s + \phi(s).$$

It follows that

$$\underbrace{r - \phi(s)}_{\in \mathcal{F}(U)} = \underbrace{s}_{\in \mathcal{G}(U)}$$

from which we deduce that $s = 0$, and hence $r = 0$. That $\mathcal{E}(U) \cong \mathcal{F}(U) \oplus \mathcal{H}(U)$ is now clear. Since U is arbitrary, $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{H}$ as desired. □

Theorem 2.4. (**Witt's Theorem**) *Let $\mathcal{A}$ be a PID algebra sheaf, let $\mathcal{E}$ be a free $\mathcal{A}$-module of rank $2n$, equipped with two symplectic $\mathcal{A}$-morphisms ω_0 and ω_1, and finally let $\mathcal{F}$ be a sub-$\mathcal{A}$-module of $\mathcal{E}$, Lagrangian with respect to both ω_0 and ω_1. Then, there exists an $\mathcal{A}$-symplectomorphism $\phi : (\mathcal{E}, \omega_0) \longrightarrow (\mathcal{E}, \omega_1)$ such that $\phi|_{\mathcal{F}} = Id_{\mathcal{F}}$.*

Proof. Let $\mathcal{G}$ be any complement of $\mathcal{F}$ in $\mathcal{E}$. By Theorem 2.3, given symplectic $\mathcal{A}$-morphisms ω_0 and ω_1, there exist Lagrangian complements $\mathcal{G}_0$ and $\mathcal{G}_1$ of $\mathcal{F}$ respectively. Again by the proof of Theorem 2.3, the restrictions ω_0', ω_1' of ω_0, ω_1 to $\mathcal{G}_0 \oplus \mathcal{F}$ and $\mathcal{G}_1 \oplus \mathcal{F}$ respectively are nondegenerate and yield $\mathcal{A}$-isomorphisms $\widetilde{\omega_0'} : \mathcal{G}_0 \longrightarrow \mathcal{F}^*$ and $\widetilde{\omega_1'} : \mathcal{G}_1 \longrightarrow \mathcal{F}^*$ respectively. Since $\mathcal{G}_0$ and $\mathcal{G}_1$ are free and of the same finite rank, there exists an $\mathcal{A}$-isomorphism $\psi : \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ such that $\widetilde{\omega_1'} \circ \psi = \widetilde{\omega_0'}$, i.e. for any sections $r \in \mathcal{G}_0(U)$ and $s \in \mathcal{F}(U)$

$$\omega_0(r, s) = \omega_1(\psi(r), s).$$

Let us extend ψ to the rest of $\mathcal{E}$ by setting it to be the identity on $\mathcal{F}$:

$$\phi := \mathrm{Id}_{\mathcal{F}} \oplus \psi : \mathcal{F} \oplus \mathcal{G}_0 \longrightarrow \mathcal{F} \oplus \mathcal{G}_1$$

and we have for any sections $r, r' \in \mathcal{G}_0(U)$ and $s, s' \in \mathcal{F}(U)$

$$\begin{aligned} \omega_1(\phi(s+r), \phi(s'+r')) &= \omega_1(s + \psi(r), s' + \psi(r')) \\ &= \omega_1(s, \psi(r')) + \omega_1(\psi(r), s') \\ &= \omega_0(s, r') + \omega_0(r, s') \\ &= \omega_0(s + r, s' + r'). \end{aligned}$$

□

We are now ready for a characterization of an $\mathcal{A}$-symplectomorphism of the form $\mathrm{I} + f$ of a symplectic orthogonally convenient $\mathcal{A}$-module $\mathcal{E}$, where f is a skewsymmetric $\mathcal{A}$-endomorphism of $\mathcal{E}$. For this purpose, we require the following result: *Given a free $\mathcal{A}$-module of finite rank $\mathcal{E}$, equipped with an $\mathcal{A}$-bilinear form ϕ, every non-isotropic free sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is a direct summand;* viz. (see [13])

$$\mathcal{E} = \mathcal{F} \perp \mathcal{F}^{\perp}.$$

We deduce from the afore-cited result that $\mathcal{F}^{\perp\perp} \simeq \mathcal{F}$. Moreover, if ϕ is *non-degenerate,* then $\mathcal{F}^{\perp}(U) \simeq \mathcal{F}(U)^{\perp}$, for every open $U \subseteq X$. Indeed, since $\mathcal{F}^{\perp}(U) \subseteq \mathcal{F}(U)^{\perp}$, then if $\mathcal{F}(U) \cap \mathcal{F}(U)^{\perp} \neq 0$, rad $\mathcal{E}(U) \neq 0$, which *contradicts* the hypothesis that $\mathcal{E}$ is non-isotropic.

Theorem 2.5. *Let $(\mathcal{E}, \omega)$ be a symplectic orthogonally convenient $\mathcal{A}$-module of rank $2n$, and f a $\mathcal{A}$-endomorphism of $\mathcal{E}$. If f is skewsymmetric and* $\mathrm{Id} + f$ *an $\mathcal{A}$-automorphism of $\mathcal{E}$, then*

(1) $f^2 = 0$;

(2) $\ker f \simeq (Im\ f)^{\perp}$;

(3) *For every open subset $U \subseteq X$, there exists a symplectic basis of $\mathcal{E}(U)$, whose first k elements (sections), $k \leq n$, form a basis of $(Im\ f)(U) := Im\ f_U \equiv f_U(\mathcal{E}(U))$, with respect to which the $\mathcal{A}(U)$-morphism*

$$(\mathrm{Id} + f)_U := \mathrm{Id}_U + f_U$$

is represented by the matrix

$$\begin{pmatrix} I_n & H \\ 0 & I_n \end{pmatrix}$$

with ${}^tH = H$.

Proof. (**1**) From Lemma 1.10, $\mathrm{Im}f$ is totally isotropic. Therefore, for any open subset U of X and sections $s, t \in \mathcal{E}(U)$,

$$\omega_U(f_U(s), f_U(t)) = 0.$$

Since

$$\begin{aligned} \omega_U((f^*)_U f_U(s), t) &= \omega_U((f_U)^* f_U(s), t) \\ &= \omega_U(f_U(s), f_U(t)) \\ &= 0 \end{aligned}$$

and ω is symplectic, it follows that

$$(f^*)_U f_U = (f_U^*) f_U = 0.$$

Thus,

$$f^* f = 0;$$

since $f^* = -f$, one reaches the desired property that $f^2 = 0$.

(**2**) Fix an open set U in X and $s \in (\ker f)(U) = \ker f_U$, see [20, p. 37, Definition 3.1]. Moreover, let $t \in \mathcal{E}(U)$; then

$$\omega_U(s, f_U(t)) = -\omega_U(f_U(s), t) = 0.$$

Thus,

$$s \in (\mathrm{Im}f)(U)^\perp \equiv f_U(\mathcal{E}(U))^\perp$$

and hence

$$(\ker f)(U) = \ker f_U \subseteq (\mathrm{Im}f)(U)^\perp = (\mathrm{Im}f)^\perp(U)$$

or

$$\ker f \subseteq (\mathrm{Im}f)^\perp \equiv f(\mathcal{E})^\perp.$$

Conversely, let $t \in (\mathrm{Im}f)^\perp(U) = (\mathrm{Im}f)(U)^\perp$. Then, for any $s \in (\mathrm{Im}f)(U) \equiv \mathrm{Im}f_U := f_U(\mathcal{E}(U)) \equiv f(\mathcal{E})(U)$, one has

$$\omega_U(t, s) = 0.$$

But $s = f_U(r)$ for some $r \in \mathcal{E}(U)$, therefore

$$\omega_U(t, f_U(r)) = -\omega_U(f_U(t), r) = 0. \qquad (2.3)$$

Since (2.3) is true for any $r \in \mathcal{E}(U)$,

$$f_U(t) = 0,$$

i.e.

$$t \in (\ker f)(U) := \ker f_U.$$

Hence,

$$(\mathrm{Im}f)^\perp(U) \subseteq (\ker f)(U)$$

or

$$(\mathrm{Im}f)^\perp \subseteq \ker f.$$

(**3**) As $\mathrm{Im}f \subseteq \ker f = (\mathrm{Im}f)^\perp$, so the sub-$\mathcal{A}$-module $\mathrm{Im}f$ is totally isotropic. Therefore, for any open $U \subseteq X$,

$$\mathrm{rank}(\mathrm{Im}f)(U) := \mathrm{rank}\ \mathrm{Im}f_U \leq n.$$

Now, let us fix an open set U in X and consider a basis $(s_1, \ldots, s_k)$, $k \leq n$, of $(\mathrm{Im} f)(U) \equiv \mathrm{Im} f_U$. By [13, Lemma 7], there exists a totally isotropic sub-$\mathcal{A}(U)$-module S of $\mathcal{E}(U)$, equipped with a basis, which we denote

$$(s_{k+1}, \ldots, s_{n+k})$$

such that

$$\omega_U(s_i, s_{n+j}) = \delta_{ij}, \qquad \text{for } i, j = 1, \ldots, k.$$

Clearly,

$$S \cap (\mathrm{Im} f)^{\perp}(U) = S \cap (\ker f)(U) = 0. \tag{2.4}$$

As a result of (2.4), the sum $S + \mathrm{Im} f_U$ is direct and $S \oplus \mathrm{Im} f_U$ is non-isotropic; therefore, one has

$$\mathcal{E}(U) = (S \oplus \mathrm{Im} f_U) \perp F$$

for some sub-$\mathcal{A}(U)$-module F of $\mathcal{E}(U)$, (cf. [13, Theorem 1]). Since $F = (S \oplus \mathrm{Im} f_U)^{\perp}$, F is contained in $(\mathrm{Im} f_U)^{\perp} = (\mathrm{Im} f)^{\perp}(U) = (\ker f)(U)$ and

$$F^{\perp} = (\mathrm{Im} f)(U) := \mathrm{Im} f_U;$$

i.e. F is an orthogonal supplementary of $(\mathrm{Im} f)(U)$ in $(\ker f)(U)$. Since F is free, non-isotropic and of rank $2n - 2k$, it can be equipped with a symplectic basis, say $(s_{k+1}, \ldots, s_n, s_{n+k+1}, \ldots, s_{2n})$, see [15]. As $s_1, \ldots, s_n \in (\ker f)(U)$, it follows that

$$(\mathrm{Id}_U + f_U)(s_j) = s_j, \qquad j = 1, \ldots, n.$$

Therefore, if H is the matrix representing f_U, $\mathrm{Id}_U + f_U$ is represented by the matrix

$$\begin{pmatrix} \mathrm{I}_n & \mathrm{H} \\ 0 & \mathrm{I}_n \end{pmatrix},$$

and this is a sympletic matrix if and only if ${}^t H = H$, ie. H is symmetric. $\square$

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