Exact nonnull distribution of wilks's Statistic : Ratio and Product of independent Components

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Abstract

The study of the noncentral matrix variate beta type distributions has been sidelined because the final expressions of the densities depend on an integral that has not been resolved in an explicit way. We derive an exact expression for the nonnull distribution of the Wilks's statistic and precise expressions for the densities of ratio and product of two independent components of matrix variates where one matrix variate has the noncentral matrix variate beta type I distribution and the other has the matrix variate beta type I distribution. We provide the expressions for the densities of the determinant of the ratio and product of these two components. These distributions play a fundamental role in various areas of statistics for example in the criteria proposed by Wilks.

Mathematical subject classification: 62H10; 62H15

Keywords: Invariant polynomials; Meijer's G-function; Noncentral matrix variate beta type I distribution; Product; Ratio; Wilks's statistic; Zonal polynomial

1. INTRODUCTION

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two independent $(p \times p)$ Wishart matrices, i.e. $\boldsymbol{A} \sim W_p(n, \boldsymbol{\Sigma})$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Omega})$, and $n, m \geq p$. The noncentral beta matrix $\boldsymbol{U}$ can be defined as $\boldsymbol{U} = (\boldsymbol{A}+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{A} (\boldsymbol{A}+\boldsymbol{B})^{-\frac{1}{2}}$, denoted as $\boldsymbol{U} \sim \boldsymbol{B}_p^I(n, m, \boldsymbol{\Omega})$. The Wilks's statistic is defined as $\Lambda = \dfrac{|\boldsymbol{A}|}{|\boldsymbol{A}+\boldsymbol{B}|} = |\boldsymbol{U}|$. It is known that the Wilks's statistic plays the same role in multivariate analysis as the F statistic plays in univariate analysis. In a recent paper, Pham-Gia [31] established the exact expression of the density of the generalized Wilks's statistic, and those of the product and ratio of two independent such statistics. However, the noncentral case was not attended to. The study of the noncentral matrix variate beta type distributions has been sidelined because the final expressions of the densities depend on an integral that has not been resolved in an explicit way. Diaz-Garzia and Gutiérrez-Jamez [9, 10] defined the nonsymmetrised matrix variate beta density, by using a procedure equivalent to finding the symmetrized density defined by Greenacre [14] (see also [3] and [12]). In this paper, we propose an exact expression for the nonnull distribution of the Wilks's statistic by using this nonsymmetrised density function of the noncentral matrix variate beta type I distribution. The nonnull distribution of Λ will be useful in investigating the power of the Wilks's test.

Furthermore, Pham-Gia [30] and Pham-Gia and Turkkan [32] pioneered the study of products and ratios of independent beta random variables, while Nadarajah [27] extended these results to the noncentral beta distribution. Those papers, however, have focussed on the univariate case and Bekker et al [2] examined operations on matrix variate beta type I variates. We will in this paper extend these ideas to the case using the nonsymmetrised matrix variate beta type I density. Therefore, this paper deals with exact expressions for the densities of the ratio ($\boldsymbol{R}$) and product ($\boldsymbol{P}$) of two independent matrix variates where one has the matrix variate beta type I density and the other has the nonsymmetrised matrix variate beta type I density. As well as the latter, we provide expressions for the densities of the determinant of the ratio and product of these two independent components. These density functions of the aforementioned determinants are derived in terms of Meijer's G-functions by inverse Mellin transforms.

The uniqueness property of the Mellin transform, for statistical distributions with bounded domain, arises from the fact that for a given density function $f(\cdot)$ if $\int_{\Omega_1} f(t)dt = 1$ and $\int_{\Omega_2} f(t)dt = 1$ then $\int_{\mathcal{S}=\Omega_1\cap\Omega_2} f(t)dt = 1$ and $f \equiv 0$ a.s. for other strips, note that $\mathcal{S}$ should be bounded. See Zemanian [34] for the uniqueness theorem in general frameworks.

The expressions are expressed in terms of Meijer's G-function, zonal polynomials, hypergeometric functions with matrix argument, or homogeneous invariant polynomials with two or more matrix arguments. The reader is referred to the papers [4, 5, 6, 7, 8, 11, 19, 20, 21] on these functions; as well as the reference books [17, 24, 26]. There are some algorithms available for calculating such functions and facilitating the use of these distributions (see [18] and [22]). Currently, there are also mathematical packages, such as MAPLE or MATHEMATICA for the computing and drawing of densities in terms of Meijer's G-function.

We find the nonnull distribution of the Wilks's statistic in section 2. In section 3 the results are presented for the densities of the ratio and its determinant of two independent matrix variates, one being matrix variate beta type I distributed and the other component has the nonsymmetrised matrix variate beta type I density; while the outcome of the product case is dealt with in section 4. Several numerical examples show that the densities can now be computed.

2. THE NONNULL DISTRIBUTION OF Λ

Let the columns of a $p \times m$ matrix $\boldsymbol{X}$ and a $p \times n$ matrix $\boldsymbol{Y}$ be distributed independently in a p-variate normal distribution with a common positive definite covariance matrix $\boldsymbol{\Sigma}$ and let $E(\boldsymbol{X}) = \boldsymbol{M}$, $E(\boldsymbol{Y}) = \boldsymbol{0}$. The presented Wilks's statistic $\Lambda = \dfrac{|\boldsymbol{Y}\boldsymbol{Y}'|}{|\boldsymbol{X}\boldsymbol{X}'+\boldsymbol{Y}\boldsymbol{Y}'|} = \dfrac{|\boldsymbol{A}|}{|\boldsymbol{A}+\boldsymbol{B}|}$ (see Theorem 1) can be used as a likelihood ratio criterion for testing whether the matrix mean $\boldsymbol{M}$ is equal to zero or not (see [29, 33]). In this regard, deriving the non-null distribution plays an important role in determining the power of the test. There is a vast variety work concerning this phenomenon in the literature. Asoh and Okamoto [1] and Gupta [15, 16] derived the nonnull distribution proposed in Theorem 1 as a product of noncentral beta variates for some special cases. None of the mentioned references derived an analytically exact density function. The present work proposes the distribution of Wilks's statistic based on Meijers' G-function in a numerical feasible form.

To derive the nonnull distribution of the Wilks's statistic we need the definition of the nonsymmetrised density of the noncentral matrix variate beta type I distribution (see [9]).

If $\boldsymbol{A} \sim W_p(n, \boldsymbol{\Sigma})$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Omega})$ are independent, $n, m \geq p$, then $\boldsymbol{U} = (\boldsymbol{A}+\boldsymbol{B})^{-\frac{1}{2}}\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{B})^{-\frac{1}{2}}$ is said to have a *noncentral beta type I distribution*, denoted as $\boldsymbol{U} \sim \boldsymbol{B}_p^I(n, m, \boldsymbol{\Omega})$, with *nonsymmetrised density function* given by

$$\left\{\beta_p\left(\frac{n}{2},\frac{m}{2}\right)\right\}^{-1}|\boldsymbol{U}|^{\frac{n}{2}-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{m}{2}-\frac{1}{2}(p+1)}\,etr\left(-\frac{1}{2}\boldsymbol{\Omega}\right)\,{}_1F_1\left(\frac{(n+m)}{2};\frac{m}{2};\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{U})\right),\ \boldsymbol{0}<\boldsymbol{U}<\boldsymbol{I}_p \tag{1}$$

where ${}_1F_1(\cdot)$ is the confluent hypergeometric function of matrix argument, $\beta_p(a,b) = \dfrac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}$ denotes the multivariate beta function, and $\Gamma_p(a)$ is the multivariate gamma function, defined as $\Gamma_p(a) = \int_{\boldsymbol{A}>\boldsymbol{0}} etr(-\boldsymbol{A})\,|\boldsymbol{A}|^{a-\frac{1}{2}(p+1)}\,d\boldsymbol{A}$ ($\mathrm{Re}(a) > \frac{1}{2}(p-1)$ and $etr(\cdot) = \exp(tr(\cdot))$). The multivariate gamma function can be expressed as

$$\Gamma_p(a) = \pi^{\frac{p(p-1)}{4}}\prod_{i=1}^{p}\Gamma\left(a-\frac{i-1}{2}\right). \tag{2}$$

(Here $\boldsymbol{C} < \boldsymbol{D}$ means that the matrix $\boldsymbol{D} - \boldsymbol{C}$ is positive definite; $\boldsymbol{C}^{\frac{1}{2}}$ is the unique positive definite square root of $\boldsymbol{C}$)

If $\boldsymbol{\Omega} = \mathbf{0},$ then $\boldsymbol{U}$ has the well known matrix variate beta type I distribution.

THEOREM 1

Let $\boldsymbol{A} \sim W_p(n, \boldsymbol{\Sigma})$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Omega})$ be independently distributed, $n, m \geq p$, and $\boldsymbol{U} = (\boldsymbol{A}+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{A} (\boldsymbol{A}+\boldsymbol{B})^{-\frac{1}{2}} \sim \boldsymbol{B}_p^I(n, m, \boldsymbol{\Omega})$.

Then the density of $\Lambda = |\boldsymbol{U}| = \dfrac{|\boldsymbol{A}|}{|\boldsymbol{A}+\boldsymbol{B}|}$ is given by

$$\frac{etr(-\frac{1}{2}\boldsymbol{\Omega})}{\Gamma_p\left(\frac{n}{2}\right)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} \Gamma_p\left(\tfrac{(n+m)}{2}, \kappa\right) C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Omega}\right) G_{p,p}^{p,0}\left(|\boldsymbol{U}| \left| \begin{array}{c} c_1, \ldots, c_p \\ d_1, \ldots, d_p \end{array}\right.\right), \tag{3}$$

$$0 < |\boldsymbol{U}| < 1$$

where $c_i = \frac{n+m}{2} + k_i - \frac{1}{2}(i+1)$, $i = 1, \ldots, p$ and $d_i = \frac{n}{2} - \frac{1}{2}(i+1)$, $i = 1, \ldots, p$, $G(\cdot)$ denotes Meijer's G-function (see [24]) and $C_\kappa(\cdot)$ is the zonal polynomial corresponding to κ (see[20]).

Furthermore

$$\Gamma_p(a, \kappa) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \Gamma\left(a + k_i - \frac{i-1}{2}\right) = \Gamma_p(a)(a)_\kappa, \qquad (\mathrm{Re}(a) \geq \frac{(p-1)}{2} - k_p) \tag{4}$$

and the generalized hypergeometric coefficient $(a)_\kappa$ is given by $(a)_\kappa = \prod_{i=1}^{m} \left(a - \frac{1}{2}(i-1)\right)_{k_i}$ where $(a)_\kappa = a(a-1)\ldots(a+k-1)$, $(a)_0 = 1$.

Proof:

From (1) and the definition of the hypergeometric function of matrix argument (see[13]) it follows that

$$\begin{aligned} &E\left[|\boldsymbol{U}|^{h-1}\right] \\ &= \frac{etr(-\frac{1}{2}\boldsymbol{\Omega})}{\beta_p\left(\frac{n}{2}, \frac{m}{2}\right)} \int_{0<\boldsymbol{U}<\boldsymbol{I}_p} |\boldsymbol{U}|^{\frac{n}{2}+h-1-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{U}|^{\frac{m}{2}-\frac{1}{2}(p+1)} {}_1F_1\left(\tfrac{(n+m)}{2}; \tfrac{m}{2}; \tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p - \boldsymbol{U})\right) d\boldsymbol{U} \\ &= \frac{etr(-\frac{1}{2}\boldsymbol{\Omega})}{\beta_p\left(\frac{n}{2}, \frac{m}{2}\right)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\left(\frac{n+m}{2}\right)_\kappa}{k! \left(\frac{m}{2}\right)_\kappa} \int_{0<\boldsymbol{U}<\boldsymbol{I}_p} |\boldsymbol{U}|^{\frac{n}{2}+h-1-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{U}|^{\frac{m}{2}-\frac{1}{2}(p+1)} C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p - \boldsymbol{U})\right) d\boldsymbol{U} \end{aligned}$$

Let $\boldsymbol{Q} = (\boldsymbol{I}_p - \boldsymbol{U})$, applying Muirhead [26, pp. 254, equation (57)] in the above expression and then simplifying it, we obtain

$$E\left[|\boldsymbol{U}|^{h-1}\right] = \frac{etr(-\frac{1}{2}\boldsymbol{\Omega})}{\Gamma_p\left(\frac{n}{2}\right)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma_p\left(\frac{n+m}{2}, \kappa\right) \Gamma_p\left(\frac{n}{2} + h - 1\right)}{k! \Gamma_p\left(\frac{n+m}{2} + h - 1, \kappa\right)} C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Omega}\right).$$

From (4), the density of $|\boldsymbol{U}|$ is uniquely determined by the inverse Mellin transform (see [24, pp. 60]).□

Remarks

1. If $\boldsymbol{\Omega} = \mathbf{0}$, then the distribution of the Wilks's statistic under the null hypothesis follows, see [31, 25].
2. For $p = 1$, result (3) simplifies to

$$f(u) = \frac{e^{-\frac{\lambda}{2}}}{\Gamma\left(\frac{n}{2}\right)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n+m}{2} + k\right)\left(\frac{\lambda}{2}\right)^k}{k!} G_{1,1}^{1,0}\left(u \left| \begin{array}{c} \frac{n+m}{2} + k - 1 \\ \frac{n}{2} - 1 \end{array}\right.\right), \quad 0 < u < 1$$

with λ the noncentrality parameter. Now, using [24, pp. 131], then the above density can be expressed as

$$f(u) = \frac{e^{-\frac{\lambda}{2}}}{\Gamma\left(\frac{n}{2}\right)} u^{\frac{n}{2}-1} (1-u)^{\frac{m}{2}-1} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n+m}{2}+k\right)\left(\frac{\lambda}{2}\right)^k (1-u)^k}{k!\Gamma\left(\frac{m}{2}+k\right)}, \quad 0<u<1,$$

which is the density of the noncentral beta distribution of type II (see [27]).
3. Figure 1 illustrates the shape of density (3) for $p=2$, $n=8, m=12$ and specific values of $\mathbf{\Omega}$.

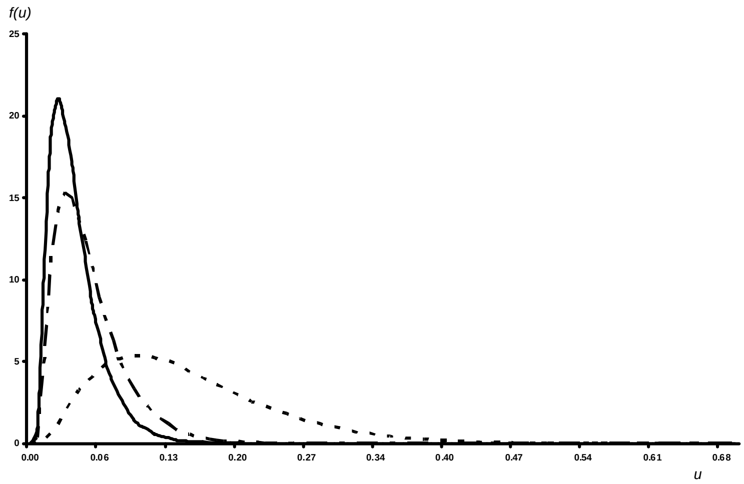

Figure 1: Density function of Λ for
(a) _ _ _ $\mathbf{\Omega}=\mathbf{0}$;(b)_ ___ _ $\mathbf{\Omega}_1 = 12\begin{bmatrix} (1.5)^2 & 1.5 \\ 1.5 & 1 \end{bmatrix}$;(c)______ $\mathbf{\Omega}_2 = 12\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

3. RATIO ($\boldsymbol{R}$) OF MATRIX VARIATE BETA TYPE I VARIATES

In this section we propose densities for the ratio and its determinant of two independent components, where one component has the matrix variate beta type I distribution and the other the nonsymmetrised matrix variate beta type I density. Pham-Gia [31] gave a complete explanation for the density of the ratio and product of two independent Wilks's statistics, but this paper considers the one component having the nonsymmetrised matrix variate beta type I distribution.

THEOREM 2

Let $\boldsymbol{U}_1 \sim \boldsymbol{B}_p^I(n_1, m_1)$ and $\boldsymbol{U}_2 \sim \boldsymbol{B}_p^I(n_2, m_2, \mathbf{\Omega})$ be independently distributed, $n_i, m_i \geq p, \;\; i=1,2$. Define $\boldsymbol{R} = \boldsymbol{U}_2^{-\frac{1}{2}} \boldsymbol{U}_1 \boldsymbol{U}_2^{-\frac{1}{2}}$, then the densities of $\boldsymbol{R}$ and $r = |\boldsymbol{R}| = |\boldsymbol{U}_1| \, |\boldsymbol{U}_2|^{-1}$ are respectively given by the following equations:

(a) For $\mathbf{0} < \boldsymbol{R} < \boldsymbol{I}_p$,

$$\begin{aligned} f(\boldsymbol{R}) &= \frac{etr(-\frac{1}{2}\mathbf{\Omega}) \, |\boldsymbol{R}|^{\frac{n_1}{2} - \frac{(p+1)}{2}}}{\prod_{i=1}^{2} \beta_p\left(\frac{n_i}{2}, \frac{m_i}{2}\right)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\kappa} \sum_{J} \sum_{\phi \in \kappa . J} \theta_{\phi}^{\kappa, J} \\ &\cdot \frac{\left(-\frac{m_1}{2} + \frac{(p+1)}{2}\right)_{\kappa} \left(\frac{n_2+m_2}{2}\right)_J \Gamma_p\left(\frac{n_1+n_2}{2}, \kappa\right) \Gamma_p\left(\frac{m_2}{2}, J\right) C_{\phi}^{\kappa, J}\left(\boldsymbol{R}, \frac{1}{2}\mathbf{\Omega}\right)}{k! j! \Gamma_p\left(\frac{n_1+n_2+m_2}{2}, \phi\right) \left(\frac{m_2}{2}\right)_J} \end{aligned} \qquad (5)$$

For $\boldsymbol{R} > \boldsymbol{I}_p$,

$$\begin{aligned} f(\boldsymbol{R}) &= \frac{etr(-\frac{1}{2}\boldsymbol{\Omega})\,|\boldsymbol{R}|^{-\frac{n_2}{2}-\frac{(p+1)}{2}}}{\prod_{i=1}^{2}\beta_p\left(\frac{n_i}{2},\frac{m_i}{2}\right)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\kappa}\sum_{J}\sum_{\phi,\lambda(J\in\phi\cdot\lambda)}\sum_{J'\equiv J}\begin{pmatrix} f \\ \iota \end{pmatrix} \\ &\cdot\frac{\left(-\frac{m_2}{2}+\frac{(p+1)}{2}\right)_{\kappa}\left(\frac{n_2+m_2}{2}\right)_J}{k!j!\left(\frac{m_2}{2}\right)_J}\theta_{J'}^{\phi,\lambda}\sum_{\sigma\in\kappa\cdot\phi\cdot\lambda}\gamma_{\kappa}^{\kappa(1,1);\sigma}\alpha_{J'}^{(\kappa),\phi,\lambda;\sigma}\frac{C_\kappa(\boldsymbol{I}_p)C_{J'}(\boldsymbol{I}_p)}{C_\sigma(\boldsymbol{I}_p)} \\ &\cdot\frac{\Gamma_p\left(\frac{n_1+n_2}{2},\varepsilon\right)\Gamma_p\left(\frac{m_1}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m_1}{2},\varepsilon\right)}C_\sigma^{\kappa,\phi,\lambda}\left(\boldsymbol{R}^{-1},\tfrac{1}{2}\boldsymbol{\Omega},-\tfrac{1}{2}\boldsymbol{\Omega}^{\frac{1}{2}}\boldsymbol{R}^{-1}\boldsymbol{\Omega}^{\frac{1}{2}}\right) \end{aligned} \tag{6}$$

where $C_{\phi}^{\kappa,J}(\cdot)$ denotes the invariant polynomials defined in Davis [8], see also Chikuse [4] and $\varepsilon \in \kappa.\lambda$.

(b) For $0 < r < \infty$,

$$f(r) = \frac{\Gamma_p\left(\frac{n_1+m_1}{2}\right)}{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i}{2}\right)}etr(-\frac{1}{2}\boldsymbol{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\tfrac{n_2+m_2}{2},\kappa\right)C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Omega}\right)G_{2p,2p}^{p,p}\left(r\,\middle|\,\begin{matrix} c_1,\ldots,c_{2p} \\ d_1,\ldots d_{2p}\end{matrix}\right) \tag{7}$$

where $c_i = \begin{cases} -\frac{n_2}{2}-\frac{(i-1)}{2} & ,i=1,\ldots,p \\ \frac{n_1+m_1}{2}-\frac{(i-p+1)}{2} & ,i=p+1,\ldots,2p \end{cases}$, $d_i = \begin{cases} \frac{n_1}{2}-\frac{(i+1)}{2} & ,i=1,\ldots,p \\ -\frac{(n_2+m_2)}{2}-k_i+\frac{(i-p-1)}{2} & ,i=p+1,\ldots,2p. \end{cases}$

Proof:

(a) From (1) and [26, pp. 259, equation (4)] it follows that for $\boldsymbol{0} < \boldsymbol{R} < \boldsymbol{I}_p$,

$$\begin{aligned} f(\boldsymbol{R}) &= K\int_{\boldsymbol{0}<\boldsymbol{U}_2<\boldsymbol{I}_p}|\boldsymbol{U}_2|^{\frac{1}{2}(p+1)}f_{\boldsymbol{U}_1}(\boldsymbol{U}_2^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{U}_2^{\frac{1}{2}})f_{\boldsymbol{U}_2}(\boldsymbol{U}_2)d\boldsymbol{U}_2 \qquad \text{(see [23])} \\ &= K\,|\boldsymbol{R}|^{\frac{n_1}{2}-\frac{(p+1)}{2}}\,etr(-\tfrac{1}{2}\boldsymbol{\Omega})\int_{0<\boldsymbol{U}_2<\boldsymbol{I}_p}|\boldsymbol{U}_2|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{R}\boldsymbol{U}_2|^{\frac{m_1}{2}-\frac{1}{2}(p+1)} \\ &\qquad\cdot\,{}_1F_1\left(\tfrac{(n_2+m_2)}{2};\tfrac{m_2}{2};\tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{U}_2)\right)d\boldsymbol{U}_2 \end{aligned} \tag{8}$$

$$\begin{aligned} &= K\,|\boldsymbol{R}|^{\frac{n_1}{2}-\frac{(p+1)}{2}}\,etr(-\tfrac{1}{2}\boldsymbol{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(-\frac{m_1}{2}+\frac{1}{2}(p+1)\right)_\kappa}{k!}\int_{0<\boldsymbol{U}_2<\boldsymbol{I}_p}|\boldsymbol{U}_2|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)} \\ &\qquad\cdot\;C_\kappa(\boldsymbol{R}\boldsymbol{U}_2)\;{}_1F_1\left(\tfrac{(n_2+m_2)}{2};\tfrac{m_2}{2};\tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{U}_2)\right)d\boldsymbol{U}_2 \\ &= K\,|\boldsymbol{R}|^{\frac{n_1}{2}-\frac{(p+1)}{2}}\,etr(-\tfrac{1}{2}\boldsymbol{\Omega})\cdot g(\boldsymbol{R},\boldsymbol{\Omega}) \end{aligned} \tag{9}$$

where $K = \left\{\prod_{i=1}^{2}\beta_p\left(\frac{n_i}{2},\frac{m_i}{2}\right)\right\}^{-1}$. Then expanding the hypergeometric function ${}_1F_1(\cdot)$ in terms of zonal polynomial and applying the same argument as [28], follows that for any $\boldsymbol{H} \in O(p)$, the orthogonal group, it can be easily seen that $g(\boldsymbol{R},\boldsymbol{\Omega}) = g(\boldsymbol{H}\boldsymbol{R}\boldsymbol{H}',\boldsymbol{H}\boldsymbol{\Omega}\boldsymbol{H}')$. Thus

$$g(\boldsymbol{R},\boldsymbol{\Omega}) = \sum_{j=0}^{\infty}\sum_{J} \frac{\left(\frac{n_2+m_2}{2}\right)_J}{j!\left(\frac{m_2}{2}\right)_J} \int\limits_{0<\boldsymbol{U}_2<\boldsymbol{I}_p} |\boldsymbol{U}_2|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}$$

$$\cdot \int\limits_{O(p)} C_\kappa\left(\boldsymbol{H}\boldsymbol{R}\boldsymbol{H}'\boldsymbol{U}_2\right) C_J\left(\tfrac{1}{2}\boldsymbol{H}\boldsymbol{\Omega}\boldsymbol{H}'(\boldsymbol{I}_p-\boldsymbol{U}_2)\right) d\boldsymbol{H}\, d\boldsymbol{U}_2$$

$$= \sum_{j=0}^{\infty}\sum_{J}\sum_{\phi\in\kappa.J} \frac{\left(\frac{n_2+m_2}{2}\right)_J C_\phi^{\kappa,J}\left(\boldsymbol{R},\frac{1}{2}\boldsymbol{\Omega}\right)}{j!\left(\frac{m_2}{2}\right)_J C_\phi(\boldsymbol{I}_p)}$$

$$\cdot \int\limits_{0<\boldsymbol{U}_2<\boldsymbol{I}_p} |\boldsymbol{U}_2|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)} C_\phi^{\kappa,J}(\boldsymbol{U}_2,\boldsymbol{I}_p-\boldsymbol{U}_2)\, d\boldsymbol{U}_2$$

using the fundamental property from Davis [7, equation (1.2)], and $(d\boldsymbol{H})$ is the normalized invariant Haar measure over the group $O(p)$ ([26], pp. 72). Applying [7, equation (3.4)] to the above intergral, substituting it in (9), then (9) simplifies to the desired result (5) with $\theta_\phi^{\kappa,J}$ as defined in [7,4].
For $\boldsymbol{R} > \boldsymbol{I}_p$, we have

$$f(\boldsymbol{R}) = K\,|\boldsymbol{R}|^{\frac{n_1}{2}-\frac{(p+1)}{2}}\, etr(-\tfrac{1}{2}\boldsymbol{\Omega}) \int\limits_{\boldsymbol{I}_p-\boldsymbol{R}\boldsymbol{U}_2>0} |\boldsymbol{U}_2|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{R}\boldsymbol{U}_2|^{\frac{m_1}{2}-\frac{1}{2}(p+1)}$$

$$\cdot {}_1F_1\left(\tfrac{(n_2+m_2)}{2};\tfrac{m_2}{2};\tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{U}_2)\right) d\boldsymbol{U}_2$$

where $K = \left\{\prod_{i=1}^{2}\beta_p\left(\frac{n_i}{2},\frac{m_i}{2}\right)\right\}^{-1}$. By performing the transformation $\boldsymbol{W} = \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{U}_2\boldsymbol{R}^{\frac{1}{2}}$ with Jacobian $J(\boldsymbol{U}_2 \to \boldsymbol{W}) = |\boldsymbol{R}|^{-\frac{1}{2}(p+1)}$ it follows that

$$f(\boldsymbol{R}) = K\,|\boldsymbol{R}|^{-\frac{n_2}{2}-\frac{(p+1)}{2}}\, etr(-\tfrac{1}{2}\boldsymbol{\Omega}) \int\limits_{0<\boldsymbol{W}<\boldsymbol{I}_p} |\boldsymbol{W}|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{W}|^{\frac{m_1}{2}-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p-\boldsymbol{R}^{-1}\boldsymbol{W}\right|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}$$

$$\cdot\ {}_1F_1\left(\tfrac{(n_2+m_2)}{2};\tfrac{m_2}{2};\tfrac{1}{2}\boldsymbol{\Omega}\left(\boldsymbol{I}_p-\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{W}\boldsymbol{R}^{-\frac{1}{2}}\right)\right) d\boldsymbol{W}$$

$$= K\,|\boldsymbol{R}|^{-\frac{n_2}{2}-\frac{(p+1)}{2}}\, etr(-\tfrac{1}{2}\boldsymbol{\Omega}) \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\kappa}\sum_{J} \frac{\left(-\frac{m_2}{2}+\frac{(p+1)}{2}\right)_\kappa \left(\frac{n_2+m_2}{2}\right)_J}{k!j!\left(\frac{m_2}{2}\right)_J}$$

$$\cdot \int\limits_{0<\boldsymbol{W}<\boldsymbol{I}_p} |\boldsymbol{W}|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{W}|^{\frac{m_1}{2}-\frac{1}{2}(p+1)} C_\kappa\left(\boldsymbol{R}^{-1}\boldsymbol{W}\right) C_J\left(\tfrac{1}{2}\boldsymbol{\Omega}\left(\boldsymbol{I}_p-\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{W}\boldsymbol{R}^{-\frac{1}{2}}\right)\right) d\boldsymbol{W}$$

$$= K\,|\boldsymbol{R}|^{-\frac{n_2}{2}-\frac{(p+1)}{2}}\, etr(-\tfrac{1}{2}\boldsymbol{\Omega}) \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\kappa}\sum_{J} \frac{\left(-\frac{m_2}{2}+\frac{(p+1)}{2}\right)_\kappa \left(\frac{n_2+m_2}{2}\right)_J}{k!j!\left(\frac{m_2}{2}\right)_J} \cdot g(\boldsymbol{\Omega},\boldsymbol{R}) \tag{10}$$

Using the binomial expansion for $C_J\left(\frac{1}{2}\boldsymbol{\Omega}\left(\boldsymbol{I}_p-\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{W}\boldsymbol{R}^{-\frac{1}{2}}\right)\right)$ (see [7, equation (2.13)]), it follows that

$$\begin{aligned} g(\boldsymbol{\Omega},\boldsymbol{R}) \;=\; & \sum_{\phi,\lambda(J\in\phi\cdot\lambda)}\sum_{J'\equiv J} \begin{pmatrix} f \\ \iota \end{pmatrix} \theta_{J'}^{\phi,\lambda} \int_{\boldsymbol{0}<\boldsymbol{W}<\boldsymbol{I}_p} |\boldsymbol{W}|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{W}|^{\frac{m_1}{2}-\frac{1}{2}(p+1)} \\ & \cdot C_\kappa(\boldsymbol{R}^{-1}\boldsymbol{W}) C_{J'}^{\phi,\lambda}\left(\tfrac{1}{2}\boldsymbol{\Omega},-\tfrac{1}{2}\boldsymbol{\Omega}\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{W}\boldsymbol{R}^{-\frac{1}{2}}\right) d\boldsymbol{W} \end{aligned} \tag{11}$$

From Chikuse [4, equation (3.11)], we have that

$$C_\kappa(\boldsymbol{R}^{-1}\boldsymbol{W})C_{J'}^{\phi,\lambda}\left(\tfrac{1}{2}\boldsymbol{\Omega},-\tfrac{1}{2}\boldsymbol{\Omega}\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{W}\boldsymbol{R}^{-\frac{1}{2}}\right) = \sum_{\sigma\in\kappa\cdot\phi\cdot\lambda}\gamma_\kappa^{\kappa(1,1);\sigma}\alpha_{J'}^{(\kappa),\phi,\lambda;\sigma}\frac{C_\kappa(\boldsymbol{I}_p)C_{J'}(\boldsymbol{I}_p)}{C_\sigma(\boldsymbol{I}_p)} \cdot C_\sigma^{\kappa,\phi,\lambda}\left(\boldsymbol{R}^{-1}\boldsymbol{W},\tfrac{1}{2}\boldsymbol{\Omega},-\tfrac{1}{2}\boldsymbol{\Omega}\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{W}\boldsymbol{R}^{-\frac{1}{2}}\right). \tag{12}$$

Substituting (12) in (11), using a similar approach as Chikuse [4, equations (3.28) and(3.29)], $\varepsilon\in\kappa.\lambda$, we have that

$$g(\boldsymbol{\Omega},\boldsymbol{R}) = \sum_{\phi,\lambda(J\in\phi\cdot\lambda)}\sum_{J'\equiv J}\begin{pmatrix} f\\ \iota\end{pmatrix}\theta_{J'}^{\phi,\lambda}\sum_{\sigma\in\kappa\cdot\phi\cdot\lambda}\gamma_\kappa^{\kappa(1,1);\sigma}\alpha_{J'}^{(\kappa),\phi,\lambda;\sigma}\frac{C_\kappa(\boldsymbol{I}_p)C_{J'}(\boldsymbol{I}_p)}{C_\sigma(\boldsymbol{I}_p)} \cdot\frac{\Gamma_p\left(\frac{1}{2}(n_1+n_2),\varepsilon\right)\Gamma_p\left(\frac{1}{2}m_1\right)}{\Gamma_p\left(\frac{1}{2}(n_1+n_2)+\frac{1}{2}m_1,\varepsilon\right)}C_\sigma^{\kappa,\phi,\lambda}\left(\boldsymbol{R}^{-1},\tfrac{1}{2}\boldsymbol{\Omega},-\tfrac{1}{2}\boldsymbol{\Omega}^{\frac{1}{2}}\boldsymbol{R}^{-1}\boldsymbol{\Omega}^{\frac{1}{2}}\right). \tag{13}$$

From (10) and (13) the desired result (6) follows.

(b) For $0<r<\infty$,

$$\begin{aligned}
&E\left[|\boldsymbol{R}|\right]^{h-1}\\
&= K\int_{0<\boldsymbol{U}_1<\boldsymbol{I}_p}|\boldsymbol{U}_1|^{\frac{n_1}{2}+h-1-\frac{(p+1)}{2}}|\boldsymbol{I}_p-\boldsymbol{U}_1|^{\frac{m_1}{2}-\frac{(p+1)}{2}}d\boldsymbol{U}_1\\
&\quad\cdot\int_{0<\boldsymbol{U}_2<\boldsymbol{I}_p}|\boldsymbol{U}_2|^{\frac{n_2}{2}-h+1-\frac{(p+1)}{2}}|\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{(p+1)}{2}}etr(-\tfrac{1}{2}\boldsymbol{\Omega})\cdot{}_1F_1\left(\tfrac{(n_2+m_2)}{2};\tfrac{m_2}{2};\tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I_p}-\boldsymbol{U}_2)\right)d\boldsymbol{U}_2\\
&= etr(-\tfrac{1}{2}\boldsymbol{\Omega})\frac{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i+m_i}{2}\right)\Gamma_p\left(\frac{n_1}{2}+h-1\right)}{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i}{2}\right)\Gamma_p\left(\frac{m_2}{2}\right)\Gamma_p\left(\frac{n_1+m_1}{2}+h-1\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n_2+m_2}{2}\right)_\kappa}{\left(\frac{m_2}{2}\right)_\kappa k!}\\
&\quad\cdot\int_{0<\boldsymbol{U}_2<\boldsymbol{I}_p}|\boldsymbol{U}_2|^{\frac{n_2}{2}-h+1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Omega}\left(\boldsymbol{I}_p-\boldsymbol{U}_2\right)\right)d\boldsymbol{U}_2
\end{aligned}$$

with K as defined before. By performing the transformation $\boldsymbol{Q}=\boldsymbol{I}_p-\boldsymbol{U}_2$, using [26, pp. 254, equation (57)] and (4), the above equation simplifies to

$$\begin{aligned}
&E\left[|\boldsymbol{R}|\right]^{h-1}\\
&= \frac{\Gamma_p\left(\frac{n_1+m_1}{2}\right)}{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i}{2}\right)}etr(-\tfrac{1}{2}\boldsymbol{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}\tfrac{1}{k!}\Gamma_p\left(\tfrac{n_2+m_2}{2},\kappa\right)C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Omega}\right)\frac{\prod_{i=1}^{p}\Gamma(d_i+h)\prod_{i=1}^{p}\Gamma(1-c_i-h)}{\prod_{i=p+1}^{2p}\Gamma(c_i+h)\prod_{i=p+1}^{2p}\Gamma(1-d_i-h)}
\end{aligned}$$

with c_i and d_i as defined. The density of r is uniquely determined by the inverse Mellin formula. (Note that the G-function does not need to have its definition domain split into $0<r<1$ and $r>1$.) $\square$

Remarks

1. If $\boldsymbol{\Omega}=\boldsymbol{0}$, then for $\boldsymbol{0}<\boldsymbol{R}<\boldsymbol{I}_p$, it follows that

$$f(\boldsymbol{R}) = \frac{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i+m_i}{2}\right)\Gamma_p\left(\frac{n_1+n_2}{2}\right)|\boldsymbol{R}|^{\frac{n_1}{2}-\frac{(p+1)}{2}}}{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i}{2}\right)\Gamma_p\left(\frac{m_1}{2}\right)\Gamma_p\left(\frac{n_1+n_2+m_2}{2}\right)}\,{}_2F_1\left(\tfrac{n_1+n_2}{2},-\tfrac{m_1}{2}+\tfrac{(p+1)}{2};\tfrac{n_1+n_2+m_2}{2};\boldsymbol{R}\right) \tag{14}$$

and for $\boldsymbol{R} > \boldsymbol{I}_p$,

$$f(\boldsymbol{R}) = \frac{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i+m_i}{2}\right)\Gamma_p\left(\frac{n_1+n_2}{2}\right)|\boldsymbol{R}|^{-\frac{n_2}{2}-\frac{(p+1)}{2}}}{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i}{2}\right)\Gamma_p\left(\frac{m_2}{2}\right)\Gamma_p\left(\frac{n_1+n_2+m_1}{2}\right)}\, {}_2F_1\left(\frac{n_1+n_2}{2}, -\frac{m_2}{2}+\frac{(p+1)}{2}; \frac{n_1+n_2+m_1}{2}; \boldsymbol{R}^{-1}\right) \tag{15}$$

where ${}_2F_1\,(.)$ is the Gauss hypergeometric function of matrix argument. Now, for the proof of the above densities, equation (8) is as follows

$$f(\boldsymbol{R}) = K\,|\boldsymbol{R}|^{\frac{n_1}{2}-\frac{(p+1)}{2}} \int\limits_{0<\boldsymbol{U}_2<\boldsymbol{I}_p} |\boldsymbol{U}_2|^{\frac{n_1+n_2}{2}-\frac{1}{2}(p+1)}\,|\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}\,|\boldsymbol{I}_p-\boldsymbol{R}\boldsymbol{U}_2|^{\frac{m_1}{2}-\frac{1}{2}(p+1)}\,d\boldsymbol{U}_2.$$

and by applying [17, pp. 36, equation (1.6.8)], equation (14) follows. Similarly (15) is obtained. (See [2].)
2. The density of $r = |\boldsymbol{R}| = |\boldsymbol{U}_1|\left|\boldsymbol{U}_2^{-1}\right|$ where $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ both *matrix variate beta type I distributed* $(\boldsymbol{\Omega} = \boldsymbol{0})$ is given by [2].
3. For $p = 1$, in (5)-(7), we have for $0 < r < 1$,

$$f(r) = \frac{\Gamma\left(\frac{n_1+m_1}{2}\right)\Gamma\left(\frac{n_1+n_2}{2}\right)}{\prod_{i=1}^{2}\Gamma\left(\frac{n_i}{2}\right)\Gamma\left(\frac{m_1}{2}\right)} e^{-\frac{\lambda}{2}} r^{\frac{n_1}{2}-1} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n_2+m_2}{2}+k\right)\left(\frac{\lambda}{2}\right)^k}{k!\,\Gamma\left(\frac{n_1+n_2+m_2}{2}+k\right)}\, {}_2F_1\left(\frac{n_1+n_2}{2}, 1-\frac{n_1}{2}; \frac{n_1+n_2+m_2}{2}+k; r\right)$$

and for $r > 1$,

$$f(r) = \frac{\Gamma\left(\frac{n_1+m_1}{2}\right)\Gamma\left(\frac{n_1+n_2}{2}\right)}{\prod_{i=1}^{2}\Gamma\left(\frac{n_i}{2}\right)\Gamma\left(\frac{n_1+n_2+m_1}{2}\right)} e^{-\frac{\lambda}{2}} r^{-\frac{n_2}{2}-1} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n_2+m_2}{2}+k\right)\left(\frac{\lambda}{2}\right)^k}{k!\,\Gamma\left(\frac{m_2}{2}+k\right)}\, {}_2F_1\left(\frac{n_1+n_2}{2}, 1-\frac{m_2}{2}-k; \frac{n_1+n_2+m_1}{2}; r^{-1}\right)$$

4. For $p = 2$, Figure 2 shows the density of $r = |\boldsymbol{R}|$ (see (7)) with $n_1 = n_2 = 8; m_1 = m_2 = 12$ and selected values of $\boldsymbol{\Omega}$.

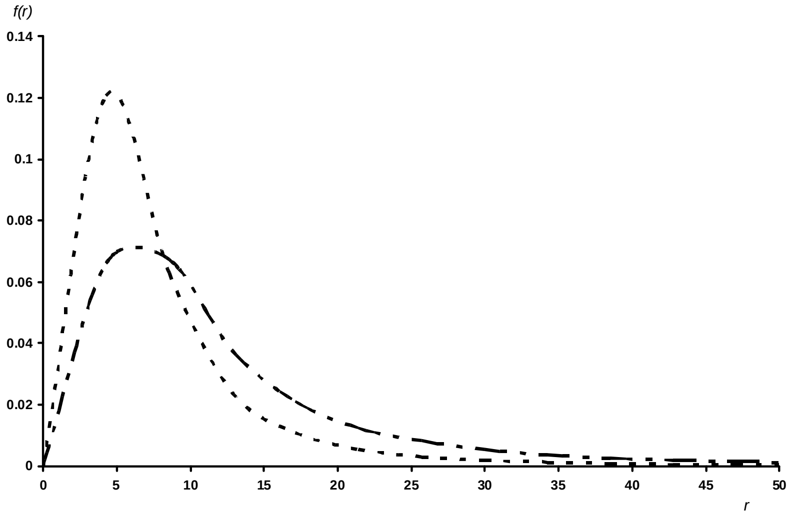

Figure 2: Density function of $|\boldsymbol{R}|$ for $p = 2$ and
(a)_ ____ _ $\boldsymbol{\Omega} = 12\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$;(b)_ _ _ $\boldsymbol{\Omega} = 12\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

4. PRODUCT (P) OF MATRIX VARIATE BETA TYPE I VARIATES

In this section we derived densities for the product and its determinant of these two independent components.

THEOREM 3

Let $\boldsymbol{U}_1 \sim \boldsymbol{B}_p^I(n_1, m_1)$ and $\boldsymbol{U}_2 \sim \boldsymbol{B}_p^I(n_2, m_2, \boldsymbol{\Omega})$ be independently distributed, $n_i, m_i \geq p,\ \ i = 1, 2$. Define $\boldsymbol{P} = \boldsymbol{U}_2^{\frac{1}{2}} \boldsymbol{U}_1 \boldsymbol{U}_2^{\frac{1}{2}}$, then the densities of $\boldsymbol{P}$ and $v = |\boldsymbol{P}|$ are respectively given by the following equations:

(a) For $\mathbf{0} < \boldsymbol{P} < \boldsymbol{I}_p$,

$$\begin{aligned} f(\boldsymbol{P}) &= \frac{\Gamma_p\left(\frac{m_1}{2}\right) etr(-\frac{1}{2}\boldsymbol{\Omega})}{\prod_{i=1}^{2} \beta_p\left(\frac{n_i}{2}, \frac{m_i}{2}\right)} |\boldsymbol{P}|^{\frac{n_1}{2} - \frac{(p+1)}{2}} |\boldsymbol{I}_p - \boldsymbol{P}|^{\frac{m_1+m_2}{2} - \frac{(p+1)}{2}} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\kappa}\sum_{J}\sum_{\phi\in\kappa.J} \theta_\phi^{\kappa,J} \\ &\quad \cdot \frac{\left(\frac{n_1+m_1-n_2}{2}\right)_\kappa \left(\frac{n_2+m_2}{2}\right)_J \Gamma_p\left(\frac{m_2}{2}, \phi\right)}{k!j!\left(\frac{m_2}{2}\right)_J \Gamma_p\left(\frac{m_1+m_2}{2}, \phi\right)} C_\phi^{\kappa,J}\left((\boldsymbol{I}_p - \boldsymbol{P}), \tfrac{1}{2}(\boldsymbol{I}_p - \boldsymbol{P})^{\frac{1}{2}} \boldsymbol{\Omega} (\boldsymbol{I}_p - \boldsymbol{P})^{\frac{1}{2}}\right). \end{aligned} \tag{16}$$

(b) For $0 < v < 1$,

$$f(v) = \frac{\Gamma_p\left(\frac{n_1+m_1}{2}\right) etr(-\frac{1}{2}\boldsymbol{\Omega})}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{m_1}{2}\right)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\tfrac{n_2+m_2}{2}, \kappa\right) C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Omega}\right) G_{2p,2p}^{2p,0}\left(v \,\middle|\, \begin{matrix} c_1, \ldots, c_{2p} \\ d_1, \ldots, d_{2p} \end{matrix}\right) \tag{17}$$

where $c_i = \begin{cases} \frac{n_1+m_1}{2} - \frac{(i+3)}{4} & , i = 1, 3, \ldots, 2p-1 \\ \frac{n_2+m_2}{2} + k_{\frac{i}{2}} - \frac{(i+2)}{4} & , i = 2, 4, \ldots, 2p \end{cases}$, $d_i = \begin{cases} \frac{n_1}{2} - \frac{(i+3)}{4} & , i = 1, 3, \ldots, 2p-1 \\ \frac{n_2}{2} - \frac{(i+2)}{4} & , i = 2, 4, \ldots, 2p. \end{cases}$

Proof:

(a)

$$\begin{aligned} & f(\boldsymbol{P}) \\ &= K \int_{\boldsymbol{P}<\boldsymbol{U}_2<\boldsymbol{I}_p} |\boldsymbol{U}_2|^{-\frac{1}{2}(p+1)} f_{\boldsymbol{U}_1}(\boldsymbol{U}_2^{-\frac{1}{2}} \boldsymbol{P} \boldsymbol{U}_2^{-\frac{1}{2}}) f_{\boldsymbol{U}_2}(\boldsymbol{U}_2) d\boldsymbol{U}_2 \qquad \text{(see [23])} \\ &= K |\boldsymbol{P}|^{\frac{n_1}{2} - \frac{(p+1)}{2}} etr(-\tfrac{1}{2}\boldsymbol{\Omega}) \int_{\boldsymbol{P}<\boldsymbol{U}_2<\boldsymbol{I}_p} |\boldsymbol{U}_2|^{\frac{n_2-n_1}{2} - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{U}_2|^{\frac{m_2}{2} - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \boldsymbol{P}\boldsymbol{U}_2^{-1}\right|^{\frac{m_1}{2} - \frac{1}{2}(p+1)} \\ &\qquad \cdot {}_1F_1\left(\tfrac{(n_2+m_2)}{2}; \tfrac{m_2}{2}; \tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p - \boldsymbol{U}_2)\right) d\boldsymbol{U}_2 \end{aligned}$$

where $K = \left\{\prod_{i=1}^{2} \beta_p\left(\frac{n_i}{2}, \frac{m_i}{2}\right)\right\}^{-1}$. Perform the transformation $\boldsymbol{T} = (\boldsymbol{I}_p - \boldsymbol{P})^{-\frac{1}{2}} (\boldsymbol{I}_p - \boldsymbol{U}_2) (\boldsymbol{I}_p - \boldsymbol{P})^{-\frac{1}{2}}$, with Jacobian $|\boldsymbol{I}_p - \boldsymbol{P}|^{\frac{1}{2}(p+1)}$, then we have that

$$\begin{aligned} & f(\boldsymbol{P}) \\ &= K |\boldsymbol{P}|^{\frac{n_1}{2} - \frac{(p+1)}{2}} etr(-\tfrac{1}{2}\boldsymbol{\Omega}) |\boldsymbol{I}_p - \boldsymbol{P}|^{\frac{(m_1+m_2)}{2} - \frac{1}{2}(p+1)} \int_{\mathbf{0}<\boldsymbol{T}<\boldsymbol{I}_p} |\boldsymbol{I}_p - (\boldsymbol{I}_p - \boldsymbol{P})\boldsymbol{T}|^{\frac{n_2-n_1-m_1}{2}} |\boldsymbol{T}|^{\frac{m_2}{2} - \frac{1}{2}(p+1)} \\ &\qquad \cdot |\boldsymbol{I}_p - \boldsymbol{T}|^{\frac{m_1}{2} - \frac{1}{2}(p+1)} {}_1F_1\left(\tfrac{(n_2+m_2)}{2}; \tfrac{m_2}{2}; \tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p - \boldsymbol{P})^{\frac{1}{2}} \boldsymbol{T} (\boldsymbol{I}_p - \boldsymbol{P})^{\frac{1}{2}}\right) d\boldsymbol{T} \end{aligned}$$

Consider the a transformation from $\boldsymbol{T} \to \boldsymbol{H}\boldsymbol{T}\boldsymbol{H}'$, $\boldsymbol{H} \in O(p)$, then expanding the hypergeometric function ${}_1F_1(\cdot)$ in terms of zonal polynomial it follows that the above integral equals

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\kappa}\sum_{J}\frac{\left(\frac{n_1+m_1-n_2}{2}\right)_\kappa\left(\frac{n_2+m_2}{2}\right)_J}{k!j!\left(\frac{m_2}{2}\right)_J}\int\limits_{\mathbf{0}<\boldsymbol{T}<\boldsymbol{I}_p}|\boldsymbol{T}|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{T}|^{\frac{m_1}{2}-\frac{1}{2}(p+1)}$$

$$\cdot\int\limits_{O(p)}C_\kappa((\boldsymbol{I}_p-\boldsymbol{P})\boldsymbol{H}\boldsymbol{T}\boldsymbol{H}')\,C_J\left(\tfrac{1}{2}(\boldsymbol{I}_p-\boldsymbol{P})^{\frac{1}{2}}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{P})^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{T}\boldsymbol{H}'\right)d\boldsymbol{H}d\boldsymbol{T}$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\kappa}\sum_{J}\sum_{\phi\in\kappa.J}\frac{\left(\frac{n_1+m_1-n_2}{2}\right)_\kappa\left(\frac{n_2+m_2}{2}\right)_J}{k!j!\left(\frac{m_2}{2}\right)_J}\int\limits_{\mathbf{0}<\boldsymbol{T}<\boldsymbol{I}_p}|\boldsymbol{T}|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{T}|^{\frac{m_1}{2}-\frac{1}{2}(p+1)}$$

$$\cdot\frac{C_\phi^{\kappa,J}\left((\boldsymbol{I}_p-\boldsymbol{P}),\frac{1}{2}(\boldsymbol{I}_p-\boldsymbol{P})^{\frac{1}{2}}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{P})^{\frac{1}{2}}\right)C_\phi^{\kappa,J}(\boldsymbol{T},\boldsymbol{T})}{C_\phi(\boldsymbol{I}_p)}d\boldsymbol{T}\quad\text{(use [7, equation (1.2)])}$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\kappa}\sum_{J}\sum_{\phi\in\kappa.J}\theta_\phi^{\kappa,J}\frac{\left(\frac{n_1+m_1-n_2}{2}\right)_\kappa\left(\frac{n_2+m_2}{2}\right)_J}{k!j!\left(\frac{m_2}{2}\right)_J}\frac{C_\phi^{\kappa,J}\left((\boldsymbol{I}_p-\boldsymbol{P}),\frac{1}{2}(\boldsymbol{I}_p-\boldsymbol{P})^{\frac{1}{2}}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{P})^{\frac{1}{2}}\right)}{C_\phi(\boldsymbol{I}_p)}$$

$$\cdot\int\limits_{\mathbf{0}<\boldsymbol{T}<\boldsymbol{I}_p}|\boldsymbol{T}|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{T}|^{\frac{m_1}{2}-\frac{1}{2}(p+1)}C_\phi(\boldsymbol{T})\,d\boldsymbol{T}\qquad\text{(use[7, equation (2.1)]}$$

Applying [26, pp.254, equation (57)], expression (16) is obtained.

(b)

$$E\left[|\boldsymbol{P}|\right]^{h-1}$$

$$=Ketr(-\tfrac{1}{2}\boldsymbol{\Omega})\int\limits_{0<\boldsymbol{U}_1<\boldsymbol{I}_p}|\boldsymbol{U}_1|^{\frac{n_1}{2}+h-1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}_1|^{\frac{m_1}{2}-\frac{1}{2}(p+1)}d\boldsymbol{U}_1$$

$$\cdot\int\limits_{0<\boldsymbol{U}_2<\boldsymbol{I}_p}|\boldsymbol{U}_2|^{\frac{n_2}{2}+h-1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}\,{}_1F_1\left(\tfrac{(n_2+m_2)}{2};\tfrac{m_2}{2};\tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{U}_2)\right)d\boldsymbol{U}_2$$

$$=etr(-\tfrac{1}{2}\boldsymbol{\Omega})\frac{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i+m_i}{2}\right)\Gamma_p\left(\frac{n_1}{2}+h-1\right)}{\prod_{i=1}^{2}\Gamma_p\left(\frac{n_i}{2}\right)\Gamma_p\left(\frac{m_2}{2}\right)\Gamma_p\left(\frac{n_1+m_1}{2}+h-1\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{(n_2+m_2)}{2}\right)_\kappa}{k!\left(\frac{m_2}{2}\right)_\kappa}$$

$$\cdot\int\limits_{0<\boldsymbol{U}_2<\boldsymbol{I}_p}|\boldsymbol{U}_2|^{\frac{n_2}{2}+h-1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}_2|^{\frac{m_2}{2}-\frac{1}{2}(p+1)}C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p-\boldsymbol{U}_2)\right)d\boldsymbol{U}$$

where K is defined as in part(a). Following the same method as in theorem 2(b), the desired result (17) is obtained. □

Remarks

1. If $\boldsymbol{\Omega}=\mathbf{0}$, then result (16) simplifies to the form obtained by [17, pp 298], known as the *hypergeometric function distribution of type* I. The density of $v=|\boldsymbol{P}|$ is given by [2].

2.For $p=1$ and using [24, pp. 131], (16) and (17) simplify to the following:

$$f(v)=\quad\frac{\Gamma\left(\frac{(n_1+m_1)}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{m_1}{2}\right)}e^{-\frac{\lambda}{2}}v^{\frac{n_2}{2}-1}(1-v)^{\frac{(m_1+m_2)}{2}-1}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{n_2+m_2}{2}+k\right)}{k!\Gamma\left(\frac{m_1+m_2}{2}+k\right)}\left(\tfrac{1}{2}\lambda(1-v)\right)^k$$

$$\cdot\;{}_2F_1\left(\tfrac{n_2+m_2-n_1}{2}+k,\tfrac{m_1}{2};\tfrac{m_1+m_2}{2}+k;1-v\right)\quad,0<v<1.$$

3. Figure 3 illustrates the density of $v=|\boldsymbol{P}|$ (see (17))for $p=2,n_1=n_2=8,m_1=m_2=12$ and selected values of $\boldsymbol{\Omega}=12\begin{bmatrix}4&2\\2&1\end{bmatrix}$.

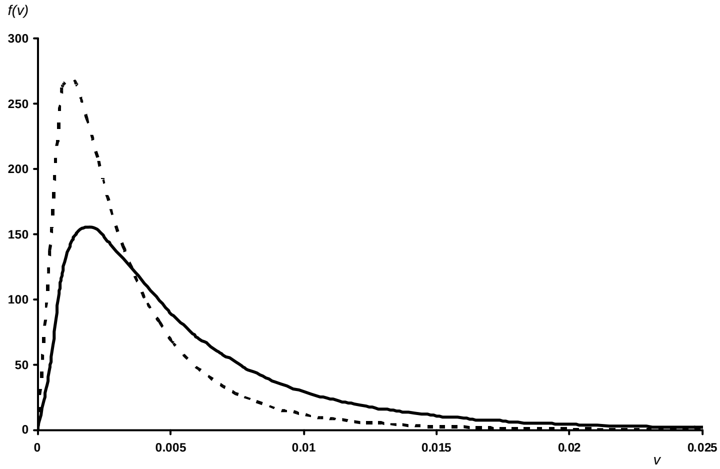

Figure 3: Density function of $|\boldsymbol{P}|$ for $p = 2$ and
(a)_ _ _ $\boldsymbol{\Omega} = 12\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$;(b)_______ $\boldsymbol{\Omega} = 12\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

5. CONCLUSION

In this paper we give an exact expression in terms of Meijer's G-function for the nonnull distribution of the Wilks's statistic based on the nonsymmetrised density of the noncentral matrix variate beta type I distribution. We establish exact expressions for the densities of the ratio ($\boldsymbol{R}$) and product ($\boldsymbol{P}$) of two independent matrix variates where one has the matrix variate beta type I density and the other has the nonsymmetrised matrix variate beta type I density. The densities of $|\boldsymbol{R}|$ and $|\boldsymbol{P}|$ are also derived for these cases. Diaz-Garzia and Gutiérrez-Jámez [9] gave further classifications of the beta matrix. When using these, other definitions of noncentral matrix variate beta type distribution, the derivation of the corresponding results will be in an analogous fashion. The availability of these expressions in this paper will stimulate research and applications.

ACKNOWLEDGEMENT

The authors thank anonymous referees for their constructive and significant comments that have helped in improving the paper. This work is based upon research supported by the National Research Foundation, South Africa and the Research and Development Programme (University of Pretoria).

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